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Non-degeneracy, Moments, Dimension, and Multifractal Analysis for Random Multiplicative Measures

(Random Multiplicative Multifractal Measures, Part II)

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Abstract. This is the second of three papers devoted to a class of random measures generated by multiplicative processes.

Part I surveys the main motivations which led B. Mandelbrot to introduce such statistically self-affine multifractal measures, from the initial limit lognormal processes to multiplicative cascades of random weights, and finally the multifractal products of pulses. A discussion contrasts the recent class of multifractal products of cylindrical pulses with the well-known canonical cascade measures.

This Part II presents the examples of Part I as particular elements of a class of random measures generated by multiplications of functions for which several fundamental problems, namely non-degeneracy, finiteness of moments, dimension of the carrier and multifractal analysis can be studied and solved. The results complete Kahane's general theory of T-martingales and are applied to new examples.

Part III will provide the proofs of the results obtained in Part II.

1. Introduction

Random singular measures obtained as limits of nonnegative multiplicative martingales were introduced in [**M1, M2, M3**] to provide models for turbulence. [**M1, M2, M3**] raised and partly solved three fundamental problems concerning the limit measure μ. Is μ non-degenerate, that is positive with positive probability? If so, under what condition the moments of high orders of $\|\mu\|$ diverge, and what is the smallest Hausdorff dimension of a set carrying all the mass of μ ? Best known is a supclass of measures constructed in [**M2, M3**], namely canonical cascade measures, CCM. See Section 6.1 of this paper or Part I [**BM2**] for a construction. For the CCM, [**KP**] confirmed the conjectures in [**M2, M3**] in definitive fashion. Recall

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also that CCM were introduced to simplify the construction in [**M1**] and allow the above mentioned questions to be answered. Returning later to the measures defined in [**M1**], whose construction involves lognormal multiplicative martingales, [**K2**] developed a theory of Gaussian multiplicative chaos. The fundamental problems were solved for certain examples reducible to CCM, and partial answers were given for certain other classes of constructions. This theory was extended in [**K3**] to non-lognormal multiplicative martingales. A Lévy stable multiplicative chaos was introduced in [**Fa2, Fa4**] that illustrates the theory with another class of examples.

At their level of generality, [**K2, K3**] and [**Fa2, Fa4**] obtain non-degeneracy for certain classes of martingales via sufficient conditions for L^p (p integer ≥ 2) convergence. Indeed the L^2 theory of these martingales is particularly manageable. Then, general sufficient conditions are given to find a lower bound for the dimension of the limit measure carrier.

Let us now return to the statistically self-similar CCM for which there exists a necessary and sufficient condition for L^1 convergence. For fine results – such as sufficient condition for L^p convergence with p close to 1 – one seeks explicit forms that would apply, if perhaps not for the general construction of [**K3**], at least for a useful large subclass. Such a class is investigated in [**WaWi**], which generalizes CCM by allowing the random weights to be governed by a Markov chain defined recursively along the branch of the b-adic tree. Another such class consists in the new examples of statistically self-similar measures introduced in [**M4**], namely the (grid free) multifractal products of cylindrical pulses, MPCP, recently extended in [**BaMu**]. MPCP are studied in [**BM1**], and their non-statistically self-similar extension, in [**B6**]. Part I ([**BM2**]) discusses CCM, MPCP and also a third class of (semi-grid free) measures that illustrate the theory in [**K3**], namely Poisson canonical cascade measures, PCCM.

To study MPCP, [**BM1**] introduces reductions that make applicable the material originally developed for CCM. Moreover, it turns out that the MPCP structure is complex enough to enable the approach in [**BM1, B6**] to derive general results for a more general subclass $\mathcal{M}$ of random measures in the theory in [**K3**].

Section 2 completes the theory of $\mathcal{M}$ provided in [**K3**]. We provide an L^p ($p \in (1, 2]$) sufficient condition for non-degeneracy in $\mathcal{M}$. Section 3 is devoted to examples, that include CCM, PCCM, MPCP and their extension in [**BaMu**]. Another topic consists in the multifractal products of pulses, MPP, that [**M4**] introduces together with MPCP. Those pulses need not be cylindrical. Section 4 deals with the almost sure simultaneous construction of uncountable families of non-degenerate measures of this type and provides a lower bound estimate for the dimension of the carrier. This problem plays a fundamental role in the multifractal analysis of statistically self-similar limit measures in $\mathcal{M}$. Section 5 gives satisfactory answers to the problems of non-degeneracy, finiteness of moments (of positive and negative orders), dimension of the carrier, and multifractal analysis for a subclass $\mathcal{M}'$ of statistically self-similar measures in $\mathcal{M}$. Section 6 applies the results in Section 5 to new constructions that illustrate $\mathcal{M}'$.

2. Non-degeneracy for infinite products of random independent functions

In this section, a sufficient condition for L^p convergence ($1 < p \leq 2$), and so non-degeneracy, is obtained for a subclass of the general construction of multiplicative chaos in [**K3**] ([**K3**], Theorem 3, provides general sufficient conditions for degeneracy, as well as a necessary and sufficient condition for L^2 convergence). This subclass includes the fundamental examples mentioned in the introduction (see also Section 3).

2.1. The general construction. In [**K3**] theory, random measures are constructed on a locally compact metric space. Here, we are interested in such random measures constructed on $\mathbb{R}$. Without loss of generality, we give their construction on $[0,1]$, and at the end of Section 3 devoted to examples we will specify when a construction has a natural extension to $\mathbb{R}$. Section 4 will obtain these measures as projections of measures constructed in the same way, but on the boundary of an homogeneous tree.

In the sequel, if K is a compact metric space, weak convergence of measures on K means weak* convergence in the dual of $C(K)$, the set of real valued continuous functions on K.

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be the probability space on which the random variables in this paper are defined.

For every Borel subset B of $\mathbb{R}^d$, denote by $\mathcal{B}(B)$ the σ-field generated by the Borel subsets of B.

Let σ be a bounded positive Borel measure on $[0,1]$. Consider a family of measurable functions $Q_\varepsilon : \big([0,1]\times\Omega, \mathcal{B}([0,1])\otimes\mathcal{B}\big) \to \big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$, $0<\varepsilon\leq 1$, and q, a nonnegative function in $L^1([0,1],\sigma)$ such that $\int_{[0,1]} q(t)d\sigma(t) > 0$.

For $\varepsilon \in (0,1]$ let $\mathcal{F}_\varepsilon$ be the σ-field generated in $\mathcal{B}$ by the family of random variables $\{Q_\lambda(t,\cdot)\}_{t\in[0,1],\ \lambda\in[\varepsilon,1]}$.

Assume that $\{Q_\varepsilon\}_{0<\varepsilon\leq 1}$ satisfies the following property

(P1) There exists a set $D \subset [0,1]$ of full σ-measure such that for all $t \in D$, $\big(Q_{s^{-1}}(t,\cdot)\big)_{s>1}$ is a right-continuous martingale with respect to $\big(\mathcal{F}_{s^{-1}}\big)_{s>1}$, with expectation 1. Moreover, with probability one, for every $\varepsilon_0 \in (0,1]$, there exists a positive integrable function $h(\cdot,\omega)$ such that $\sup_{\varepsilon_0\leq\varepsilon\leq 1} Q_\varepsilon(t,\omega) \leq h(t,\omega)$ for σ-almost every $t\in[0,1]$.

Denote by μ_ε the measure on $[0,1]$ whose density with respect to σ is

$$\frac{d\mu_\varepsilon}{d\sigma}(t) = Q_\varepsilon(t)\, q(t).$$

It follows from [**K3**] that with probability one, the measures μ_ε converges weakly, as $\varepsilon \to 0$, to a non-negative measure μ. It also follows that if a given point $t\in[0,1]$ is not an atom of σ, then t is not an atom of μ almost surely.

Since we can substitute to σ the measure defined by $q(t).d\sigma(t)$, without loss of generality we assume that $q(t) = 1$ in the sequel. Sometimes the measures μ_ε and μ will be denoted respectively by $Q_\varepsilon\cdot\sigma$ and $Q\cdot\sigma$.

REMARK 2.1. (1) Here, the martingales are indexed by parameters ε tending to 0 in connexion with the fact that in the examples, letting ε tend to 0 influences the values of the martingale (μ_ε) only at small scales. In fact, in [**K3**] the martingale is parametrized by a discrete parameter $n \in \mathbb{N}^*$ and this martingale (Q_n) is taken so that $\mathbb{E}(Q_n(t)) = q(t)$.

The fact that the parameter ε ranges in an interval upper bounded by 1 is an arbitrary choice. This upper bound could be replaced by any other $T > 0$ in the definition.

(2) The existence of the function h in **(P1)** is required to ensure, via the dominated convergence theorem, that for every $f \in C([0,1])$, the family of random variables $\left(\int_{[0,1]} f(t) d\mu_{s^{-1}}(t) \right)_{s>1}$ forms a right-continuous martingale. Then, the approach in [**K3**] works in the continuous parameter context.

2.2. The subclass $\mathcal{M}$. We add the following assumptions:

(P2) the family $\{Q_\varepsilon\}$ possesses a factorization

$$Q_{\varepsilon'} = Q_\varepsilon Q_{\varepsilon,\varepsilon'} \quad (0 < \varepsilon' \leq \varepsilon \leq 1),$$

with $Q_{\varepsilon,\varepsilon} = 1$, and $\{Q_{\varepsilon,\varepsilon'}\}$ having the same measurability property as $\{Q_\varepsilon\}$.

For $\varepsilon \in (0,1]$ and every nontrivial subinterval I of $[0,1]$ let $\overline{\mathcal{F}}_\varepsilon^I$ be the σ-field generated in $\mathcal{B}$ by the families of random variables $\{Q_{\varepsilon,\lambda}(t,\cdot)\}_{t\in I,\ \lambda\in(0,\varepsilon)}$. Denote $\overline{\mathcal{F}}_\varepsilon^{[0,1]}$ by $\overline{\mathcal{F}}_\varepsilon$.

(P3) For every $0 < \varepsilon \leq 1$, $\mathcal{F}_\varepsilon$ and $\overline{\mathcal{F}}_\varepsilon$ are independent.

(P4) There exists $\beta > 0$ such that for every $\varepsilon \in (0,1]$ and every family $\mathcal{G}$ of nontrivial subintervals of $[0,1]$ of common length ε such that $d(I,J) \geq \beta\varepsilon$ for every $I \neq J \in \mathcal{G}$, the σ-algebra's $\overline{\mathcal{F}}_\varepsilon^I$, $I \in \mathcal{G}$, are mutually independent ($d(I,J) = \inf\{|t-s| :\ t \in I,\ s \in J\}$).

If properties **(P1)** to **(P4)** are satisfied, we say that $Q \cdot \sigma$ belongs to the class $\mathcal{M}$.

If, moreover, the stationary property

(P5) The probability distribution of $Q_\varepsilon(t,\cdot)$ does not depend on t (more precisely t in the set D introduced in **(P1)**)

is satisfied, then for $p \geq 0$ define (independently of $t \in D$)

$$\theta_Q(p) = \limsup_{\varepsilon\to 0} \frac{\log \mathbb{E}\big(Q_\varepsilon(t)^p\big)}{\log(1/\varepsilon)}.$$

Notice that due to the martingale property of $\big(Q_\varepsilon(t)\big)$, $\mathbb{E}\big(Q_\varepsilon(t)^p\big)$ is monotonic. Consequently, $\theta_Q(p)$ is also equal to $\limsup_{n\to\infty} \frac{\log_b \mathbb{E}\left(Q_{b^{-n}}(t)^p\right)}{n}$ for every $b > 1$.

2.3. Toward non-degeneracy in $\mathcal{M}$. L^p convergence results. In this section, we assume that 1 is not an atom of σ. For every integer $b \geq 2$ and $n \geq 0$, let $A^n = \{0,\dots,b-1\}^n$ and for $w \in A^n$ let I_w denote the b-adic semi-open to the right interval naturally encoded by w.

THEOREM 2.2 (L^p **convergence**).

(1) *Assume that properties* **(P1)** *to* **(P4)** *hold. Let* $p \in (1,2]$. *Assume there exists an integer* $b \geq 2$ *such that*

$$\sum_{n\geq 0}\left(\sum_{w\in A^n}\sigma(I_w)^{p-1}\int_{I_w}\mathbb{E}\big(Q_{b^{-n-1}}(t)^p\big)\,d\sigma(t)\right)^{1/p}<\infty. \tag{2.1}$$

Then $\mu_\varepsilon([0,1])$ *converges to* $\mu([0,1])$, *as* $\varepsilon\to 0$, *almost surely and in* L^p *norm.*

(2) *Assume property* **(P1)** *holds.* $\mu_\varepsilon([0,1])$ *converges to* $\mu([0,1])$, *as* $\varepsilon \to 0$, *almost surely and in* L^2 *norm, if and only if*

$$\sup_{\varepsilon\in(0,1)}\int_{[0,1]^2}\mathbb{E}\big(Q_\varepsilon(s)Q_\varepsilon(t)\big)\,d\sigma(s)d\sigma(t)<\infty.$$

In Theorem 2.2, if σ is the Lebesgue measure ℓ, the sufficient condition (2.1) becomes

$$\sum_{n\geq 0}b^{-n(1-1/p)}\left(\int_{[0,1]}\mathbb{E}\big(Q_{b^{-n}}(t)^p\big)\,dt\right)^{1/p}<\infty, \tag{2.2}$$

which holds for some $b\geq 2$ if and only if it holds for all $b \geq 2$.

Following the [**BrMiP**] setting (see also the references therein), given an integer $b\geq 2$ we define the multifractal function of σ by

$$\varphi_\sigma(p)=\limsup_{n\to\infty}\frac{1}{n}\log_b\sum_{w\in A^n}\sigma(I_w)^p.$$

In fact, this quantity does not depend on $b \geq 2$ for $p\geq 0$. If **(P5)** holds, then a simpler sufficient condition than (2.1) for L^p convergence is given by the following corollary.

Corollary 2.3. *Assume properties* **(P1)** *to* **(P5)** *hold. Let* $p\in(1,2]$. *Assume*

$$\varphi_\sigma(p)+\theta_Q(p)<0\quad\big(1-p+\theta_Q(p)<0 \text{ if } \sigma=\ell\big). \tag{2.3}$$

Then $\mu_\varepsilon([0,1])$ *converges to* $\mu([0,1])$, *as* $\varepsilon\to 0$, *almost surely and in* L^p *norm.*

Remark 2.4. (1) The second part of Theorem 2.2 is found and pointed out as “particularly manageable” and useful in [**K2, K3**]. Indeed, it provides enough information in many cases [**K2, Fa4, K5, Fa5**].

The first part of Theorem 2.2 now provides a simple sufficient condition for L^p convergence ($p\in(1,2]$) in a subclass of constructions. This result generalizes the one obtained in [**B6**] and recalled in the next section. Moreover, when considering old or recent statistically self-similar examples, for $p\in(1,2]$ one recovers the same sufficient (and necessary) condition derived by another approach using the self-similarity (see [**KP**], [**BM1**], [**BaMu**]). This result is based on a combination of an inequality by von Bahr and Essen ([**vBahrE**]) and properties **(P2)** to **(P4)**. These properties capture those used in several particular examples of such martingales to derive L^p convergence via the [**vBahrE**] inequality ([**Bi2**],[**B5**],[**B6**],[**Fa6**]). For example (2.3) is derived in [**Fa6**] in the particular case where the martingale density $Q_{b^{-n}/2}$ is the CCM one. We notice that the construction of [**Fa2, Fa4**] belongs to $\mathcal{M}$.

(2) As will be shown in proving Theorem 4.2 in Part III ([**B7**]), Theorem 2.2 can be extended to similar complex-valued martingales. This is done in [**Bi3**] and [**B6**] in particular cases.

(3) There is almost no hope to find a general criterion avoiding a condition of high frequences decorrelation like **(P4)**. Indeed, consider the example of generalized Riesz products with random phases studied in [**BCM**]: b is an integer ≥ 2, W a nonnegative integrable 1-periodic function, and $(\phi_n)_{n\geq 0}$ a sequence of uniformly distributed in $[0,1)$ independent random variables. There, Q_ε is given by

$$Q_\varepsilon(t) = \prod_{k=0}^{n} W\big(b^k(t+\phi_k)\big) \quad (\varepsilon \in (b^{-n-1}, b^{-n}]).$$

Conditions **(P1)** to **(P3)** hold but **(P4)** does not. For $\sigma = \ell$, non-degeneracy holds if and only if the martingale $\mu_\varepsilon([0,1])$ is almost surely equal to 1. Moreover, this fact is characterized in terms of the vanishing of certain Fourier coefficients of W.

(4) When σ is the Lebesgue measure and **(P5)** holds, our result (Corollary 2.3) is of the same kind as the one obtained in [**WaWi**] for certain measures in the class described in the introduction of this paper.

3. Examples of martingales satisfying conditions (P1) to (P4).

3.1. Products of Pulses associated with point processes.

Definition 3.1 (**Cylindrical pulse**). Given a nontrivial subinterval I of $\mathbb{R}$, a cylindrical pulse based on I is a simple function $P : \mathbb{R} \to \mathbb{R}_+$ of the form

$$P(t) = W(I)\mathbf{1}_I(t) + \mathbf{1}_{I^c}(t),$$

where $W(I)$ is a constant and I^c denotes the complement of I. Given a point $M = (t,\lambda) \in \mathbb{R} \times \mathbb{R}_+^*$, P_M will denote a cylindrical pulse based on the interval $I_M := [t-\lambda, t+\lambda)$.

Definition 3.2 (**Pulse**). Given a nontrivial subinterval $I = [a,b)$ of $\mathbb{R}$ and a nonnegative function $\widetilde{W} \in L^1([0,1])$, the pulse based on $(I, \widetilde{W})$ is the function $P : \mathbb{R} \to \mathbb{R}_+$ defined by

$$P(t) = \widetilde{W}\Big(\frac{t-a}{b-a}\Big)\mathbf{1}_{[a,b)}(t) + 1_{[a,b)^c}(t).$$

Given a point $M = (t,\lambda) \in \mathbb{R} \times \mathbb{R}_+^*$, $\widetilde{P}_M$ will denote the pulse based on $(I_M, \widetilde{W})$.

Let S be a locally finite subset of $\mathbb{R} \times \mathbb{R}_+^*$, either deterministic, or defined by a random Poisson point process with locally bounded positive Borel intensity measure Λ.

Let $\{B_k\}_{k\geq 1}$ be a partition of $\mathbb{R} \times (0,1]$ into Borel subsets whose Λ-measure is positive and finite.

If S is deterministic, define N_k to be the cardinality $\# S \cap B_k$ of $S \cap B_k$. If $N_k \geq 1$, denote by $M_{k,1}, \dots, M_{k,N_k}$ the elements of $S \cap B_k$. The numbers N_k and the $M_{k,n}$, $1 \leq n \leq N_k$, are considered as constant random variables.

In order to obtain a Poisson point process S with intensity Λ ([**Ki**]), let $\Lambda_{|B_k}$ denote the restriction of Λ to $\mathcal{B}(B_k)$ and choose a sequence $(M_{k,n})_{n\geq 1}$ of B_k-valued random variables with common distribution $\frac{\Lambda_{|B_k}}{\Lambda(B_k)}$. Then denote by N_k a Poisson variable with parameter $\Lambda(B_k)$. Assume that the previous random variables are mutually independent. Finally, define $S = \{M_{k,n};\ 1 \leq k,\ 1 \leq n \leq N_k\}$.

Products of cylindrical pulses. This subclass of $\mathcal{M}$ includes CCM, PCCM and MPCP, which are discussed in Part I [**BM2**] and also in Section 6 later in this paper.

Let W be a non-negative integrable random variable, and fix $(W_{k,n})_{n\geq 1}$ a sequence of copies of W. It will be assumed that $\mathbb{E}(W) > 0$ when the set S is deterministic.

Assume that all the random variables $M_{k,n}$, N_k and $W_{k,n}$, $k, n \geq 1$, are mutually independent.

Now, for $M = (t_M, \lambda_M) = M_{k,n} \in S$ denote by P_M the cylindrical pulse based on I_M such that $W(I_M) = W_{k,n}$, also denoted W_M.

Then obtain $\{Q_\varepsilon\}_{0<\varepsilon\leq 1}$, which satisfies properties **(P1)** to **(P4)** (with $\beta = 2$), as

$$Q_\varepsilon(t) = \frac{\prod_{M\in S\cap\{\lambda\geq\varepsilon\}} P_M(t)}{\mathbb{E}\left(\prod_{M\in S\cap\{\lambda\geq\varepsilon\}} P_M(t)\right)}.$$

This can be reformulated as follows: for all $\varepsilon \in (0,1]$ and $t \in \mathbb{R}$, define the truncated cone $\mathcal{C}_\varepsilon(t) = \{(s,\lambda) \in \mathbb{R} \times (0,1];\ t-\lambda < s \leq t+\lambda,\ \varepsilon \leq \lambda < 1\}$. One has

$$\begin{aligned} Q_\varepsilon(t) &= \frac{\prod_{M\in S\cap\mathcal{C}_\varepsilon(t)} W_M}{\mathbb{E}\left(\prod_{M\in S\cap\mathcal{C}_\varepsilon(t)} W_M\right)} \\ &= \begin{cases} \big(\mathbb{E}(W)\big)^{-\#\, S\cap\mathcal{C}_\varepsilon(t)} \displaystyle\prod_{M\in S\cap\mathcal{C}_\varepsilon(t)} W_M & (S \text{ deterministic}), \\ \exp\big(-\Lambda\left(\mathcal{C}_\varepsilon(t)\right)(\mathbb{E}(W)-1)\big) \displaystyle\prod_{M\in S\cap\mathcal{C}_\varepsilon(t)} W_M & (S\ \ \text{Poisson}). \end{cases} \end{aligned}$$

REMARK 3.3. (1) We see that changing W into γW with $\gamma > 0$ does not affect the value of the density when S is deterministic. Consequently, normalizing with the term $(\mathbb{E}(W))^{-\#\, S\cap\mathcal{C}_\varepsilon(t)}$ is equivalent to changing W into $W/\mathbb{E}(W)$, i.e. assuming $\mathbb{E}(W) = 1$.

Also, when S is a Poisson point process, no normalization is necessary when $\mathbb{E}(W) = 1$.

(2) Given $\gamma > 0$, one can change the interval $I_M = [t_M - \lambda_M, t_M + \lambda_M)$ into $I_M = [t_M - \gamma\lambda_M, t_M + \gamma\lambda_M)$ without changing the structure of the construction. This only affects β which is changed into $\gamma\beta$. Then $\mathcal{C}_\varepsilon(t) = \{(s,\lambda) \in \mathbb{R} \times (0,1];\ t - \gamma\lambda < s \leq t + \gamma\lambda,\ \varepsilon \leq \lambda < 1\}$.

(3) For the computations of the expectations of products of functions associated with Poisson processes involved in all this section, the reader is referred to the proof of Lemma 7.1 in Part III ([**B7**]).

Products of non-cylindrical pulses. Fix $\widetilde{W}$ a nonnegative function in $L^1([0,1),\ell)$, and extend it on $\mathbb{R}$ by 1-periodicity. Assume that $\widetilde{W}$ is non identically 0 when S is deterministic.

If S is deterministic, consider $(\phi_M)_{M\in S}$ a sequence of independent random phases which are uniformly distributed in $[0,1)$. Then for $M\in S$ define $\widetilde{P}_M$ as the pulse based on $\big(I_M,\widetilde{W}(\cdot+\phi_M)\big)$.

If S is the Poisson point process with intensity Λ, define $\widetilde{P}_M$ to be the pulse based on $(I_M,\widetilde{W})$.

Then obtain $\{\widetilde{Q}_\varepsilon\}_{0<\varepsilon\leq 1}$, which satisfies properties **(P1)** to **(P4)** (with $\beta=2$), as

$$\widetilde{Q}_\varepsilon(t)=\frac{\prod_{M\in S\cap\{\lambda\geq\varepsilon\}}\widetilde{P}_M(t)}{\mathbb{E}\left(\prod_{M\in S\cap\{\lambda\geq\varepsilon\}}\widetilde{P}_M(t)\right)}. \tag{3.1}$$

The expression (3.1) simplifies to be

$$\widetilde{Q}_\varepsilon(t)=\left(\int_{[0,1]}\widetilde{W}(u)\,du\right)^{-\#\,S\cap\mathcal{C}_\varepsilon(t)}\prod_{M\in S\cap\mathcal{C}_\varepsilon(t)}\widetilde{W}\left(\frac{t+\phi_M-t_M+\lambda_M}{2\lambda_M}\right)$$

when S is deterministic and the property

$$\forall\ \ M\in S,\ 1/(2\lambda_M)\in\mathbb{N} \tag{3.2}$$

holds; the expression (3.1) simplifies to be

$$\widetilde{Q}_\varepsilon(t)=\exp\left[-\Lambda\left(\mathcal{C}_\varepsilon(t)\right)\left(\int_{[0,1]}\widetilde{W}(u)\,du-1\right)\right]\prod_{M\in S\cap\mathcal{C}_\varepsilon(t)}\widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)$$

when S is a Poisson point process and the property

$$\Lambda \text{ is invariant by horizontal translations} \tag{3.3}$$

holds.

Remark 3.4. The choice of $(B_k,N_k,(M_{k,n})_{n\geq1})_{k\geq1}$, $((W_{k,n})_{n\geq1})_{k\geq1}$ and $((\phi_{M_{k,n}})_{n\geq1})_{k\geq1}$ affects neither the probability distribution of the stochastic processes $(Q_\varepsilon(t))_{\varepsilon\in(0,1],\ t\in\mathbb{R}}$ and $(\widetilde{Q}_\varepsilon(t))_{\varepsilon\in(0,1],\ t\in\mathbb{R}}$, nor those of the other random variables defined in this paper.

Combining both constructions together. When S is deterministic, if the ϕ_Ms are chosen independent of the W_Ms, one also can consider the product $\widehat{Q}_\varepsilon=Q_\varepsilon\widetilde{Q}_\varepsilon$ to get a more elaborate element of $\mathcal{M}$.

When S is a Poisson point process, Q_ε, $\widetilde{Q}_\varepsilon$ and their product $\widehat{Q}_\varepsilon$ have been considered in [**M4**] when $\mathbb{E}(W)=\int_{[0,1]}\widetilde{W}(t)\,dt=1$ and the intensity of S is

$$\Lambda_\rho(dtd\lambda)=\frac{\rho}{2}\frac{dtd\lambda}{\lambda^2}\quad(\rho>0).$$

The element $\widehat{Q}\cdot\ell$ of $\mathcal{M}$ so obtained is what [**M4**] calls *multifractal products of pulses*, MPP (see Section 6).

For a general choice of the intensity Λ and the pair $(W, \widetilde{W})$, the correct combination is not the product $Q_\varepsilon \widetilde{Q}_\varepsilon$ but the martingale obtained with the products of pulses $P_M(t)\widetilde{P}_M(t)$

$$\widehat{Q}_\varepsilon(t) = \frac{\prod_{M\in S\cap\{\lambda\geq\varepsilon\}} P_M(t)\widetilde{P}_M(t)}{\mathbb{E}\left(\prod_{M\in S\cap\{\lambda\geq\varepsilon\}} P_M(t)\widetilde{P}_M(t)\right)}.$$

Under (3.3) this simplifies to be

$$\widehat{Q}_\varepsilon(t) = \frac{\displaystyle\prod_{M\in S\cap\mathcal{C}_\varepsilon(t)} W_M \widetilde{W}\left(\frac{t-t_M+\lambda_M}{2\lambda_M}\right)}{\exp\left[\Lambda\left(\mathcal{C}_\varepsilon(t)\right)\left(\mathbb{E}(W)\displaystyle\int_{[0,1]}\widetilde{W}(u)\,du - 1\right)\right]}.$$

Now let us restate Theorem 2.2 for these constructions.

The measure $\widehat{Q}_\varepsilon \cdot \sigma$ is also denoted by μ_ε, and the limit measure $\widehat{Q}\cdot\sigma$ by μ.

For $t\in\mathbb{R}$, define $\mathcal{C}(t)$ as being $\bigcup_{0<\varepsilon<1}\mathcal{C}_\varepsilon(t)$.

For $p\geq 0$, define

$$\begin{cases}\theta(p) = \log\mathbb{E}(W^p) - p\log\mathbb{E}(W)\\ \widetilde{\theta}(p) = \log\int_{[0,1]}\widetilde{W}(u)^p\,du - p\log\int_{[0,1]}\widetilde{W}(u)\,du.\end{cases}$$

Theorem 3.5 (L^p **convergence, deterministic** S). *Assume (3.2) holds if* $\widetilde{W}\not\equiv 1$.

(1) *Let* $p\in(1,2]$. *Suppose there exists an integer* $b\geq 2$ *such that*

$$\sum_{n\geq 0}\left(\sum_{w\in A^n}\sigma(I_w)^{p-1}\int_{I_w}\exp\left((\theta(p)+\widetilde{\theta}(p))\#\,S\cap\mathcal{C}_{b^{-n-1}}(t)\right)\,d\sigma(t)\right)^{1/p}<\infty.$$

Then $\mu_\varepsilon([0,1])$ *converges to* $\mu([0,1])$, *as* $\varepsilon\to 0$, *almost surely and in* L^p *norm.*

(2) $\mu_\varepsilon([0,1])$ *converges to* $\mu([0,1])$, *as* $\varepsilon\to 0$, *almost surely and in* L^2 *norm, if and only if*

$$\int_{[0,1]^2}\exp\left((\theta(2)+\widetilde{\theta}(2))\#\,S\cap\mathcal{C}(t)\cap\mathcal{C}(s)\right)\,d\sigma(t)d\sigma(s)<\infty.$$

If $\#\,S\cap\mathcal{C}_{b^{-n}}(t)$ is equivalent to n uniformly in t, we have $\theta_Q(p) = \left(\theta(p)+\widetilde{\theta}(p)\right)/\log(b)$. Then condition (2.3) simplifies to be

$$\varphi_\sigma(p) + \left(\theta(p)+\widetilde{\theta}(p)\right)/\log(b) < 0;$$

when $\widehat{Q}_\varepsilon$ is the martingale associated with CCM, this coincides with the sufficient condition in [**KP**] when $\sigma=\ell$ and Theorem B in [**Fa6**] for a general σ.

If S is a Poisson point process, for $p\geq 0$ define

$$\widehat{\theta}(p) = \mathbb{E}(W^p)\int_{[0,1]}\widetilde{W}(u)^p\,du - 1 - p\left(\mathbb{E}(W)\int_{[0,1]}\widetilde{W}(u)\,du - 1\right).$$

Theorem 3.6 (L^p **convergence, Poisson point process** S). *Assume (3.3) holds if* $\widetilde{W}\not\equiv 1$.

(1) *Let* $p \in (1,2]$. *Suppose there exists an integer* $b \geq 2$ *such that*

$$\sum_{n\geq 0}\left(\sum_{w\in A^n}\sigma(I_w)^{p-1}\int_{I_w}\exp\left(\widehat{\theta}(p)\Lambda\left(\mathcal{C}_{b^{-n-1}}(t)\right)\right)\,d\sigma(t)\right)^{1/p}<\infty.$$

Then $\mu_\varepsilon([0,1])$ *converges to* $\mu([0,1])$, *as* $\varepsilon \to 0$, *almost surely and in* L^p *norm.*

(2) $\mu_\varepsilon([0,1])$ *converges to* $\mu([0,1])$, *as* $\varepsilon \to 0$, *almost surely and in* L^2 *norm, if and only if*

$$\int_{[0,1]^2}\exp\left(\widehat{\theta}(2)\Lambda\left(\mathcal{C}(t)\cap\mathcal{C}(s)\right)\right)\,d\sigma(t)d\sigma(s)<\infty.$$

When (3.3) holds, the probability distribution of $\widehat{Q}_\varepsilon(t)$ does not depend on t. Then condition (2.3) becomes:

$$\varphi_\sigma(p)+\overline{\alpha}\widehat{\theta}(p)<0, \tag{3.4}$$

where $\overline{\alpha}$ is a parameter that only depends on the geometry of the Poisson point process and measures the size of the expected number of points in $\mathcal{C}_\varepsilon(t)$ with respect to the same number in the statistically self-similar case for which $\Lambda=\Lambda_\rho$ with $\rho=1$:

$$\overline{\alpha}=\limsup_{\varepsilon\to 0}\frac{\Lambda\big(\mathcal{C}_\varepsilon(t)\big)}{\log(1/\varepsilon)}.$$

In the particular case of MPCP, one recovers the sufficient condition found in [**BM1**].

Remark 3.7. Theorems 3.5 and 3.6 are both corollaries of Theorem 2.2. When W and $\widetilde{W}$ are positive, Theorem 3.6 can be obtained as a corollary of Theorem 3.5 by conditioning on S and using the Jensen inequality. In fact, if one defines $\widehat{\theta}(p)=\theta(p)+\widetilde{\theta}(p)$ and $\Lambda(B)=\#S\cap B$ when S is deterministic, Theorems 3.5 and 3.6 are formally the same.

Theorem 3.6 and (3.4) are established in [**B6**] when $\widetilde{W}\equiv 1$. When S is any locally finite deterministic set, the result is new (when S is the geometric realization of an homogeneous b-adic tree and $\widetilde{W}\equiv 1$, Theorem 3.5 is a consequence of results in [**Bi3**] for martingales in the branching random walk) .

3.2. Extension of products of cylindrical pulses: log-infinitely divisible cascades.

This class of random measures is constructed in [**BaMu**]. It illustrates [**K3**] theory. Also it is contained in $\mathcal{M}$ and includes MPCP generated with positive random weights.

Let us recall that the characteristic function $\mathbb{E}(e^{i\xi X})$ of a real valued infinitely divisible random variable X takes the form $e^{\varphi_{\pi,m,s}(\xi)}$ with

$$\varphi_{\pi,m,s}(\xi)=im\xi-\frac{s^2}{2}\xi^2+\int_{\mathbb{R}}\left(e^{i\xi u}-1-i\xi\sin(u)\right)\pi(du),$$

where $m,s\in\mathbb{R}$ and the nonnegative Borel measure π is called the Lévy measure of X and is so that $\int_{|u|\leq 1}u^2\pi(du)<\infty$ and $\pi\big((-1,1)^c\big)<\infty$.

Products of cylindrical pulses associated with Poisson point processes (see Section 3.1) have the following property when $W>0$: for every $t\in\mathbb{R}$ and $\varepsilon\in(0,1)$, the characteristic function of the random variable $\log Q_\varepsilon(t)$ is given by

$$\mathbb{E}\left(e^{i\xi\log Q_\varepsilon(t)}\right)=\exp\left[\Lambda\big(\mathcal{C}_\varepsilon(t)\big)\left(-i\xi\big(\mathbb{E}(W)-1\big)+\mathbb{E}(e^{i\xi\log W})-1\right)\right].$$

It is straithforward to verify that this is the characteristic function of an infinitely divisible random variable. When the Poisson intensity $\Lambda=\rho\Lambda_1$ ($\rho>0$, $\Lambda_1(dtd\lambda)=dtd\lambda/\lambda^2$), that is when one constructs MPCP, this property is an immediate consequence of the i.i.d. property of the Ws and the self-similarity property of Λ_1.

[**BaMu**] exhibits the following fact. Let π be the measure on $\mathbb{R}$ defined by $\pi(du)=\mathbb{P}_{\log W}(du)$. The log-density $\log Q_\varepsilon(t)$ of MPCP is equal to $P\big(\mathcal{C}_\varepsilon(t)\big)$, where the mapping P is defined on the Borel subsets B of $\mathbb{R}\times\mathbb{R}_+^*$ of finite Λ_1-measure by

$$P(B)=-\rho\Lambda_1(B)(\mathbb{E}(W)-1)+\sum_{M\in S\cap B}\log(W_M).$$

It follows that in the sense (specified below) of [**RaRo**], P is an "independently scattered infinitely divisible" random measure on $\mathbb{R}\times\mathbb{R}_+^*$; moreover, the Lévy measure associated with P is $\rho\pi$.

The construction of the more general "log-infinitely divisible cascades" performed in [**BaMu**] uses the following result of [**RaRo**]: If π is the Lévy measure of an infinitely divisible random variable, one can associate with π and every $(m,s)\in\mathbb{R}^2$ a random function on the elements of $\mathcal{B}(\mathbb{R}\times\mathbb{R}_+^*)$ of finite Λ_1-measure, namely $P_{\pi,m,s}$, which is called "independently scattered infinitely divisible random measure" on $\mathbb{R}\times\mathbb{R}_+^*$ because it possesses the following properties:
(1) for every $B\in\mathcal{B}(\mathbb{R}\times\mathbb{R}_+^*)$ such that $\Lambda_1(B)<\infty$,

$$\mathbb{E}\left(e^{i\xi P_{\pi,m,s}(B)}\right)=e^{\varphi_{\pi,m,s}(\xi)\Lambda_1(B)}\quad(\xi\in\mathbb{R});$$

(2) for every finite family $\{B_i\}$ of pairwise disjoint elements of $\mathcal{B}(\mathbb{R}\times\mathbb{R}_+^*)$ such that $\Lambda_1(B_i)<\infty$, the random variables $P_{\pi,m,s}(B_i)$ are mutually independent.

Now let J be the interval $\{q\in\mathbb{R}:\ \int_{|u|\geq1}e^{qu}\,\pi(du)<\infty\}$. Define the convex function $\psi_{\pi,m,s}(\cdot)$ to be equal to $\varphi_{\pi,\mu,s}(-i\cdot)$ on J and $+\infty$ outside of J. By construction,

$$\mathbb{E}\left(e^{qP_{\pi,m,s}(B)}\right)=e^{\psi_{\pi,m,s}(q)\Lambda_1(B)}$$

for every $q\in J$ and every Borel subset B of $\mathbb{R}\times\mathbb{R}_+^*$ of finite Λ_1-measure.

If, moreover, $1\in J$ and (m,s) is chosen so that $\psi_{\pi,m,s}(1)=0$, [**BaMu**] obtains the positive martingale defined by

$$Q_\varepsilon(t)=e^{P_{\pi,m,s}\left(\mathcal{C}_\varepsilon(t)\right)}$$

and considers $Q\cdot\ell$. It is not clear that in general there exists a version of the process $\big(Q_\varepsilon(t)\big)$ which satisfies **(P1)** or the right-continuity involved in Remark 2.1(2). Such a property holds for example when $\int_{|u|\leq1}|u|\pi(du)<\infty$.

Another remarkable point in [**BaMu**] is that the cone $\mathcal{C}_\varepsilon(t)$ can be replaced by a modified one, yielding a nice "exact scaling property" for the limit measure $Q\cdot\ell$.

After the construction provided in [**BaMu**], it is immediate to perform the previous construction with respect to any locally bounded positive Poisson intensity

Λ. If **(P1)** (or the right-continuity involved in Remark 2.1(2)) does not hold for any version of Q, we consider the almost sure weak limit measure μ of a sequence μ_{ε_n} such that $\varepsilon_n \to 0$ as $n \to \infty$. In this case, Q_ε is redefined to be equal to Q_{ε_n} on $(\varepsilon_{n+1}, \varepsilon_n]$, and so it satisfies **(P1)**. Remarking that Q_ε also satisfies properties **(P2)** to **(P4)** (again with $\beta = 2$), Theorem 2.2 becomes:

THEOREM 3.8 (L^p **convergence, log-infinitely divisible cascades**).

(1) *Let $p \in (1, 2]$. Suppose there exists an integer $b \geq 2$ such that*

$$\sum_{n\geq 0} \left(\sum_{w\in A^n} \sigma(I_w)^{p-1} \int_{I_w} \exp\left(\psi_{\pi,m,s}(p)\Lambda\big(\mathcal{C}_{b^{-n-1}}(t)\big)\right) d\sigma(t) \right)^{1/p} < \infty.$$

Then $\mu_\varepsilon([0, 1])$ converges to $\mu([0, 1])$, as $\varepsilon \to 0$, almost surely and in L^p norm.

(2) *$\mu_\varepsilon([0, 1])$ converges to $\mu([0, 1])$, as $\varepsilon \to 0$, almost surely and in L^2 norm, if and only if*

$$\int_{[0,1]^2} \exp\left(\psi_{\pi,m,s}(2)\Lambda\left(\mathcal{C}(t) \cap \mathcal{C}(s)\right)\right) d\sigma(t)d\sigma(s) < \infty.$$

When (3.3) holds, the probability distribution of $Q_\varepsilon(t)$ does not depend on t. Then condition (2.3) becomes:

$$\varphi_\sigma(p) + \overline{\alpha}\psi_{\pi,m,\sigma}(p) < 0.$$

Here again, for $p \in (1, 2]$ and $\sigma = \ell$, we recover the sufficient (and almost necessary) condition for convergence in L^p for $\mu_\varepsilon([0, 1])$ when $\Lambda = \Lambda_1$ ([**BaMu**]).

REMARK 3.9. (1) When $\varphi_{\pi,m,s}$ is the characteristic function of a Gaussian random variable, the continuous base free martingale $Q_\varepsilon \cdot \sigma$ illustrates [**K2**] theory as well as [**M1**].

MPCP belong to constructions involving infinitely divisible laws without Gaussian part. The way they can be perceived as associated with Gaussian $P_{\pi,m,s}(B)$ is conditionally on the Poisson point process when W is lognormal.

(2) The scaling of moments property in the construction of [**BaMu**] possesses a weaker version in absence of self-similarity [**CRA**].

(3) Since the examples of functions $Q_\varepsilon(t)$ associated with Poisson point processes or intensities are in fact defined on the whole real line, if σ is a locally bounded positive Borel measure defined on $\mathbb{R}$, an almost sure weak limit $Q \cdot \sigma_{|K}$ of $Q_\varepsilon \cdot \sigma$, as $\varepsilon \to 0$, is defined on every compact subset of $\mathbb{R}$. By choosing an unbounded increasing sequence of positive numbers (a_n) such that $\sigma(\{-a_n, a_n\}) = 0$ for all n, we can define almost surely on $\mathbb{R}$ a measure $Q \cdot \sigma$ whose restriction to each $K_n = [-a_n, a_n]$ is $Q \cdot \sigma_{|K_n}$. Finally, the measure $Q \cdot \sigma$ is the vague limit, as $\varepsilon \to 0$, of the measures $Q_\varepsilon \cdot \sigma$ on $\mathbb{R}$.

4. SIMULTANEOUS CONVERGENCE OF UNCOUNTABLE FAMILIES IN $\mathcal{M}$

We are given a bounded positive Borel measure σ on $[0, 1]$.

The simultaneous construction of uncountable families of non-degenerate martingale limit measures such as μ in Section 2 is a natural problem. It arises for example when studying the multifractal analysis of CCM or MPCP. Indeed, in order to get almost surely the whole multifractal spectrum of the measure, it is necessary to associate such a family of "Gibbs" measures with each sample of the construction. Moreover, one has to find simultaneous lower bounds for the lower Hausdorff dimensions of these measures. This is done in [**B3**] for CCM and [**BM1**] for MPCP.

The same problem arises in [**BFa1**], which describes how many times different points are covered by the random arcs in the Dvoretzky covering of the circle, and also how many times different points are covered by the random intervals in the Poisson covering of the real line.

In Section 2, the fact that a given measure-valued martingale μ_ε converges almost surely, as $\varepsilon \to 0$, is a consequence of the nonnegative martingale convergence theorem in $\mathbb{R}$ (see [**K3**] for details). Theorem 2.2 provides a sufficient condition for the limit to be non-degenerate. This result makes it possible to study simultaneously countable families of such constructions, but tells nothing for uncountable families.

Roughly speaking, given an uncountable family $Q_\varepsilon(t,\omega)(\gamma)$ of functions satisfying properties **(P1)** to **(P4)**, the parameter γ ranging in an uncountable set Γ, the problem of simultaneous convergence reduces to showing that for any subinterval I of $[0,1]$, with probability one, for every $\gamma\in\Gamma$, $Q_\varepsilon(\gamma)\cdot\sigma(I)$ converges as $\varepsilon\to 0$.

So we are led to study the simultaneous convergence of uncountable families of real valued martingales $Y(\gamma)$. This problem appears in the context of multiplicative martingales related to the dyadic tree structure in [**JoLeN**]. It is then encountered in [**Bi2, Bi3**] in the context of martingales in the branching random walk. These works inspired [**B3**] for the simultaneous construction of Gibbs measures. All the results of simultaneous convergence in these papers are intimately related to the regularity of $\gamma\mapsto Y(\gamma)$. Also, a minimal regularity is required. Here, we shall limit ourselves to the case where the dependence is analytic. For other kinds of regularity, the reader is referred to [**Bi2**], [**B4, B5, B6**]. We also point out that without assumption **(P4)** [**K7**] constructs simultaneously some families of non-degenerate measures associated with lognormal $Q_\varepsilon(t,\cdot)(\gamma)$, $\gamma\in[0,1]$. In this context, the lognormality plays a crutial role in solving the problem.

We consider a measurable subset Γ of $\mathbb{R}^d$ $(d\geq 1)$ and a family of measurable functions $Q_\varepsilon:\ \big([0,1]\times\Gamma\times\Omega,\mathcal{B}([0,1])\otimes\mathcal{B}(\Gamma)\otimes\mathcal{B}\big)\to\big(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+)\big)$, $0<\varepsilon\leq 1$, such that for every $\gamma\in\Gamma$ the family $\big\{(Q_\varepsilon(\cdot,\gamma,\cdot)\big\}_{\varepsilon\in(0,1]}$ satisfies properties **(P1)** to **(P4)** (and **(P1)** with the same subset D for all $\gamma\in\Gamma$ if D differs from $[0,1]$). Thus we get a family of measure-valued martingales $\big((\mu_\varepsilon^\gamma=Q_\varepsilon(\cdot,\gamma,\cdot)\cdot\sigma)_{\varepsilon\in(0,1]}\big)_{\gamma\in\Gamma}$. The assumption on the regularity of this process with respect to γ will be specified in the statement of Theorem 4.2.

It turns out that solving the problem of the simultaneous convergence of measures μ_ε^γ involves the construction of an associated family of measures on an homogeneous tree. We introduce these connected measures in the next section.

From now on, we assume that σ is atomless.

4.1. Associated measures on an homogeneous tree. We need some new notations.

Let b be an integer ≥ 2.

For any integer $m \geq 0$ we denote by A^m the set of finite words of length m on the alphabet $\{0, \ldots, b-1\}$ (by convention, $A^0 = \{\emptyset\}$). We use $|w|$ to denote the length m of $w \in A^m$ and I_w to denote the closed b-adic subinterval $[\sum_{i=1}^m w_i b^{-i}, b^{-m} + \sum_{i=1}^m w_i b^{-i}]$ of $[0,1]$ naturally encoded by $w = w_1 \cdots w_m$. Let $A^* = \bigcup_{m=0}^\infty A^m$ and $\partial A^* = \{0, \ldots, b-1\}^{\mathbb{N}}$. The set $A^* \cup \partial A^*$ is equipped with the concatenation operation. For $w \in A^*$, $\mathcal{A}_w = w\partial A^*$ denotes the cylinder determined by w, i.e. $\mathcal{A}_w = \{ww' : \ w' \in \partial A^*\}$. Let $\mathcal{A}^*$ be the σ-field of ∂A^* generated by all cylinders.

For every $\tilde{t} \in \partial A^*$ and $n \geq 1$, denote by $\mathcal{A}_n(\tilde{t})$ the cylinder of the n^{th} generation containing $\tilde{t}$.

For every $t \in [0,1)$ and $n \geq 1$, denote by $I_n(t)$ the closure of the b-adic interval of the n^{th} generation, semi-open to the right, which contains t.

For every $w \in A^*$, denote the center of I_w by t_w. Also denote by f_w the affine increasing function that maps $[0,1]$ onto I_w.

Let d be the usual ultrametric distance on ∂A^* defined by $d(\tilde{t}, \tilde{s}) = b^{-k}$ where $k = \sup\{i \geq 1 : \ \tilde{t}_1 \cdots \tilde{t}_i = \tilde{s}_1 \cdots \tilde{s}_i\}$ (by convention $\sup \emptyset = 0$).

Let π be the mapping from ∂A^* to $[0,1]$ defined by

$$\pi(\tilde{t}) = \sum_{i=1}^{\infty} \frac{\tilde{t}_i}{b^i} \qquad (\tilde{t} = \tilde{t}_1 \cdots \tilde{t}_i \cdots \in \partial A^*).$$

Let $\tilde{\ell}$ denote the unique measure on $(\partial A^*, \mathcal{A}^*)$ such that for all $w \in A^*$, $\tilde{\ell}(\mathcal{A}_w) = b^{-|w|}$.

Since σ is atomless, there exists an unique measure $\widetilde{\sigma}$ on $(\partial A^*, \mathcal{A}^*)$ such that $\sigma = \widetilde{\sigma} \circ \pi^{-1}$ and $\widetilde{\sigma}(\mathcal{A}_w) = \sigma(I_w)$ for all $w \in A^*$.

If $Q \cdot \sigma$ belongs to $\mathcal{M}$, for $\varepsilon \in (0,1]$ and $\tilde{t} \in \partial A^*$ define $\widetilde{Q}_\varepsilon(\tilde{t}) = Q_\varepsilon\big(\pi(\tilde{t})\big)$. Also let $\widetilde{\mu}_\varepsilon$ be the measure on $(\partial A^*, \mathcal{A}^*)$ whose density with respect to $\widetilde{\sigma}$ is equal to $\widetilde{Q}_\varepsilon(\tilde{t})$. We also write $\widetilde{\mu}_\varepsilon = \widetilde{Q}_\varepsilon \cdot \widetilde{\sigma}$. For $w \in A^*$, the restriction of $\widetilde{\mu}_{b^{-|w|}\varepsilon}$ to $\mathcal{A}_w$ can be written as

$$d\widetilde{\mu}_{b^{-|w|}\varepsilon} = \widetilde{Q}_{b^{-|w|}}(\tilde{t}).d\widetilde{\mu}_\varepsilon^{\mathcal{A}_w},$$

where $\widetilde{\mu}_\varepsilon^{\mathcal{A}_w}$ is the measure on $(\mathcal{A}_w, w\mathcal{A}^*)$ whose density with respect to $\widetilde{\sigma}$ is

$$\frac{d\widetilde{\mu}_\varepsilon^{\mathcal{A}_w}}{d\widetilde{\sigma}}(\tilde{t}) = Q_{b^{-|w|}, b^{-|w|}\varepsilon}\big(\pi(\tilde{t})\big)$$

(see **(P2)** for the definition of $Q_{\varepsilon,\varepsilon'}$). We also define on I_w the measure $\mu_\varepsilon^{I_w}$ whose density with respect to σ is

$$\frac{d\mu_\varepsilon^{I_w}}{d\sigma}(t) = Q_{b^{-|w|}, b^{-|w|}\varepsilon}(t).$$

Suppose now that for every $w \in A^*$, the family $\big(Q_\varepsilon^w(t,\omega) = Q_{b^{-|w|}, b^{-|w|}\varepsilon}(f_w(t), \omega)\big)$, $0 < \varepsilon \leq 1$, satisfies **(P1)**. Due to [**K3**], with probability one, for every $w \in A^*$, $\widetilde{\mu}_\varepsilon^{\mathcal{A}_w}$ converges weakly, as $\varepsilon \to 0$, to a measure $\widetilde{\mu}^{\mathcal{A}_w}$. Denote $\widetilde{\mu}^{\mathcal{A}_\emptyset}$ by $\widetilde{\mu}$. By construction we have

$$\mu_\varepsilon = \widetilde{\mu}_\varepsilon \circ \pi^{-1},$$

and

$$\mu_\varepsilon^{I_w} = \widetilde{\mu}_\varepsilon^{\mathcal{A}_w} \circ \pi^{-1}, \quad \mu^{I_w} = \widetilde{\mu}^{\mathcal{A}_w} \circ \pi^{-1}.$$

Also, fundamental relations for the sequel arise:

The general functional equation. For all $n > m \geq 1$

$$\mu_{b^{-n}}([0,1]) = \sum_{w\in A^m} \mu_{b^{-n}}(I_w) = \sum_{w\in A^m} \int_{I_w} Q_{b^{-m}}(t)\, \mu^{I_w}_{b^{m-n}}(dt). \tag{4.1}$$

If, with probability one, $t \mapsto Q_{b^{-m}}(t)$ has only jump discontinuities, then

$$\mu(I_w) = \int_{I_w} Q_{b^{-m}}(t)\, \mu^{I_w}(dt) \quad \forall\ w \in A_m. \tag{4.2}$$

(Proof: σ being atomless, due to [**K3**], with probability one, the b-adic points of $[0,1]$ are not atoms of μ, so $\mu(I_w) = \lim_{n\to\infty} \mu_{b^{-n}}(I_w)$; moreover, again by [**K3**], conditionally on $\mathcal{F}_{b^{-m}}$, almost surely the countable family of jump points of $t \mapsto Q_{b^{-m}}(t)$ is of μ^{I_w}-measure 0, since μ^{I_w} is independent of $\mathcal{F}_{b^{-m}}$ by **(P3)**, so $\lim_{n\to\infty} \mu_{b^{-n}}(I_w) = \int_{I_w} Q_{b^{-|w|}}\, \mu^{I_w}(dt)$.)

It follows that

$$\|\mu\| = \sum_{w\in A^m} \mu(I_w) = \sum_{w\in A^m} \int_{I_w} Q_{b^{-m}}(t)\, \mu^{I_w}(dt) \quad \forall\ m \geq 1. \tag{4.3}$$

4.2. Simultaneous convergence result. We return to the simultaneous convergence of the family μ^γ_ε; $\widetilde{\mu}^\gamma_\varepsilon$ denotes the measure on $(\partial A^*, \mathcal{A}^*)$ associated with μ^γ_ε; $Q_\varepsilon(t,\gamma)$ denotes the random variable $Q_\varepsilon(t,\gamma,\cdot)$. Our assumption is the following:

(A1): Γ is a non-empty open set or a singleton. Moreover, there exists an integer $b \geq 2$ such that for every compact subset K of Γ
(i) for every $w \in A^*$ the family

$$\left(\gamma \in K \mapsto \int_{I_w} Q_{s^{-1}}(t,\gamma)\, d\sigma(t)\right)_{s>1}$$

is a right-continuous martingale in $C(K)$.

(ii) There exists an open subset U_K of $\mathbb{C}^d$ such that $K \subset U_K$, and for every $m,n \in \mathbb{N}$, $Q_{b^{-m}} : [0,1]\times K\times\Omega \to \mathbb{R}_+$ and $Q_{b^{-m},b^{-m-n}} : [0,1]\times K\times\Omega \to \mathbb{R}_+$ possess respectively a measurable extension $\widehat{Q}_{b^{-m}}$ and $\widehat{Q}_{b^{-m},b^{-m-n}}$ from $[0,1]\times U_K\times\Omega$ to $\mathbb{C}$; $\widehat{Q}_{b^{-m},b^{-m}} \equiv 1$. Moreover, with probability one, for every $w \in A^*$ and $n \geq |w|$

$$z \in U_K \mapsto \int_{I_w} \widehat{Q}_{b^{-n}}(t,z,\omega)\, d\sigma(t)$$

exists and is analytic. One denotes $\widehat{Q}_{b^{-m}}(t,z,\cdot)$ and $\widehat{Q}_{b^{-m},b^{-m-n}}(t,z,\cdot)$ by $\widehat{Q}_{b^{-m}}(t,z)$ and $\widehat{Q}_{b^{-m},b^{-m-n}}(t,z)$ respectively. Also the following properties hold:

(P'1) for every $(t,z) \in D\times U_K$ and $m \geq 0$, $\mathbb{E}\big(\widehat{Q}_{b^{-m}}(t,z)\big)$ is defined and equal to 1;

(P'2) for every $m,n \geq 0$, $\widehat{Q}_{b^{-m-1}} = \widehat{Q}_{b^{-m}}\widehat{Q}_{b^{-m},b^{-m-1}}$ and $\widehat{Q}_{b^{-m},b^{-m-n-1}} = \widehat{Q}_{b^{-m},b^{-m-n}}\widehat{Q}_{b^{-m-n},b^{-m-n-1}}$;

(P'3) For every $m \geq 0$ and $n \geq 1$, the σ-algebra's $\sigma\big(\widehat{Q}_{b^{-k}}(t,z) : t \in [0,1],\ z \in U_K,\ 0 \leq k \leq m\big)$ and $\sigma\big(\widehat{Q}_{b^{-m},b^{-k}}(t,z) : t \in [0,1],\ z \in U_K,\ k > m\big)$ are independent, as well as $\sigma\big(\widehat{Q}_{b^{-m},b^{-m-k}}(t,z) : t \in [0,1],\ z \in U_K,\ 0 \leq k \leq n\big)$ and $\sigma\big(\widehat{Q}_{b^{-m-n},b^{-m-k}}(t,z) : t \in [0,1],\ z \in U_K,\ k > n\big)$;

(P'4) For every $m\geq 0$, and every family $\mathcal{G}$ of nontrivial subintervals of $[0,1]$ of common length b^{-m} such that $d(I,J)\geq \beta b^{-m}$ for every $I\neq J\in\mathcal{G}$, the σ-algebra's $\sigma\big(\widehat{Q}_{b^{-m},b^{-k}}(t,z):\ t\in I,\ z\in U_K,\ k>m\big)$, $I\in\mathcal{G}$, are mutually independent.

(iii) U_K being chosen as in (i) and (ii), for every compact subset K' of U_K and $w\in A^*$, there exists a number $p\in(1,2]$ such that

$$(4.4)\qquad \sum_{n\geq 1}\sup_{z\in K'}\left(\sum_{v\in A^n}\sigma(I_{wv})^{p-1}\int_{I_{wv}}\mathbb{E}\Big(|\widehat{Q}_{b^{-|w|-n-1}}(t,z)|^p\Big)\,d\sigma(t)\right)^{1/p}<\infty.$$

Remark 4.1. If assumption **(A1)** holds with the integer b then it holds for every integer of the form b^N, $N\geq 1$.

Theorem 4.2. *Suppose* **(A1)** *holds. With probability one, for all $\gamma\in\Gamma$, the measure $\widetilde{\mu}_\varepsilon^\gamma$ converges weakly, as $\varepsilon\to 0$, to a nonnegative measure $\widetilde{\mu}^\gamma$, and μ_ε^γ converges to the measure $\mu^\gamma=\widetilde{\mu}^\gamma\circ\pi^{-1}$. Moreover, if $t\in[0,1]\mapsto Q_\varepsilon(t,\gamma)$ is positive almost surely for all $\gamma\in\Gamma$, then, with probability one, for all $\gamma\in\Gamma$, the support of μ^γ (resp. $\widetilde{\mu}^\gamma$) is* $\mathrm{supp}(\sigma)$ *(resp.* $\mathrm{supp}(\widetilde{\sigma})$*).*

4.3. Simultaneous lower bounds for dimensions. Given a subset E of $[0,1]$, define its σ-Hausdorff dimension as

$$\dim_\sigma(E)=\inf\{d\geq 0:\ \mathcal{H}^{d,\sigma}(E)=0\}$$

where

$$\mathcal{H}^{d,\sigma}(E)=\lim_{\delta\to 0^+}\inf\{\sum\sigma(U_i)^d\ :\ E\subset\bigcup_i U_i,\ |U_i|\leq\delta\}.$$

Define similarly $\dim_{\widetilde{\sigma}}(\widetilde{E})$ for any subset $\widetilde{E}$ of ∂A^*. Of course, when $\sigma=\ell$, one recovers the usual Hausdorff dimension.

In this section we give sufficient conditions for computing a lower bound for the lower σ-Hausdorff dimension of the measures μ^γ constructed in the previous section: the lower σ-Hausdorff dimension of a positive measure μ on $[0,1]$ is defined as $\dim_\sigma(\mu)=\inf\{\dim_\sigma(B):\ B\in\mathcal{B}([0,1]),\ \mu(B)>0\}$.

We make the following assumptions **(A2)**$(i)(ii)(iii)(iv)(v)$.

(A2) (i) Assumptions (i) and (ii) of **(A1)** are strengthened as follows: If K is a compact subset of Γ, for every $m,n\geq 0$, the probability distribution of $z\in U_K\mapsto\widehat{Q}_{b^{-m},b^{-m-n}}(t,z)$ does not depend on $t\in D$. Moreover, for every $w\in A^*$ and $n\geq 1$,

$$\left(\gamma\in K\mapsto\int_{I_w}Q_{b^{-|w|},b^{-|w|}s^{-1}}(t,\gamma)\,d\sigma(t)\right)_{s>1}$$

is a right-continuous martingale in $C(K)$, and

$$z\in U_K\mapsto\int_{I_w}\widehat{Q}_{b^{-|w|},b^{-|w|-n}}(t,z)\,d\sigma(t)$$

exists and is analytic.

Next, for $z\in U_K$ and $p\geq 1$ define

$$\begin{cases}\theta(z,p)=\limsup_{n\to\infty}\frac{1}{n}\log_b\,\mathbb{E}\big(|\widehat{Q}_{b^{-n}}(t,z)|^p\big)\\ \widehat{\theta}^{(b)}(z,p)=\limsup_{n\to\infty}\sup_{m\geq 1}\frac{\log\,\mathbb{E}\big(|\widehat{Q}_{b^{-n},b^{-n-m}}(t,z)|^p\big)}{\log b^m}\end{cases}$$

$\theta(z,p)$ is a convex function of p for a fixed z. It does not depend on b.

Also, for $p\geq 1$ define

$$\widehat{\varphi}_\sigma^{(b)}(p)=\limsup_{n\to\infty}\sup_{\substack{w\in A^n,\\ \sigma(I_w)>0}}\sup_{m\geq 1}\frac{1}{m}\log_b\sum_{v\in A^m}\left(\sigma(I_{wv})/\sigma(I_w)\right)^p.$$

It is an exercise to show that $\theta(z,p)\leq\widehat{\theta}^{(b)}(z,p)$ and $\varphi_\sigma(p)\leq\widehat{\varphi}_\sigma^{(b)}(p)$.

Remark 4.3. Suppose that σ possesses a self-similar structure, for example σ is a quasi-Bernoulli measure (see [**BrMiP**]) depending on the c-adic grid, $c\geq 2$. Then, the number $\widehat{\varphi}_\sigma^{(b)}(p)$ introduced for technical reasons tends to $\varphi_\sigma(p)$ as $b=c^N$ tends to ∞. This remark and assumption $(ii)(\alpha)$ below together suggest that in general b must be chosen as large as possible.

(ii) Assumption (iii) of **(A1)** is replaced by the following one: for every compact subset K' of U_K, there exists $p\in(1,2]$ such that

(α)

$$\widehat{\varphi}_\sigma^{(b)}(p)+\sup_{z\in K'}\widehat{\theta}^{(b)}(z,p)<0.$$

(β) for every $\delta>0$ there exists $n_0(\delta)\geq 1$ such that for every $n\geq n_0(\delta)$ and $z\in K'$

$$\sup_{m\geq 1}\frac{\log\,\mathbb{E}\big(|\widehat{Q}_{b^{-n},b^{-n-m}}(t,z)|^p\big)}{\log b^m}\leq\widehat{\theta}^{(b)}(z,p)+\delta.$$

Notice that assumption $(ii)(\alpha)$ implies that for every $\gamma\in K$, one has $\varphi_\sigma(p)+\theta(\gamma,p)<0$. Since this function of p is convex and its value at $p=1$ is $\varphi_\sigma(1)+\theta(\gamma,1)=0$, and also $\theta(\gamma,\cdot)$ is non-decreasing, we deduce that $\varphi'_\sigma(1^+)+\frac{\partial\theta}{\partial p}(\gamma,1^+)<0$ for every $\gamma\in K$ and $\varphi'_\sigma(1^+)<0$. Consequently,

$$\underline{D}(\gamma,\sigma)=1+\frac{\theta'_\gamma(1^+)}{\varphi'_\sigma(1^+)}\in(0,1],$$

where

$$\theta'_\gamma(1^+):=\frac{\partial\theta}{\partial p}(\gamma,1^+).$$

(iii) For every compact subset K of Γ such that K is the closure of its interior, the function $\gamma\mapsto\frac{\partial\theta}{\partial p}(\gamma,1^+)$ is continuously differentiable on the interior of K.

(iv) For every compact subset K of Γ,

(α) There exists $p_K>1$ such that for every $\delta>0$, there exists $n_0(\delta)\geq 1$ such that for all $\gamma\in K$ and $q\in[1,p_K]$

$$\frac{1}{n}\log_b\mathbb{E}\big(Q_{b^{-n}}(t,\gamma)^q\big)\leq\theta(\gamma,q)+\delta.$$

(β)

$$\theta(\gamma,1+x)=\theta'_\gamma(1^+)x+o(x)\quad(\gamma\in K,\ x>0)$$

where $o(x)$ is uniform over $\gamma\in K$.

(v) For every compact subset K of Γ, if $\eta>0$ is small enough, for every $n\geq 1$ and $w\in A^n$, there exists a random variable $M_w(\eta)$ such that for all $\gamma\in K$

$$\sup_{t\in I_w} Q_{b^{-n}}(t,\gamma)^{1+\eta}\leq M_w(\eta)\,(Q_{b^{-n}}(t_w,\gamma))^{1+\eta}$$

and either of the following properties (α) or (β) applies

$(\alpha)(1)$ $M_w(\eta)$ is independent of $\gamma\mapsto Q_{b^{-n}}(t_w,\gamma)$ and $\mathbb{E}\big(M_w(\eta)\big)=\exp(o(n))$ uniformly over $w\in A^n$. (2) If K is the closure of its interior, for every $k\geq 0$ the function $\gamma\mapsto Q_{b^{-k},b^{-k-1}}(t_w,\gamma)$ is almost surely continuously differentiable over the interior of K, and

$$\sup_{1\leq i\leq d}\ \sup_{\gamma\in\mathrm{Int(K)}}\ \sup_{0\leq k\leq n}\ \mathbb{E}\left(Q_{b^{-k},b^{-k-1}}(t_w,\gamma)^\eta\left|\frac{\partial Q_{b^{-k},b^{-k-1}}}{\partial\gamma_i}(t_w,\gamma)\right|\right)=\exp\big(o(n)\big);$$

$(\beta)(1)$ For every $h>0$, $\mathbb{E}\big(M_w(\eta)^h\big)=\exp(o(n))$ uniformly over $w\in A^n$. (2) If K is the closure of its interior, for every $k\geq 0$ the function $\gamma\mapsto Q_{b^{-k},b^{-k-1}}(t_w,\gamma)$ is almost surely continuously differentiable over the interior of K, and for every $h'>1$ close enough to 1

$$\sup_{1\leq i\leq d}\ \sup_{\gamma\in\mathrm{Int(K)}}\ \sup_{0\leq k\leq n}\ \mathbb{E}\left(Q_{b^{-k},b^{-k-1}}(t_w,\gamma)^{\eta h'}\left|\frac{\partial Q_{b^{-k},b^{-k-1}}}{\partial\gamma_i}(t_w,\gamma)\right|^{h'}\right)=\exp\big(o(n)\big).$$

Then let $(\widetilde{\mathbf{A}}\mathbf{2})$ be the collection of assumptions $\mathbf{(A2)}(i)(ii)(iii)(iv)$ and $\big((v)(\alpha)(1)$ or $(v)(\beta)(1)\big)$.

Finally, for $\gamma\in\Gamma$ such that $\widetilde{\mu}^\gamma\neq 0$, define $\mathcal{P}(\gamma)$ and $\mathcal{P}'(\gamma)$ as the following properties

$$\mathcal{P}(\gamma):\begin{cases}\liminf\limits_{n\to 0}\dfrac{\log\,\widetilde{\mu}^\gamma\big(\mathcal{A}_n(\tilde t)\big)}{\log\,\widetilde{\sigma}\big(\mathcal{A}_n(\tilde t)\big)}\geq\underline{D}(\gamma,\sigma) & \widetilde{\mu}^\gamma-\text{almost everywhere}\\ \liminf\limits_{n\to 0}\dfrac{\log\,\mu^\gamma\big(I_n(t)\big)}{\log\,\sigma\big(I_n(t)\big)}\geq\underline{D}(\gamma,\sigma) & \mu^\gamma-\text{almost everywhere,}\end{cases}$$

$$\mathcal{P}'(\gamma):\begin{cases}\liminf\limits_{n\to 0}\dfrac{\log\,\widetilde{\sigma}\big(\mathcal{A}_n(\tilde t)\big)}{\log\,\widetilde{\ell}\big(\mathcal{A}_n(\tilde t)\big)}\geq-\varphi'_\sigma(1^+) & \widetilde{\mu}^\gamma-\text{almost everywhere}\\ \liminf\limits_{n\to 0}\dfrac{\log\,\sigma\big(I_n(t)\big)}{\log\,\ell\big(I_n(t)\big)}\geq-\varphi'_\sigma(1^+) & \mu^\gamma-\text{almost everywhere.}\end{cases}$$

THEOREM 4.4.

(1) *Suppose* $\mathbf{(A2)}$ *holds. Then* $\mathbf{(A1)}$ *holds. Let* $\mathcal{C}$ *be* Γ *if* Γ *is a singleton or a* C^1 *curve in* Γ. *With probability one, for all* $\gamma\in\mathcal{C}$, *conditionally on* $\widetilde{\mu}^\gamma\neq 0$, $\mathcal{P}(\gamma)$ *holds and* $\min(\dim_{\widetilde{\sigma}}(\widetilde{\mu}^\gamma),\dim_\sigma(\mu^\gamma))\geq\underline{D}(\gamma,\sigma)$.

Moreover, if the controls in $\mathbf{(A2)}(v)(\alpha)(2)$ *and* $\mathbf{(A2)}(v)(\beta)(2)$ *also hold with* $\eta=0$ *then, with probability one, for all* $\gamma\in\mathcal{C}$, *conditionally on* $\widetilde{\mu}^\gamma\neq 0$, $\mathcal{P}'(\gamma)$ *holds and* $\min(\dim_{\widetilde{\ell}}(\widetilde{\mu}^\gamma),\dim_\ell(\mu^\gamma))\geq-\varphi'_\sigma(1^+)-\theta'_\gamma(1^+)$.

(2) *Suppose* ($\widetilde{\mathbf{A}}\mathbf{2}$) *holds. Then* **(A1)** *holds. With probability one, there exists* $\Gamma(\omega) \subset \Gamma$ *such that* $\Gamma \setminus \Gamma(\omega)$ *is of null Lebesgue measure and for every* $\gamma \in \Gamma(\omega)$ *such that* $\widetilde{\mu}^\gamma \neq 0$, $\mathcal{P}(\gamma)$ *and* $\mathcal{P}'(\gamma)$ *hold as well as the conclusions of (1) concerning the dimensions.*

Remark 4.5. (1) Assumption **(A2)**(ii) strengthens **(A1)**(iii) because in the proof of Theorem 4.4 we need a uniform control of some moment of order > 1 of the random variables $\widetilde{\sigma}(\mathcal{A}_w)^{-1}\widetilde{\mu}_\varepsilon^{\gamma,\mathcal{A}_w}(\mathcal{A}_w)$.

(2) In the case where K is a singleton $\{\gamma\}$, **(A2)** reduces to **(A2)**$(i)(ii)$ and $\big((v)(\alpha)(1)$ or $(v)(\beta)(1)\big)$. In fact, in this case one also has the remarkable following property: with probability one, conditionally on $\mu = \mu^\gamma \neq 0$,

$$(4.5) \qquad -\varphi'_\sigma(1^+) \leq \liminf_{n\to 0} \frac{\log\, \sigma\big(I_n(t)\big)}{\log\, \ell\big(I_n(t)\big)} \leq \limsup_{n\to 0} \frac{\log\, \sigma\big(I_n(t)\big)}{\log\, \ell\big(I_n(t)\big)} \leq -\varphi'_\sigma(1^-)$$

μ-almost everywhere, so the property (4.5) true σ-almost everywhere (see [**H**] for example) is also true $Q\cdot\sigma$-almost everywhere.

Also [**K2**] and [**Fa4**] obtained the same kind of lower bounds for $\dim_\sigma(\mu)$ for certain choices of Q and σ.

(3) Theorem 4.4 extends Theorem 10 in [**B6**] which is concerned only with the particular families of products of cylindrical pulses associated with a Poisson point process and $\sigma = \ell$. Nevertheless, we mention that results in [**B6**] involve a C^1 regularity with respect to γ rather than analyticity.

(4) Theorem 9 of [**B6**] for products of cylindrical pulses also obtains an upper bound for the dimension of a single non-degenerate measure $\mu = Q\cdot\sigma$. Also simultaneous upper bounds are given for families of measures associated with statistically self-similar Poisson point processes (Theorem 5 of [**B6**]), and in this case they coincide with the lower bounds. It is not easy to state general conditions under which such an estimate could be derived.

4.4. Application to the multifractality of $Q_\varepsilon(t)$. The virtue of the infinite products of (random) functions we are interested in is that often they converge to multifractal limit measures. It is at least the case for statistically self-similar constructions. When there is no self-affinity, there is no reason why the limit measure should not be multifractal. But such objects are technically very difficult to deal with. This is partly due to the difficulty to control moments of negative orders of pieces of the measure in these cases, and until now the best that can be done is to compute μ-almost everywhere the Hölder exponent of the limit measure μ, yielding only one point of the multifractal spectrum. Nevertheless, returning to the construction of these infinite products, the mutifractality of the limit object must have a conterpart in terms of the high variability of the density when one looks at partial product of functions after a large number of multiplications. In this section, we illustrate this idea with a simple, but significative and nontrivial example. Let Λ be a Poisson intensity invariant by horizontal translations: $\Lambda = \ell\otimes\nu$. Recall that $\overline{\alpha}$ was defined in Section 3 as $\overline{\alpha} = \limsup_{\varepsilon\to 0} \frac{\Lambda(\mathcal{C}_\varepsilon)}{\log(1/\varepsilon)}$. Define

$$\widehat{\alpha} = \inf_{b\geq 2} \limsup_{n\to\infty} \sup_{m\geq 1} \frac{\Lambda(\mathcal{C}_{b^{-n-m}} \setminus \mathcal{C}_{b^{-n}})}{\log\, b^m}.$$

One has $\overline{\alpha} \leq \widehat{\alpha}$. Then let $Q_\varepsilon(t,a)$ be the product of cylindrical pulses associated with a Poisson point process with intensity Λ and a W almost surely equal to a constant $a > 0$.

Suppose $\Lambda(\mathcal{C}_\varepsilon) \to \infty$ as $\varepsilon \to 0$ and for $\beta \in \mathbb{R}$ define

$$F_\beta = \{t \in [0,1] :\ \lim_{\varepsilon\to 0} \frac{\log Q_\varepsilon(t,a)}{\Lambda(\mathcal{C}_\varepsilon)} = \beta\}.$$

Theorem 4.6.

(1) *Suppose* $\limsup_{\varepsilon\to 0} \varepsilon\nu([\varepsilon,1]) < \infty$, $\widehat{\alpha} = 0$ *and* $a \neq 1$. *If* $a < 1$ *(resp.* $a > 1$*), with probability one, for all* $\beta \in (-\infty, 1-a]$ *(resp.* $[1-a,\infty)$*),*

$$\dim\, F_\beta = 1;$$

moreover, if $\beta > 1-a$ *(resp.* $\beta < 1-a$*) then* $F_\beta = \emptyset$.

(2) *Suppose* $\limsup_{\varepsilon\to 0} \varepsilon\nu([\varepsilon,1]) < \infty$, $\overline{\alpha} = \widehat{\alpha} \in (0,\infty)$ *and* $a \neq 1$. *With probability one, for all* $\gamma \in \mathbb{R}$ *such that* $1 + \overline{\alpha}(a^\gamma - 1 - \gamma a^\gamma \log a) \geq 0$,

$$\dim\, F_{a^\gamma \log(a) + 1 - a} = 1 + \overline{\alpha}(a^\gamma - 1 - \gamma a^\gamma \log a);$$

moreover, if $\gamma \in \mathbb{R}$ *is such that* $1 + \overline{\alpha}(a^\gamma - 1 - \gamma a^\gamma \log a) < 0$ *then* $F_{a^\gamma \log(a)+1-a} = \emptyset$.

This result is in fact a consequence of [**BFa1**] because in the particular case of Theorem 4.6, $\log Q_\varepsilon(t,a)/\Lambda(\mathcal{C}_\varepsilon)$ is closely related to the number of intervals $[t_M - \lambda_M, t_M + \lambda_M)$ associated with points M of the Poisson process S such that $\lambda_M \geq \varepsilon$ and $t \in [t_M - \lambda_M, t_M + \lambda_M)$. One of the main tools in proving Theorem 4.6 is the family of measures μ^γ associated with the family $Q_\varepsilon(t, a^\gamma)$ with $\Gamma = \mathbb{R}$ and $\sigma = \ell$. It turns out that with probability one, μ^γ exists as a non-degenerate limit for all γ such that $1 + \overline{\alpha}(a^\gamma - 1 - \gamma a^\gamma \log a) > 0$ by Theorem 4.2, and a lower bound for the dimensions of sets F_β follows from Theorem 4.4 after showing that $\mu^\gamma(F_{a^\gamma \log(a)+1-a}) > 0$.

Theorem 4.6(1) is illustrated by $\nu = \sum_{n\geq 2} \delta_{\frac{1}{n\log(n)}}$, and Theorem 4.6(2) by $\nu = \sum_{n\geq 2} \delta_{\frac{1}{n}}$.

More general results are obtained in [**BFa2**] for the examples of Section 3.

5. Self-similarity in $\mathcal{M}$; non-degeneracy, moments, dimension of the carrier, and multifractality

In all this section we assume that σ is the Lebesgue measure ℓ. We consider an element $Q \cdot \ell$ of the subclass $\mathcal{M}$ such that $Q_1(t) = 1$ almost surely for all $t \in [0,1]$ and which satisfies the stationary condition **(P5)** with $D = [0,1]$. Moreover we suppose that Q_ε satisfies the following self-similarity property in distribution:

(P6) There exists an integer $b \geq 2$ such that, for all $w \in A^*$, the stochastic processes $\big(Q_\varepsilon(t)\big)_{t\in[0,1],\ \varepsilon\in(0,1]}$ and $\big(Q_{b^{-|w|}, b^{-|w|}\varepsilon}\big(f_w(t)\big)\big)_{t\in[0,1],\ \varepsilon\in(0,1]}$ have the same probability distribution, where f_w is the increasing affine function that maps $[0,1]$ onto I_w. Moreover, the probability distribution of the stochastic process $\big(Q_{b^{-1}}(t)\big)_{t\in[k/b,(k+1)/b)}$ does not depend on $k \in \{0, \cdots b-1\}$.

In this case we say that the limit measure μ is weakly statistically self-similar, because the affinities involved in **(P6)** depend on the base b. The subclass of $\mathcal{M}$ of such weakly statistically self-similar measures μ is denoted by $\mathcal{M}'$.

Define on $\mathbb{R}$ the concave function

$$\begin{aligned}\tau(q) &= -1+q-\log_b \mathbb{E}\Big(\mathbf{1}_{\{Q_{b^{-1}}(t)>0\}}Q_{b^{-1}}(t)^q\Big)\\ &= -1+q-\lim_{\varepsilon\to 0}\frac{1}{\log(1/\varepsilon)}\log \mathbb{E}\big(\mathbf{1}_{\{Q_\varepsilon(t)>0\}}Q_\varepsilon(t)^q\big)\end{aligned}$$

with the convention $0\times\infty = 0$ (the second equality above holds because of **(P2)**,**(P3)**,**(P5)** and **(P6)**).

In [**ManNoR**], a class of self-similar constructions which intersects $\mathcal{M}'$ is also considered. For the elements of this class, [**ManNoR**] discusses the L^2 convergence and also gives necessary conditions for non-degeneracy and finiteness of moments of positive orders in terms of τ that extend those of [**M2, M3, KP**] for CCM. Of course, one expects the strong counterpart of such necessary conditions to hold, that is these conditions to be also sufficient. This is the subject of this section in the class $\mathcal{M}'$.

It is shown in [**BM1**] that, with some effort, it is possible for the study of MPCP to use arguments developed for the study of CCM (in particular in [**KP**]). The key point is to relate the more difficult functional equation satisfied by MPCP (6.2) to the simple one satisfied by CCM (6.1). The importance of these functional equations is due to the self-similarity of the constructions. It turns out that the functional equation satisfied by MPCP and the approach in [**BM1**] to relate MPCP to CCM are general enough to derive general results on non-degeneracy, moments, and multifractal analysis for elements of $\mathcal{M}'$. Before stating these results, we specify the general functional equation (4.3) in Proposition 5.1 and use it to derive from the theory in [**K3**] a key argument in the study of non-degeneracy.

5.1. Self-similarity of the limit measure and a fine point in [K3] theory. Remember the definitions of Section 4.1. Given two random variables X and Y, their identity in distribution is denoted by $X \overset{d}{\equiv} Y$.

An immediate consequence of properties **(P3)** and **(P6)** is the following proposition.

Proposition 5.1 (**Self-similarity**). *Fix $w\in A^*$. With probability one, for every $\varepsilon\in(0,b^{-|w|}]$,*

$$\mu_\varepsilon(I_w)=\int_{I_w} Q_{b^{-|w|}}(t)\,d\mu^{I_w}_{b^{|w|}\varepsilon}(t),$$

where $t\mapsto Q_{b^{-|w|}}(t)$ is independent of the $\mu_\varepsilon^{I_w}$'s. Moreover, for all $f\in C(I_w)$:

(1) $\int_{I_w} f(t)\,\mu_\varepsilon^{I_w}(dt)\overset{d}{\equiv}|I_w|\int_{[0,1]} f\circ f_w(t)\,\mu_\varepsilon(dt)$ *for all $\varepsilon\in(0,1]$. In particular, $\|\mu_\varepsilon^{I_w}\|\overset{d}{\equiv}|I_w|\,\|\mu_\varepsilon\|$.*

(2) $\int_{I_w} f(t)\,\mu^{I_w}(dt)\overset{d}{\equiv}|I_w|\int_{[0,1]} f\circ f_w(t)\,\mu(dt)$. *In particular, $\|\mu^{I_w}\|\overset{d}{\equiv}|I_w|\,\|\mu\|$.*

Now let us establish an important fact that will play a fundamental role in proving sufficient condition for non-degeneracy. Since the random functions $Q_{b^{-k},b^{-k-1}}(t)$,

$k \geq 1$, are mutually independent, it follows from [**K3**] (Theorem 4) that the operator on non-negative measures on ∂A^*

$$L : \rho \mapsto \mathbb{E}(\widetilde{Q} \cdot \rho)$$

is a projection. (By definition if $f \in C(\partial A^*)$ then

$$\int_{\partial A^*} f(\tilde{t})\, \mathbb{E}(\widetilde{Q}.\rho)(d\tilde{t}) = \mathbb{E}\big(\int_{\partial A^*} f(\tilde{t})\, \widetilde{Q}.\rho(dt)\big).)$$

Here, because of **(P6)**, the probability distribution of $\widetilde{\mu}(\mathcal{A}_w)$ depends only on $|w|$. Moreover, since ∂A^* is totally disconnected, we have $\|\widetilde{\mu}\| = \|\mu\| = \sum_{w \in A^m} \widetilde{\mu}(\mathcal{A}_w)$ for all $m \geq 0$. Consequently, $\mathbb{E}\big(\mu(\mathcal{A}_w)\big) = \mathbb{E}(\|\mu\|) b^{-|w|}$ for all $w \in A^*$ and

$$\mathbb{E}(\widetilde{\mu}) = \mathbb{E}(\|\mu\|)\tilde{\ell}. \tag{5.1}$$

The operator L being a projection, this yields

$$\mathbb{E}(\|\mu\|) \in \{0,1\}. \tag{5.2}$$

5.2. Non-degeneracy, moments, and dimension of the carrier. Our results involve certain of the following conditions, which are inspired from the study of MPCP [**BM1**]. Let $t \in [0,1)$ and for every $n \geq 0$ let I_n be a closed b-adic subinterval of $[0,1]$ of the n^{th} generation.

$(\mathbf{C_1})$ $\mathbb{E}\big(\sup_{s \in I_n} Q_{b^{-n}}(s)\big) = \varphi(n)$, where $\varphi(n) = o(n)$;

$(\mathbf{C_2(q)})$ $q \in \mathbb{R}_+$ and

$$\mathbb{E}\left(\sup_{s \in I_n} Q_{b^{-n}}(s)^q\right) \leq e^{\varphi_q(n)} \mathbb{E}\left(Q_{b^{-n}}(t)^q\right),$$

where $\varphi_q(n) = o(n)$;

$(\mathbf{C_3(q)})$ $q \in \mathbb{R}_+$ and

$$\mathbb{E}\left(\inf_{s \in I_n} Q_{b^{-n}}(s)^q\right) \geq e^{-\varphi_q(n)} \mathbb{E}\left(Q_{b^{-n}}(t)^q\right),$$

where $\varphi_q(n) = o(n)$;

$(\mathbf{C_3'(q)})$ $q \in \mathbb{R}_+$, and there exists a random variable Q_n and a stochastic process $t \in I_n \mapsto \overline{Q}_n(t)$ such that $Q_{b^{-n}}(t) = Q_n \overline{Q}_n(t)$ for every $t \in I_n$, Q_n and $\overline{Q}_n$ are independent, and there exists a function $\varphi_q(n) = o(n)$ such that for every $t \in I_n$

$$\mathbb{E}\left(Q_n^q\right) \mathbb{E}\left(\overline{Q}_n(t)\right)^q \geq e^{-\varphi_q(n)} \mathbb{E}\left(Q_{b^{-n}}(t)^q\right);$$

$(\mathbf{C_4(q)})$ $Q_{b^{-1}} > 0$, $q < 0$, and there exists $n \geq 1$ such that

$$\mathbb{E}\left(\big(\inf_{s \in I_n} Q_{b^{-n}}(s)\big)^q\right) < +\infty.$$

Remark 5.2. In the terminology used in the study of certain dynamical systems, the above conditions can be viewed as a kind of principle of bounded distorsion in the mean.

Let μ be in $\mathcal{M}'$.

Theorem 5.3 (**Non-degeneracy**).

(1) *Suppose* $(\mathbf{C_1})$ *holds. If* $\tau'(1^-)>0$ *then the martingale* $\mu_\varepsilon([0,1])$ *converges to* $\mu([0,1])$*, as* $\varepsilon\to 0$*, almost surely and in* L^1 *norm. In particular* μ *is non-degenerate. Moreover, if* Q_ε *is positive,* $\mathbb{P}(\mu\neq 0)=1$.

(2) *Suppose there exists* $h<1$ *such that* $(\mathbf{C_2(h)})$ *holds. If* μ *is non-degenerate then* $\tau'(1^-)\geq 0$.

THEOREM 5.4 (**Moments of positive orders**). *Let* $h>1$.

(1) *Suppose* $\tau(h)>0$*. If* $h\in(1,2]$*, or* $(\mathbf{C_2(q)})$ *holds for every* $q\in\{h\}\bigcup(2,h)\cap\mathbb{N}$ *, then* $0<\mathbb{E}\big(\mu([0,1])^h\big)<+\infty$.

(2) *Suppose* $0<\mathbb{E}\big(\mu([0,1])^h\big)<+\infty$ *and* $(\mathbf{C_3(h)})$ *or* $(\mathbf{C'_3(h)})$ *holds. Then* $\tau(h)\geq 0$.

THEOREM 5.5 (**Moments of negative orders**). *Suppose* μ *is non-degenerate. Let* $q<0$*. Suppose* $\tau(q)>-\infty$ *and* $(\mathbf{C_4(q)})$ *holds. Then* $\mathbb{E}\left(\mu([0,1])^q\right)<+\infty$.

THEOREM 5.6 (**Dimension of the carrier**). *Assume there exists* $h\in(1,2]$ *such that* $\tau(h)>0$*. Assume also that* $(\mathbf{C_2(h)})$ *holds for every* $h<1$ *close enough to 1. With probability one, conditionally on* $\mu\neq 0$,

$$\lim_{n\to 0}\frac{\log\,\widetilde{\mu}\big(\mathcal{A}_n(\tilde{t})\big)}{-n\log(b)}=\tau'(1)\quad \widetilde{\mu}-\text{almost everywhere}$$

and

$$\lim_{n\to 0}\frac{\log\,\mu\big(I_n(t)\big)}{-n\log(b)}=\tau'(1)\quad \mu-\text{almost everywhere.}$$

In particular, $\widetilde{\mu}$ *(resp.* μ*) is carried by a Borel subset of* ∂A^* *(resp.* $[0,1]$*) of Hausdorff dimension* $\tau'(1)$*; moreover, any Borel subset of Hausdorff dimension less than* $\tau'(1)$ *has a null* $\widetilde{\mu}$*-measure (resp.* μ*-measure).*

REMARK 5.7. (1) At this level of generality, it is difficult to obtain a more specified result for finiteness of moments of negative orders. Indeed, CCM and MPCP already show very different issues with respect to this question. This will be specified in Section 6.3.

(2) For the class of measures it considers, [**WaWi**] introduces a "size-biasing" approach for non-degeneracy and finiteness of moments of positive orders problems. This method proved to be powerful for CCM. It was used in [**BM1**] and [**BCM**] to show the necessity of $\tau'(1)>0$ for non-degeneracy under some assumptions. It seems difficult to exploit the "size-biasing" approach and derive a result of the type of Theorem 5.3(2) in full generality in $\mathcal{M}'$. Indeed, the example of MPCP shows that this method uses very specific properties of the construction (see [**BM1**] or the proof of Theorem 6.6 in [**B7**]). Nevertheless it will be used to derive the necessity of this condition for products of non-cylindrical pulses in Sections 6.1 and 6.2. The size-biasing method is not adapted to the finiteness of moments of positive orders problem for MPCP.

(3) If one assumes only $\tau(h)>0$ in Theorem 5.6, one obtains almost surely μ-almost everywhere $\tau'(1)$ as lower bound for the logarithmic density of μ; this is Theorem 4.4.

(4) It is certainly possible to extend the results of this section to the case where the measure $\widetilde{\ell}$ on ∂A^* is replaced by an ergodic invariant Markov measure. Indeed, for this choice of σ, problems of non-degeneracy and moments of positive order are solved in [**Fa6**] when Q_ε is the martingale associated with CCM.

(5) More connections with previous works are provided in Section 6.

5.3. Multifractal analysis. In this section we consider a non-degenerate element μ of $\mathcal{M}'$ and assume $\tau'(1) > 0$.

The Hausdorff and packing dimensions of a subset of $\mathbb{R}$ (resp. ∂A^*) are considered with respect to the usual distance (resp. d), and denoted respectively by dim and Dim (see [**F2**] for definitions).

For $t \in [0,1]$ and $r > 0$ let $I(t,r)$ denote the interval $[t-r, t+r] \cap [0,1]$.

The multifractal analysis of $\widetilde{\mu}$ and μ aims at computing the Hausdorff and packing dimensions of sets of points where these measures possess a given Hölder regularity. Recall that there are two main points of view in studying this problem, namely the box-multifractal analysis, and the centered multifractal analysis. Both coincide for $\widetilde{\mu}$, and among the sets of regularity of particular interest, we select the following.

For every $\alpha \geq 0$, define

$$\begin{cases} \overline{\widetilde{E}}_\alpha = \{\tilde{t} \in \partial A^* : \limsup_{n\to\infty} \dfrac{\log \tilde{\mu}(\mathcal{A}_n(\tilde{t}))}{-n \log b} = \alpha\}, \\ \underline{\widetilde{E}}_\alpha = \{\tilde{t} \in \partial A^* : \liminf_{n\to\infty} \dfrac{\log \tilde{\mu}(\mathcal{A}_n(\tilde{t}))}{-n \log b} = \alpha\}, \\ \widetilde{E}_\alpha = \overline{\widetilde{E}}_\alpha \bigcap \underline{\widetilde{E}}_\alpha. \end{cases}$$

For μ, box and centered multifractal analyses differ. In this paper, the first one will be concerned with the sets ($\alpha \geq 0$)

$$\begin{cases} \overline{E}_\alpha = \{t \in [0,1) : \limsup_{n\to\infty} \dfrac{\log \mu(I_n(t))}{-n \log b} = \alpha\}, \\ \underline{E}_\alpha = \{t \in [0,1) : \liminf_{n\to\infty} \dfrac{\log \mu(I_n(t))}{-n \log b} = \alpha\}, \\ E_\alpha = \overline{E}_\alpha \bigcap \underline{E}_\alpha; \end{cases}$$

the other one with

$$\begin{cases} \overline{F}_\alpha = \{t \in [0,1] : \limsup_{r\to 0^+} \dfrac{\log \mu(I(t,r))}{\log r} = \alpha\}, \\ \underline{F}_\alpha = \{t \in [0,1] : \liminf_{r\to 0^+} \dfrac{\log \mu(I(t,r))}{\log r} = \alpha\}, \\ F_\alpha = \overline{F}_\alpha \bigcap \underline{F}_\alpha. \end{cases}$$

Also define the so-called large deviation spectrum of $\widetilde{\mu}$ (for more on large deviation spectra, see [**R, F2, L-VVoj, Z**])

$$\alpha \geq 0 \mapsto \widetilde{f}(\alpha) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log \#\{w \in A^n : \ b^{-n(\alpha+\varepsilon)} \leq \tilde{\mu}(\mathcal{A}_w) \leq b^{-n(\alpha-\varepsilon)}\}}{n \log(b)}.$$

and the analog of φ_μ (see Section 2.3 for the definition of φ_μ) for $\widetilde{\mu}$:

$$\widetilde{\varphi}_{\widetilde{\mu}}(q) = \limsup_{n\to\infty} \frac{1}{n} \log_b \sum_{w\in A^n} \widetilde{\mu}(\mathcal{A}_w)^q.$$

Remark 5.8. (1) The reader is referred to [**BrMiP, Be4, O2, BeBh, H, BeBhH, BBeP**] for theoretical results on box and centered multifractal analyses of measures. The function φ_μ has two counterparts in the centered multifractal formalism, namely b_μ and B_μ, which will be involved in Theorem 5.12. These functions are introduced in [**O2**] in order to respectively estimate Hausdorff and packing dimensions of the sets S_α, $S \in \{\overline{F}, \underline{F}, F\}$, defined above.

(2) The random measures in the class $\mathcal{M}'$ we are interested in are not covered by theoretical results mentioned above. It is important to explain why. [**BrMiP**] and [**O2**], respectively devoted to box-multifractal analysis and centered multifractal analysis, compute the dimensions of the above sets for classes of measures μ such that for every $q \in \mathbb{R}$, there exists a Gibbs measure μ_q and a constant $C_q > 0$ such that for every t in the closed support of μ and $n \geq 1$

$$\frac{1}{C_q} \leq \frac{\mu\big(I_n(t)\big)^q b^{-n\varphi_\mu(q)}}{\mu_q\big(I_n(t)\big)} \leq C_q \tag{5.3}$$

in [**BrMiP**] and

$$\frac{1}{C_q} \leq \frac{\mu\big(I(t, b^{-n})\big)^q b^{-nB_\mu(q)}}{\mu_q\big(I(t, b^{-n})\big)} \leq C_q \tag{5.4}$$

in [**O2**].

[**Be4**], [**BeBh**] and [**BeBhH**] find weaker sufficient conditions than in [**BrMiP**] and [**O2**] to compute these dimensions. In particular, their results also cover classes of measures such that for some q there is no Gibbs measure as above. It is in practice impossible to decide whether the random measures in $\mathcal{M}'$, like CCM and MPCP, satisfy these conditions. On the other hand, it will be seen in Part III [**B7**] (Lemma 5.2 and 5.3) that when μ belongs to $\mathcal{M}'$, it satisfies properties closely related but weaker than (5.3) and (5.4).

(3) [**BBeP**] establishes connections between the approach in [**BrMiP**] and [**O2**] and gives sufficient conditions, namely "neighboring boxes conditions", that make it possible to perform both multifractal analyses simultaneously. Results are obtained for quasi-Bernoulli measures as well as CCM. We will adopt this approach in proving Theorems 5.11 and 5.12 in Part III.

(4) Computing the Hausdorff and packing dimensions of the sets $\overline{F}_\alpha \bigcap \underline{F}_\beta$ for $\alpha \leq \beta$ would give the best multifractal description. This was done in a deterministic context for example in [**OW**].

Define $J = \{q \in \mathbb{R} : \ \mathbb{E}\big(\mathbf{1}_{\{\|\mu\|>0\}}\|\mu\|^q\big) < \infty\}$.

Define $\mathcal{J} = \{q \in \mathbb{R} : \ \tau(q) > -\infty, \ \tau'(q)q - \tau(q) > 0\}$. We notice that if $\tau > -\infty$ in a neighborhood of $[0,1]$, then $\mathcal{J}$ contains a neighborhood of $[0,1]$.

Remark 5.9. If the condition $(\mathbf{C_2(q)})$ is satisfied for every $q \in \mathcal{J} \cap (2,\infty)$, then $\mathcal{J} \cap \mathbb{R}_+ \subset J$. This is due to Theorem 5.4 and the fact that condition $\tau'(q)q - \tau(q) > 0$ implies $\tau(q) > 0$ for $q > 1$. If $Q_{b^{-1}} > 0$ and the condition $(\mathbf{C_4(q)})$ is satisfied for every $q \in \mathcal{J} \cap \mathbb{R}_-$, then $\mathcal{J} \cap \mathbb{R}_- \subset J$. This is due to Theorem 5.5.

The following conditions **(C)** and **(C')** will be assumed in Theorem 5.11 and 5.12 respectively:

(C)(1) $Q_{b^{-1}} > 0$.

(C)(2) The condition $(\mathbf{C_2(q)})$ is satisfied for every $q \in \mathcal{J} \setminus [1,2]$, and the condition $(\mathbf{C_4(q)})$ is satisfied for every $q \in \mathcal{J} \cap \mathbb{R}_-$.

(C)(3) For every compact subinterval K of $\mathcal{J}$, if $\eta>0$ is small enough then for every $n\geq 1$ and $w\in A^n$, there exists a random variable $M_w(\eta)$ such that for all $q\in K$

$$\left(\sup_{t\in I_w} Q_{b^{-n}}(t)^{(1+\eta)q}\right)\left(\sup_{t\in I_w} Q_{b^{-n}}(t)^{-\eta q}\right)\leq M_w(\eta)Q_{b^{-n}}(t_w)^q \tag{5.5}$$

and either (α) $M_w(\eta)$ is independent of $Q_{b^{-n}}(t_w)$ and $\mathbb{E}\big(M_w(\eta)\big)=\exp(o(n))$, or (β) $\mathbb{E}\big(M_w(\eta)^h\big)=\exp(o(n))$ for all $h>1$.

If $v\in A^*$ let $i(v)$ stand for the unique integer such that $I_v=[i(v)b^{-|v|},(i(v)+1)b^{-|v|}]$. Then for $v,w\in A^*$ such that $|v|=|w|$ define $\delta(v,w)=|i(v)-i(w)|$.

(C')(1): **(C)**(1).
(C')(2): **(C)**(2).
(C')(3) For every $\varepsilon>0$ and every compact subinterval K of $\mathcal{J}\cap\mathbb{R}_-^*$ (resp. $\mathcal{J}\cap\mathbb{R}_+$), if $\eta>0$ is small enough then
(i) for every $n\geq 1$ and every pair $(v,w)\in(A^n)^2$ such that $\delta(v,w)\leq b'=3$ (resp. $b'=4b+2$) there exists a random variable $M_{v,w}(\eta)$ such that for all $q\in K$

$$\left(\sup_{t\in I_w} Q_{b^{-n}}(t)^{(1+\eta)q}\right)^{1+\eta}\left(\sup_{t\in I_v} Q_{b^{-n}}(t)^{-\eta q}\right)\leq M_{v,w}(\eta)Q_{b^{-n}}(t_w)^{(1+\eta)q}Q_{b^{-n}}(t_v)^{-\eta q} \tag{5.6}$$

and either (α) $M_{v,w}(\eta)$ is independent of $\big(Q_{b^{-n}}(t_w)Q_{b^{-n}}(t_v)\big)$ and $\mathbb{E}\big(M_{v,w}(\eta)\big)=\exp(o(n))$, or (β) $\mathbb{E}\big(M_{v,w}(\eta)^h\big)=\exp(o(n))$ for all $h>1$. Moreover, $o(n)$ is uniform over these pairs (v,w).

(ii) In every neighborhood of 1^+ there exists h' such that for all $q\in K$

$$\sum_{\substack{v,w\in A^n\\ 0<\delta(v,w)\leq b'}}\left(\mathbb{E}\left(Q_{b^{-n}}\big(t_w\big)^{(1+\eta)qh'}Q_{b^{-n}}(t_v)^{-\eta qh'}\right)\right)^{1/h'}=O\left(b^{-n\left(\tau(q)-q-\eta\varepsilon/4\right)}\right), \tag{5.7}$$

the O being uniform over $n\geq 1$ and $q\in K$. Moreover,

$$\sup_{\substack{q\in K,k\geq 1\\ v,w\in A^n,\ 0<\delta(v,w)\leq b'}}\frac{\mathbb{E}\left(\left|\frac{dQ_{b^{-k},b^{-k-1}}(t_w)^{(1+\eta)q}Q_{b^{-k},b^{-k-1}}(t_v)^{-\eta q}}{dq}\right|^{h'}\right)}{\mathbb{E}\left(Q_{b^{-k},b^{-k-1}}(t_w)^{(1+\eta)qh'}Q_{b^{-k},b^{-k-1}}(t_v)^{-\eta qh'}\right)}<\infty. \tag{5.8}$$

REMARK 5.10. Properties (5.5), (5.6) and (5.7) are some kinds of principles of bounded distorsion, and although reasonable, are supplementary assumptions. It is the price to pay to obtain general results. Nevertheless, it will be seen that they are satisfied by CCM and certain products of functions, and also by MPCP under strong assumptions on the random weight W. The fact that weaker hypotheses on W are assumed in [**BM1**] to derive the multifractal analysis of MPCP comes from a direct use of the specificities of the construction, that is the properties of the Poisson point process.

Multifractal analysis on ∂A^***.** Define the Legendre transform of a function h from $\mathbb{R}$ to $\mathbb{R}\cup\{-\infty\}$ by

$$h^* : \alpha \mapsto \inf_{q\in\mathbb{R}} \alpha q - h(q).$$

Define $\alpha_{\min} = \inf\, \tau'(\mathcal{J})$ and $\alpha_{\max} = \sup\, \tau'(\mathcal{J})$.

THEOREM 5.11. *Suppose $\tau > -\infty$ in a neighborhood of $[0,1]$ and* **(C)** *holds. With probability one:*

(1) $-\widetilde{\varphi}_{\widetilde{\mu}}(q) = \tau(q)$ *for all* $q \in \mathcal{J}$.

(2) *For every* $q \in \mathcal{J}$ *and* $S \in \{\overline{\widetilde{E}}, \underline{\widetilde{E}}, \widetilde{E}\}$,

$$\dim\, S_{\tau'(q)} = Dim\, S_{\tau'(q)} = \widetilde{f}(\tau'(q)) = \tau'(q)q - \tau(q) = \tau^*\big(\tau'(q)\big).$$

(3) *One has* $0 < \alpha_{\min} \le \alpha_{\max} < \infty$. *If* $\tau^*(\alpha_{\min})$ *(resp.* $\tau^*(\alpha_{\max})) = 0$, *then* $F_\alpha = \emptyset$ *for all* $\alpha \in (0, \alpha_{\min})$ *(resp.* $(\alpha_{\max}, \infty)$) *and* $F \in \{\overline{\widetilde{E}}, \underline{\widetilde{E}}, \widetilde{E}\}$.

Multifractal analysis on $[0,1]$**.**

THEOREM 5.12. *Suppose $\tau > -\infty$ in a neighborhood of $[0,1]$ and* **(C')** *holds. With probability one:*

(1) $\varphi_\mu(q) = b_\mu(q) = B_\mu(q) = -\tau(q)$ *for every* $q \in \mathcal{J}$.

(2) *For every* $q \in \mathcal{J}$ *and* $S \in \{\overline{E}, \underline{E}, E, \overline{F}, \underline{F}, F\}$,

$$\dim\, S_{\tau'(q)} = Dim\, S_{\tau'(q)} = \tau'(q)q - \tau(q) = \tau^*\big(\tau'(q)\big).$$

(3) *One has* $0 < \alpha_{\min} \le \alpha_{\max} < \infty$. *If* $\tau^*(\alpha_{\min})$ *(resp.* $\tau^*(\alpha_{\max})) = 0$, *then* $S_\alpha = \emptyset$ *for all* $\alpha \in (0, \alpha_{\min})$ *(resp.* $(\alpha_{\max}, \infty)$) *and* $S \in \{\overline{E}, \underline{E}, E, \overline{F}, \underline{F}, F\}$.

REMARK 5.13. Of course, because of Theorem 5.11, assertions (2) and (3) of Theorem 5.12 hold for $S \in \{\overline{E}, \underline{E}, E\}$ under **(C)**.

6. PRODUCTS OF PULSES ASSOCIATED WITH STATISTICALLY SELF-SIMILAR POINT PROCESSES

In this section we apply results of Section 5 to fundamental examples that belong to the class of Section 3.1. The log-infinitely divisible cascades of [**BaMu**] will be specifically studied in [**BaBMu**].

6.1. Geometric b-adic tree. We consider the following geometric realization of the b-adic tree:

$$S = \Big\{ \Big(\frac{k+1/2}{b^n}, \frac{1}{2b^n} \Big) :\ n \ge 1,\ k = 0, \dots, b^n - 1 \Big\},$$

and the combination of the associated constructions in Section 3.1. We get the martingale $\widehat{Q}_\varepsilon = Q_\varepsilon \widetilde{Q}_\varepsilon$. Taking $\widetilde{W} \equiv 1$ yields the construction of CCM. Taking $W \equiv 1$ yields a new construction which is a kind of completely decorrelated counterpart of the generalized Riesz products with random phases described in Remark 2.3. In order to fulfill property **(P6)**, we redefine this martingale as follows: $\widehat{Q}_\varepsilon := \widehat{Q}_{\varepsilon/2}$. Then properties **(P1)** to **(P6)** are fulfilled and $\mu = \widehat{Q} \cdot \ell$ belongs to $\mathcal{M}'$.

The associated function τ is given by

$$\tau(q) = -1 + q - \log_b \left(\mathbb{E}(\mathbf{1}_{\{W>0\}} W^q)\right) - \log_b \left(\mathbb{E}(\mathbf{1}_{\{\widetilde{W}>0\}} \widetilde{W}^q)\right),$$

where

$$\mathbb{E}(\mathbf{1}_{\{\widetilde{W}>0\}} \widetilde{W}^q) = \int_{[0,1]} \mathbf{1}_{\{\widetilde{W}(t)>0\}} \widetilde{W}(t)^q \, dt$$

(we choose the normalization $\mathbb{E}(W) = \int_{[0,1]} \widetilde{W}(t)\, dt = 1$).

We shall make one of the following assumptions on $\widetilde{W}$: if $\widetilde{W}$ is positive, let

$$\psi(n) = \sum_{k=0}^{n} \sup_{\substack{t,s\in[0,1] \\ |t-s|\le b^{-n}}} \left|\log\left(\widetilde{W}(t)\right) - \log\left(\widetilde{W}(s)\right)\right|.$$

(H1) $\widetilde{W}$ is positive and $\psi(n) = \log\big(o(n)\big)$.

(H2) $\widetilde{W}$ is positive and $\psi(n) = o(n)$.

Notice that under each of these principles of bounded distorsion the function $\widetilde{W}$ is continuous.

The following proposition makes it possible to apply Theorem 5.3, 5.4, 5.5, 5.6, 5.11 and 5.12 to the measure μ.

Proposition 6.1.

(1) $(\mathbf{C_1})$ *holds if* **(H1)** *holds.*
(2) $(\mathbf{C_2(q)})$ *holds if* **(H2)** *holds.*
(3) $(\mathbf{C_3(q)})$ *holds if* **(H2)** *holds.*
(4) $(\mathbf{C_4(q)})$ *holds if* $W > 0$ *and* $\mathbb{E}(W^q) < \infty$.
(5) *Suppose* $\mathbb{E}(W^q) < \infty$ *for* q *in a neighborhood of* $[0,1]$ *and* **(H2)** *holds. Then* **(C)** *and* **(C')** *hold.*

Remark 6.2. Theorem 5.3(2) can be improved as follows:

Theorem 6.3. *Suppose* $\widetilde{W} > 0$ *and* $\psi(n) = o(\sqrt{n\log\log n})$. *If* $\tau'(1^-) = 0$ *then* μ *is degenerate.*

6.2. Statistically self-similar Poisson point processes. We consider a Poisson point process whose intensity is either given by

$$\Lambda_\rho(dtd\lambda) = \frac{\rho}{2}\frac{dtd\lambda}{\lambda^2} \quad (\rho > 0)$$

or

$$\widetilde{\Lambda}_\rho = \ell \otimes \sum_{n\ge 1} \frac{\rho}{2}\log(b) b^n \delta_{b^{-n}} \quad (\rho > 0),$$

as well as the combination of the associated constructions in Section 3.1. We get the martingale $\widehat{Q}_\varepsilon$ and the limit measure $\mu = \widehat{Q}\cdot\ell$ which belongs to $\mathcal{M}'$. The so obtained subclass of $\mathcal{M}'$ is the Multifractal products of pulses, MPP, introduced in [**M4**]. Taking $\widetilde{W} \equiv 1$ yields either MPCP or PCCM of Part I ([**BM2**]). Taking $W \equiv 1$ emphasizes a phenomenon already observed in the discussion of Part I: associating a deterministic object, here $\widetilde{W}$, with the realizations of a Poisson point process, suffices to create random multifractal measures.

For both intensities, the associated function τ does not depend on b and is given by

$$\tau(q) = -1 + q\left[1 + \rho\left(\mathbb{E}(W)\mathbb{E}(\widetilde{W}) - 1\right)\right] - \rho\left(\mathbb{E}(\mathbf{1}_{\{W>0\}}W^q)\mathbb{E}(\mathbf{1}_{\{\widetilde{W}>0\}}\widetilde{W}^q) - 1\right).$$

Proposition 6.4.

(1) $(\mathbf{C_1})$ *holds if* **(H1)** *holds.*
(2) $(\mathbf{C_2(q)})$ *holds if* **(H2)** *holds.*
(3) $(\mathbf{C_3(q)})$ *holds if* **(H2)** *holds.*
(4) $(\mathbf{C_4(q)})$ *holds if* $W > 0$ *and* $\mathbb{E}(W^q) < \infty$.
(5) *Suppose that all the moments of* W *are finite and* **(H2)** *holds. Then* **(C)** *and* **(C')** *hold.*

Remark 6.5. (1) The comment in Section 2 of Part I concerning the self-similarity property of PCCM and MPCP holds for the combination of cylindrical and non-cylindrical pulses here: in Proposition 5.1(1)(2), if $\Lambda = \widetilde{\Lambda}_\rho$ we can replace the b-adic intervals I_w by nontrivial intervals of length a negative integer power of b, and if $\Lambda = \Lambda_\rho$, we can take any nontrivial subinterval of $[0, 1]$.

(2) As for the martingale considered in the previous section, Theorem 5.3(2) can be improved as follows:

Theorem 6.6. *Suppose there exists* $\gamma > 0$ *such that* $\mathbb{E}\big((1 + W)|\log W|^{2+\gamma}\big)$ *is finite. Suppose, moreover, that* $\widetilde{W} > 0$ *and there exists* $\varepsilon > 0$ *such that* $\psi(n) = O(n^{\frac{1}{2}-\varepsilon})$. *If* $\tau'(1^-) = 0$ *then* μ *is degenerate.*

(3) Proposition 6.4(5) shows that the general point of view adopted in Section 5 is applicable here, only under a strong assumption on W. Moreover, it turns out that exploiting directly the specificities of our particular examples makes it possible to avoid the verification of **(C)** and **(C')** in dealing with multifractal analysis. Indeed

Theorem 6.7. *Assume* μ *is non-degenerate,* **(H2)** *holds and* $\mathbb{E}(W^q) < \infty$ *for* q *in a neighborhood of* $[0, 1]$. *Then the conclusions of Theorems 5.11 and 5.12 hold.*

6.3. Remarks and comments. (Complementary information on CCM can be found in [**P5**]).

Remark 6.8 (Non-degeneracy). The NSC $\tau'(1^-) > 0$ for non-degeneracy is obtained in [**KP**] for CCM. It is obtained for MPCP in [**BM1**] when $W > 0$ and $\mathbb{E}\big((1+W)|\log W|^{2+\gamma}\big) < \infty$ for some $\gamma > 0$. Notice that now the case $\mathbb{P}(W = 0) > 0$ is taken into account for MPCP in Theorem 5.3.

For CCM, the functional equation (4.3) takes a simpler form whose expression in probability distribution is

$$\|\mu\| = Y \overset{d}{\equiv} \frac{1}{b\mathbb{E}(W)} \sum_{i=0}^{b-1} W_i Y(i), \tag{6.1}$$

where the random variables W_i, $Y(i)$, $0 \le i \le b-1$, are mutually independent, with $W_i \overset{d}{\equiv} W$ and $Y_i \overset{d}{\equiv} Y$. Under the condition $\tau'(1^-) > 0$, the construction of CCM provides nontrivial with finite first moment non-negative solutions of (6.1).

Non-degeneracy of μ is connected to the more general question of the possible existence of nontrivial nonnegative solutions for (6.1). This problem was studied successively in [**DL**], [**Gu**], [**Li2**] and [**Li3**] (in [**DL**], the equation takes the slightly general form presented in [**M2**]; [**Li2**] and [**Li3**] consider the generalized equation when the b-adic tree is replaced by a Galton-Watson tree; solutions with finite first moment for the generalized equation are obtained via martingale construction in [**P2**] and [**Bi1**]).

It turns out that if $\mathbb{E}\big(W|\log(W)|\big) < \infty$, (6.1) possesses a non-trivial non-negative solution if and only if there exists $\alpha \in (0,1]$ such that $\tau(\alpha) = 0$ and $\tau'(\alpha) \geq 0$ (α is unique). Moreover, if such an α exists: (i) if $\alpha < 1$, $\mathbb{E}(Y^\beta) < \infty$ for all $\beta \in (0,\alpha)$ and $\mathbb{E}(Y^\alpha) = \infty$. (ii) If $\alpha = 1$ and $\tau'(1) = 0$, $\mathbb{E}(Y^\beta) < \infty$ for all $\beta \in (0,1)$ and $\mathbb{E}(Y) = \infty$. (iii) If $\alpha = 1$, $\tau'(1) > 0$ and $\mathbb{E}(W^h) < \infty$ for some $h > 1$, the solutions have finite first moment and coincide with the probability distributions of positive multiples of $\|\mu\|$.

For PCCM and MPCP, (4.3), can be rewritten as follows

$$\|\mu\| = b^{-m\delta(V-1)} \sum_{w\in A^m} Q_{T^{I_w}} \int_{I_w} Q_{B^{I_w}\cap \mathcal{C}_{b^{-m}}(t)}\ \mu^{I_w}(dt) \quad \forall\ m \geq 1, \tag{6.2}$$

where by definition for every bounded Borel subset B of the upper half-plane

$$Q_B = \prod_{M\in B\cap S} W_M,$$

$$T^{I_w} = \bigcap_{t\in I_w} \mathcal{C}_{b^{-|w|}}(t),$$

and

$$B^{I_w} = \Big(\bigcup_{t\in I_w} \mathcal{C}_{b^{-|w|}}(t)\Big) \setminus T^{I_w}.$$

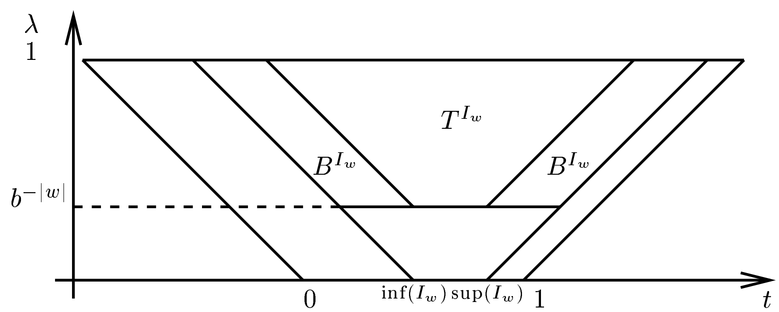

FIGURE 1. : Illustration of the sets defined above.

REMARK 6.9 (Moments of order > 1). Under some assumptions, Theorem 5.4 concludes on the necessity of $\tau(p) \geq 0$ as well as the sufficiency of $\tau(p) > 0$ for a non-degenerate $\|\mu\|$ to have a finite moment of order $p \geq 1$. The NSC $\tau(p) > 0$ is obtained in [**KP**] for CCM. Theorem 5.4 is obtained for MPCP in [**BM1**] when $W > 0$ (result extended in [**BaMu**]), but the (strict) positivity plays no role in this problem. The proof of Theorem 5.4(2) for MPCP in [**BM1**] uses the fact that MPCP satisfy property $(\mathbf{C_3'(q)})$ for $q \geq 0$. Our proof in [**B7**] will use the different property $(\mathbf{C_3(q)})$.

Notice that if $\mu \in \{\text{PCCM}, \text{MPCP}\}$ is non degenerate, $\|\mu\| \in L^2$ if and only if $\tau(2) > 0$, by Theorem 3.6(3).

For CCM, Guivarc'h ([**Gu**]) showed that if the law of $\log(W)$ is non lattice and if divergence of high moments holds, i.e. there exists an $h > 1$ (necessarily unique) such that $\tau(h) = 0$, then $\lim_{t\to\infty} t^h \mathbb{P}(Y \geq t)$ exists and is positive.

Remark 6.10 (Dimension of the carrier). Theorem 5.6 was conjectured in [**M2, M3**] and proved in [**KP**] for CCM, under the assumption **(H)**: $\mathbb{E}(\|\mu\| \,|\log \|\mu\||) < \infty$ (it holds automatically if $\tau(h) > 0$ for some $h > 1$). The tool is the often called "Peyrière probability" measure $\mathcal{Q}$ on $(\Omega \times [0,1], \mathcal{B} \otimes \mathcal{B}([0,1])$ defined as

$$\mathcal{Q}(A) = \mathbb{E}\left(\int_{[0,1]} \mathbf{1}_A(\omega, t)\, \mu(dt)\right),$$

which is appropriate for studying the logarithmic densities of μ, almost surely, μ-almost everywhere via the law of large numbers ([**KP**] involved a weaker martingale property). [**WaWi**] shows that **(H)** holds if and only if $\mathbb{E}(W(\log\, W)^2) < \infty$. [**K4**] obtained the correct dimension with no assumption apart from the non-degenderacy (that is avoiding the use of $\mathcal{Q}$). The approach is based on some operations on independent operators such as L in Section 5.1 (the reader can also consult [**Fa3, K5, Fa5, Fa6**] for other examples of use of these operations). Theorem 5.6 for MPCP is a consequence of the study in [**BM1**].

Remark 6.11 (Non-degeneracy, moments of positive orders and dimension for constructions on other structures). [**P2**] generalized results concerning these three problems to the case of CCM like measures constructed on Galton-Watson tree and their projections on $[0,1]$ (see also [**LiRo, Li1, Li2, Li3**]). Similar results are obtained for such constructions on colored trees or graphs and their projections on $\mathbb{R}^d$, [**P3, P4, Be1, Be2, Be3**].

As recalled in Section 1, [**K2**] obtained similar results for the three problems in the context of Gaussian multiplicative chaos by relating the CCM constructed on ∂A^* with certain Gaussian random weights to "Gaussian multiplicative chaos" on some other classical metric spaces. [**K2**] also studies moments of even orders for general Gaussian structures. [**Fa1**] studies this problem for moments of odd and even orders. Lower bound for the dimension of the carrier are found in [**K2**] and [**Fa2, Fa4**].

[**Fa6**] also obtained a definitive answer to these three problems when the construction is done with CCM density and the measure $\widetilde{\ell}$ on ∂A^* is replaced by an ergodic invariant (under the shift operation) Markov measure (see also the remark at the end of Section 5.2).

Remark 6.12 (Moments of negative orders). Theorem 5.5 gives a simple sufficient condition for the finiteness of a moment of negative order for $\|\mu\|$ when the density martingale is positive. Moreover, this result suffices to deal with the multifractal analysis of non-degenerate limits of such densities. But at its level of generality Theorem 5.5 does not capture the versatility of the question it deals with. Indeed, CCM and MPCP exhibit very different behaviors, due to the difference between their respective auto-correlation structures. Let us state complete results for CCM (W is normalized to satisfy $\mathbb{E}(W) = 1$), and for MPCP when $W > 0$.

Theorem 6.13 (**Moments of negative orders for CCM**). *Assume μ is non-degenerate. Fix $h > 0$.*

(1) *Case* $\mathbb{P}(W>0)=1$*: if* $\mathbb{E}(W^{-h})<\infty$ *then* $\mathbb{E}(\|\mu\|^{-bh})<\infty$*. Conversely, if* $\mathbb{E}(\|\mu\|^{-h})<\infty$ *then* $\mathbb{E}(W^{-h'})<\infty$ *for all* $h'\in(0,h/b)$*.*

(2) *Case* $\mathbb{P}(W=0)>0$*: if* $b^{1+h}\big(\mathbb{P}(\mu=0)\big)^{\frac{b-1}{b}}\mathbb{E}(\mathbf{1}_{\{W>0\}}W^{-h})<1$*, then* $\mathbb{E}(\mathbf{1}_{\{\|\mu\|>0\}}\|\mu\|^{-h'})<\infty$ *for all* $h'\in(0,h)$*. Conversely, if* $\mathbb{E}(\mathbf{1}_{\{\|\mu\|>0\}}\|\mu\|^{-h})<\infty$ *then* $b^{1+h}\big(\mathbb{P}(\mu=0)\big)^{\frac{b-1}{b}}\mathbb{E}(\mathbf{1}_{\{W>0\}}W^{-h})<1$*.*

THEOREM 6.14 (**Moments of negative orders for MPCP**). *Assume* μ *is non-degenerate and* $W>0$*. Fix* $h>0$*.* $\mathbb{E}(\|\mu\|^{-h})<\infty$ *if and only if* $\mathbb{E}(W^{-h})<\infty$*.*

In the case $\mathbb{P}(W=0)>0$, no result have been obtained yet.

Moments of negative orders for CCM were initially studied in [**K5**] when $W>0$. It seems that the result of [**K5**] (included in a series of lectures) remained unknown to other authors during almost ten years. This result claims that if $\mathbb{P}(0\le W\le x)=O(x^h)$ as $x\to 0$, then $\mathbb{P}(0\le Y\le x)=O(x^{bh})$ as $x\to 0$. [**CoKo**] and [**HoWa**] obtained the existence of all moments of negative orders under the strong hypothesis $\operatorname{essinf}(W)>0$. [**Mol**] obtained a result comparable to the one of [**K5**], namely the first assertion of Theorem 6.13 when $W>0$. Independently, [**B1, B2**] obtained $\mathbb{E}(W^{-h'})<\infty$ for all $h'\in(0,h/b)$ if and only if $\mathbb{E}(Y^{-h'})<\infty$ for all $h'\in(0,h)$. [**B1, B2**] also obtained results in the case $\mathbb{P}(W=0)>0$, in particular the fact that in this case it is necessary that moments of negative high orders always diverge. Theorem 6.13(2) is due to Liu [**Li4, Li5**] who considers, among others, the problem of moments of negative orders in the more general context when the functional equation is based on a supercritical Galton–Watson tree structure instead of a b-adic structure.

For MPCP, Theorem 6.14 is established in [**BM1**]. Its extension to PCCM is immediate.

REMARK 6.15 (Multifractal analysis). Theorem 5.11 for CCM and some of their extensions is established in [**B3**] (except that [**B3**] is only concerned with the Hausdorff spectrum associated with the level sets $\widetilde{E}_\alpha$; but this spectrum is essential since it gives the correct lower bound for other spectra). Theorem 5.11 for the Hausdorff spectrum associated with the sets F_α is established in [**BM1**] for MPCP when $W>0$. Theorem 5.11 for CCM follows numerous works ([**K5, HoWa, F1, Mol, O1, ArPa, B2**]) on the subject, whose major default is that they only give, under more or less strong hypotheses, for every fixed $q\in\mathcal{J}\cap J$, almost surely the dimension of $E_{\tau'(q)}$.

Complications arise when the density martingale has positive probability to vanish. Indeed, due to Theorem 5.4, one always has $(\mathcal{J}\cap\mathbb{R}_+)\subset J$. Due to Theorem 5.5, one always has $(\mathcal{J}\cap\mathbb{R}_-)\subset J$ for positive densities. But due to Theorem 6.13(2), it is possible that $(\mathcal{J}\cap\mathbb{R}_-)\setminus J\neq\emptyset$. For example, take $\varepsilon>0$, $b=2$, $p_0=1/2+\varepsilon$, $p_1=1/2-(3\varepsilon)/2$, and $\varepsilon\delta_0+p_0\delta_{1/(2p_0)}+p_1\delta_{1/(2p_1)}$ as probability distribution for W. If ε is small enough, $\tau'(1)>0$, $\mathcal{J}=\mathbb{R}_-$, but necessarily $\mathbb{E}(\mathbf{1}_{\{W>0\}}W^{-h})$ tend to ∞ as $h\to\infty$, so $J\cap\mathbb{R}_-$ is bounded. Then, one reaches the inequality $\varphi(q)\le-\tau(q)$ only on $\mathcal{J}\cap J$, and a piece of the spectra is missing.

[**B3**] also studies the endpoints of the Hausdorff spectrum and obtains

THEOREM 6.16 (**Endpoints of spectra for CCM**). *Assume the hypothesis of Theorem 5.11.*

(1) $\alpha_{\min}$.

(i) If there exists $q_0 \in \mathbb{R}_+$ *such that* $\tau^*(\tau'(q_0)) = 0$, *then* $\tau'(q_0) = \alpha_{\min}$. *Moreover, If* $\mathbb{E}(W^{\delta q_0}) < \infty$ *for some* $\delta > 1$ *then, with probability one, conditionally on* $\mu \neq 0$, $E_{\alpha_{\min}} \neq \emptyset$, *and for* $F \in \{\overline{E}, \underline{E}, E\}$, *one has*

$$\dim F_{\alpha_{\min}} = Dim\, F_{\alpha_{\min}} = \widetilde{f}(\alpha_{\min}) = 0.$$

(ii) If $\mathbb{R}_+ \subset \mathcal{J}$ *and* $\tau^*(\alpha_{\min}) > 0$ *then, with probability one, conditionally on* $\mu \neq 0$, $E_{\alpha_{\min}} \neq \emptyset$. *Moreover, if* $F \in \{\overline{E}, \underline{E}, E\}$, *one has*

$$\dim F_{\alpha_{\min}} = Dim\, F_{\alpha_{\min}} = \widetilde{f}(\alpha_{\min}) = \tau^*(\alpha_{\min}),$$

and $F_\alpha = \emptyset$ *for all* $\alpha \in (0, \alpha_{\min})$.

(2) $\alpha_{\max}$.

(i) If there exists $q_0 \in \mathbb{R}_-$ *such that* $\tau^*(\tau'(q_0)) = 0$, *then* $\tau'(q_0) = \alpha_{\max}$. *Moreover, if* $\mathbb{E}(\mathbf{1}_{\{W>0\}} W^{\delta q_0}) < \infty$ *for some* $\delta > 1$ *and* $q_0 \in J$ *then, with probability one, conditionally on* $\mu \neq 0$, $E_{\alpha_{\max}} \neq \emptyset$ *and for* $F \in \{\overline{E}, \underline{E}, E\}$, *one has*

$$\dim F_{\alpha_{\max}} = Dim\, F_{\alpha_{\max}} = \widetilde{f}(\alpha_{\max}) = 0$$

and $F_\alpha = \emptyset$ *for all* $\alpha \in (\alpha_{\max}, \infty)$.

(ii) If $\mathbb{R}_- \subset \mathcal{J}$ *and* $\tau^*(\alpha_{\max}) > 0$ *then, with probability one, conditionally on* $\mu \neq 0$, $E_{\alpha_{\max}} \neq \emptyset$ *and* $\dim E_{\alpha_{\max}} \geq \tau^*(\alpha_{\max})$. *Moreover, if* $W > 0$, *then for* $F \in \{\overline{E}, \underline{E}, E\}$, *one has*

$$\dim F_{\alpha_{\max}} = Dim\, F_{\alpha_{\max}} = \widetilde{f}(\alpha_{\max}) = \tau^*(\alpha_{\max}),$$

and $F_\alpha = \emptyset$ *for all* $\alpha \in (\alpha_{\max}, \infty)$.

It is shown in [**OsWa**] that if the conditions of Theorem 6.16(1)(i) (resp. Theorem 6.16(2)(i) with $q_0 \in J$) hold, then the function φ_μ exists as a limit and is linear on $[q_0, \infty)$ (resp. $(-\infty, q_0]$) with slope $-\alpha_{\min}$ (resp. $-\alpha_{\max}$) (this improves a first study in [**Mol**]).

Theorem 5.12 is established in [**BBeP**] for CCM. The technique establishes a bridge between the [**BrMiP**] multifractal formalism using boxes, and Olsen's centered multifractal formalism ([**O2**]). In particular, CCM satisfy both multifractal formalisms. For MPCP with $W > 0$, the result is proved in [**BM1**], also by using results for the b-adic grid. But no relation between the various formalisms is established in [**BM1**]. The proof of Theorem 5.12 will use the approach developed in [**BBeP**].

References

[ArPa] M. Arbeiter and N. Patzschke, *Random self-similar multifractals*, Math. Nachr., **181** (1996), 5–42.

[BaMu] E. Bacry and J.-F. Muzy, *Log-infinitely divisible multifractal processes*, Commun. Math. Phys., **236** (2003), 449–475.

[BaBMu] E. Bacry, J. Barral and J.-F. Muzy, in preparation.

[B1] J. Barral, *Continuité, moments d'ordres négatifs, et analyse multifractale des cascades multiplicatives de Mandelbrot.* Ph.D. Thesis, Orsay, 1997.

[B2] J. Barral, *Moments, continuité, et analyse multifractale des martingales de Mandelbrot*, Probab. Theory Relat. Fields, **113** (1999), 535–569.

[B3] J. Barral, *Continuity of the multifractal spectrum of a statistically self-similar measure*, J. Theoretic. Probab., **13** (2000), 1027–1060.

[B4] J. Barral, *Differentiability of multiplicative processes related to branching random walks*, Ann. Inst. H. Poincaré Prob. Statist., **36** (2000), 407–417.

[B5] J. Barral, *Generalized vector multiplicative cascades*, Adv. Appl. Probab. **33** (2001), 874–895.

[B6] J. Barral, *Poissonian products of random weights: uniform convergence and related measures*, Rev. Math. Ibero-americana, **19** (2003), 813–856.

[B7] J. Barral, *Techniques for the study of infinite products of independent random functions (Random multiplicative multifractal measures, Part III)*, these proceedings.

[BBeP] J. Barral, F. Ben Nasr and J. Peyrière, *Comparing multifractal formalisms: the Neighboring Boxes Conditions*, Asian J. Math., **7** (2003), 149–166.

[BCM] J. Barral, M.-O. Coppens and B.B. Mandelbrot, *Multiperiodic multifractal martingale measures*, J. Math. Pures Appl., **82** (2003), 1555–1589.

[BFa1] J. Barral and A.H. Fan, *Covering numbers of different points in Dvoretzky covering*, submitted for publication.

[BFa2] J. Barral and A.H. Fan, *Asymptotic behavior of densities of certain multiplicative chaos*, in preparation.

[BM1] J. Barral and B.B. Mandelbrot, *Multifractal products of cylindrical pulses*, Probab. Theory Relat. Fields, **124** (2002), 409–430.

[BM2] J. Barral and B.B. Mandelbrot, *Introduction to infinite products of random independent functions (Random multiplicative multifractal measures, Part I)*, these proceedings.

[Be1] F. Ben Nasr, *Mesures aléatoires de Mandelbrot associées à des substitutions*, C. R. Acad. Sci. Paris Sér. I Math., **304** (1987), 255–258.

[Be2] F. Ben Nasr, *Ensembles aléatoires self-affines en loi*, Bull. Sci. Math., **116** (1992), no. 1, 111–119.

[Be3] F. Ben Nasr, *Dimension de Hausdorff de certains fractals aléatoires*, Sém. Théor. Nombres Bordeaux (2), **4** (1992), no. 1, 129–140.

[Be4] F. Ben Nasr, *Analyse multifractale de mesures*, C. R. Acad. Sci. Paris, **319**, Série I (1994), 807–810.

[BeBh] F. Ben Nasr and I. Bhouri, *Spectre multifractal de mesures boréliennes sur* $\mathbb{R}^d$, C. R. Acad. Sci. Paris Sér. I Math., **325** (1997), no. 3, 253–256.

[BeBhH] F. Ben Nasr, I. Bhouri and Y. Heurteaux, *The validity of the multifractal formalism: results and examples*, Adv. Math., **165** (2002), 264–284.

[Bi1] J.D. Biggins, *Martingale convergence in the branching random walk* J. Appl. Probability **14** (1977), 25–37.

[Bi2] J.D. Biggins, *Uniform convergence of martingales in the one-dimensional branching random walk*, Selected Proceedings of the Sheffield Symposium on Applied Probability, 1989. Edited by I.V. Basawa and R.L. Taylor. IMS Lecture Notes– Monograph Series 18, 1991, pp. 159–173.

[Bi3] J.D. Biggins, *Uniform convergence of martingales in the branching random walk*, Ann. Prob., **20** (1992), 137–151.

[Bil] P. Billingsley, *Ergodic Theory and Information*, Wiley, New York, 1965.

[BrMiP] G. Brown, G. Michon and J. Peyrière, *On the multifractal analysis of measures*, J. Stat. Phys., **66** (1992), no. 3/4, 775–790.

[CRA] P. Chainais, R. Riedi and P. Abry, *On non scale invariant infinitely divisible cascades*, Actes du GRETSI, Paris, 2003.

[CoKo] P. Collet and F. Koukiou, *Large deviations for multiplicative chaos*, Commun. Math. Phys., **147** (1992), 329-342.

[DL] R. Durrett and T. Liggett, *Fixed points of the smoothing transformation*, Z. Wahrsch. verw. Gebiete **64** (1983), 275–301.

[F1] K.J. Falconer, *The multifractal spectrum of statistically self-similar measures*, J. Theor. Prob., **7** (1994), 681–702.

[F2] K.J. Falconer, *Techniques in Fractal Geometry*, Wiley, New York, 1997.

[Fa1] A.H. Fan, *Une condition suffisante d'existence du moment d'ordre m (entier) du chaos multiplicatif*, Ann. Sci. Math. Québec, **10** (1986), no. 2, 119–120.

[Fa2] A.H. Fan, *Chaos additif et chaos multiplicatif de Lévy*, C. R. Acad. Sci. Paris Sér. I Math., **308** (1989), no. 5, 151–154.

[Fa3] A.H. Fan, *Equivalence et orthogonalité des mesures aléatoires engendrées par des martingales homogènes*, Studia Mathematica, **98** (1991), 249–266.

[Fa4] A.H. Fan, *Sur les chaos de Lévy stables d'indice* $0 < \alpha < 1$, Ann. Sci. Math. Québec, **21** (1997), no. 1, 53–66.

[Fa5] A.H. Fan, *How many intervals cover a point in Dvoretzky covering?*, Israël J. Math., **131** (2002), 157–184.

[Fa6] A.H. Fan, *On Markov-Mandelbrot martingales*, J. Math. Pures Appl., **81** (2002), 967–982.

[FaK] A.H. Fan and J.-P. Kahane, *How many intervals cover a point in random dyadic covering?*, Portugaliae Mathematica, **58** (2001), 59–75.

[Gu] Y. Guivarc'h, *Sur une extension de la notion de loi semi-stable*, Ann. Inst. H. Poincaré, Probab. et Statist. **26** (1990), 261–285.

[H] Y. Heurteaux, *Estimations de la dimension inférieure et de la dimension supérieure des mesures*, Ann. Inst. H. Poincaré, Probab. et Statist. **34** (1998), 309–338.

[HoWa] R. Holley and E.C. Waymire, *Multifractal dimensions and scaling exponents for strongly bounded random fractals*, Ann. Appl. Probab., **2** (1992), 819–845.

[JoLeN] A. Joffe, L. Le Cam and J. Neveu, *Sur la loi des grands nombres pour des variables aléatoires de Bernoulli attachées à un arbre dyadique*, C. R. Acad. Sci. Paris, **277** (1973), 963–964.

[K1] J.-P. Kahane, *Sur le modèle de turbulence de Benoî̂t Mandelbrot*, C. R. Acad. Sci. Paris, **278** (1974), 621–623.

[K2] J.-P. Kahane, *Sur le chaos multiplicatif*, Ann. Sci. Math. Québec, **9** (1985), 105–150.

[K3] J.-P. Kahane, *Positive martingales and random measures*, Chi. Ann. of Math., **8B1** (1987), 1–12.

[K4] J.-P. Kahane, *Multiplications aléaroires et dimensions de Hausdorff*, Ann. Inst. Henri Poincaré, **23** (1987), 289–296.

[K5] J.-P. Kahane, *Produits de poids aléatoires et indépendants et applications.* In: Fractal Geometry and Analysis, J. Bélair and S. Dubuc (eds.), 1991, pp. 277–324.

[K6] J.-P. Kahane, *Random coverings and multiplicative processes.* In : Fractal Geometry and Stochastics, II (Greifswald/Koserow, 1998), pp. 125–146, Progr. Probab., 46, Birkhäuser, Basel, 2000.

[K7] J.-P. Kahane, *Multiplicative chaos and multimeasures*, Complex analysis, operators, and related topics, 115–126, Oper. Theory Adv. Appl., 113, Birkhäuser, Basel, 2000.

[KP] J.-P. Kahane and J. Peyrière, *Sur certaines martingales de Benoît Mandelbrot*, Adv. Math, **22** (1976), 131–145. Translation in [**M5**].

[Ki] J.F.C. Kingman, *Completely random measures*, Pacific J. Math., **21** (1967), 59–78.

[L-VVoj] J. Lévy-Véhel and R. Vojak, *Mutual multifractal analysis of sequences of Choquet capacities: preliminary results*, Adv. Appl. Math., **20** (1998), 1–43.

[Li1] Q. Liu, *Sur une équation fonctionnelle et ses applications: une extension du théorème de Kesten-Stigum concernant des processus de branchement*, Adv. in Appl. Probab., **29** (1997), 353–373.

[Li2] Q. Liu, *Fixed points of a generalized smoothing transformation and applications to the branching random walk*, Adv. in Appl. Probab., **30** (1998), 85–112.

[Li3] Q. Liu, On generalized multiplicative cascades, Stoch. Proc. Appl., **86** (2000), 263–286.

[Li4] Q. Liu, *Asymptotic properties and absolute continuity of laws stable by random weighted mean*, Stoch. Proc. Appl., **95** (2001), 83–107.

[Li5] Q. Liu, *An extension of a functional equation of Poincaré and Mandelbrot*, Asian J. Math., **6** (2002), 145–168.

[LiRo] Q. Liu and A. Rouault, *On two measures defined on the boundary of a branching tree.* In: Classical and Modern Branching Processes (Minneapolis, MN, 1994), IMA Vol. Math. Appl., **84**, Springer-Verlag, New York, 1997, pp. 187–201.

[M1] B.B. Mandelbrot, *Possible refinement of the lognormal hypothesis concerning the distribution of energy in intermittent turbulence.* In: Statistical Models and Turbulence (La Jolla, California). Edited by M. Rosenblatt and C. Van Atta. Lectures Notes in Physics 12. Springer–Verlag, New York, 1972, pp. 333–351. Reprint in [**M5**].

[M2] B.B. Mandelbrot, *Intermittent turbulence in self-similar cascades: divergence of hight moments and dimension of the carrier*, J. Fluid. Mech., **62** (1974), 331–358. Reprint in [**M5**].

[M3] B.B. Mandelbrot, *Multiplications aléatoires itérées et distributions invariantes par moyennes pondérées*, C. R. Acad. Sci. Paris, **278** (1974), 289–292 and 355–358. Reprint in [**M5**].

[M4] B.B. Mandelbrot, *Multifractal products of pulses*, unpublished memorandum, 1996.

[M5] B.B. Mandelbrot, *Multifractals and $1/f$ Noise*, Springer–Verlag, New York, 1999.

[ManNoR] P. Mannersalo, I. Norros and R. Riedi, *Multifractal products of stochastic processes: construction and basic properties*, Adv. Appl. Prob., **34** (2002), 888–903.

[Mol] G.M. Molchan, *Scaling exponents and multifractal dimensions for independent random cascades*, Commun. Math. Phys., **179** (1996), 681–702.

[N] J. Neveu, *Martingales à Temps Discret*, Masson et C$^{\text{ie}}$, Paris, 1972.

[O1] L. Olsen, *Random geometrically graph directed self-similar multifractals*, Pitman Res. Notes Math. Ser., **307**, 1994.

[O2] L. Olsen, *A multifractal formalism*, Adv. Math., **116** (1995), 92–195.

[OW] L. Olsen and S. Winter, *Normal and non-normal points of self-similar sets and divergence points of self-similar measures*, J. London Math. Soc., **67** (2003), 103–122.

[OsWa] M. Ossiander and E.C. Waymire, *Statistical estimation for multiplicative cascades*, Ann. Stat., **28** (2000), 1533–1560.

[P1] J. Peyrière, *Turbulence et dimension de Hausdorff*, C. R. Acad. Sci. Paris, **278** (1974), 567–569.

[P2] J. Peyrière, *Calculs de dimensions de Hausdorff*, Duke Math. J., **44** (1977), 591–601.

[P3] J. Peyrière, *Processus de naissance avec interaction des voisins, évolution de graphes*, Ann. Inst. Fourier (Grenoble), **31** (1981), no. 4, vii, 187–218.

[P4] J. Peyrière, *Mesures singulières associées à des découpages aléatoires d'un hypercube*, Colloq. Math., **51** (1987), 267–276.

[P5] J. Peyrière, *Recent results on Mandelbrot multiplicative cascades*, Progress in Probability, **46** (2000), Birkhäuser–Verlag, Basel, pp. 147–159.

[RaRo] B.S. Rajput and J. Rosinski, *Spectral representations of infinitely divisible processes*, Probab. Theory Related Fields, **82** (1989), 451–487.

[R] R. Riedi, *An improved multifractal formalism and self-similar measures*, J. Math. Anal. Appl., **189** (1995), 462–490.

[vBahrE] B. von Bahr and C.-G. Esseen, *Inequalities for the r^{th} absolute moment of sum of random variables,* $1 \le r \le 2$, Ann. Math. Statist., **36** (1965), 299–303.

[WaWi] E.C. Waymire and S.C. Williams, *A cascade decomposition theory with applications to Markov and exchangeable cascades*, Trans. Amer. Math. Soc., **348** (1996), no. 2, 585–632.

[Z] G. Zohar, *Large deviation formalsim for multifractals*, Stoch. Proc. Appl., **79** (1999), 229–242.

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