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INTERGALACTIC VOIDS AND FILAMENTS, AND WHY THEY NECESSARILY OCCUR IN FRACTAL MODELS

by

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1. Introduction.

Before a natural shape can be explained, it is usually necessary to describe it by an analytic formula that generates geometric "simulations". Ordinarily, the resulting "mere description" is only a black box, from which one can remove nothing beyond what has been knowingly put in. However, a good geometric description may suggest things to look for and to measure, and make predictions about what one should expect to observe.

Two of my fractal models of the distribution of galaxies are sketched below. In either case the input *does not* knowingly include either clustering, or voids, or "disconnected pseudo-filaments". Yet, clusters, voids and disconnected pseudo-filaments are invariably and inevitably part of the output. The present text is a summary of, and an introduction to, Chapters 9 and 32 to 35 of my book *The Fractal Geometry of Nature*, W. H. Freeman, 1982.

Speaking loosely, my models' basic input is scale invariance. Part of the output is that the mass in a sphere of radius R takes the form $M(R) = FR^D$. The empirical counterpart of this relation was pioneered by Carpenter and de Vaucouleurs. The D is a numerical parameter — typically *not* an integer — and the F is a random prefactor. In the loose language of yesterday, D was an anomalous exponent. In the precise language of my new fractal geometry of nature, D is a fractal dimension. In the most general spatial fractal (e.g., in those yielded by my second and more versatile model — to be described below), D can take any value between 0 and 3. Loosely speaking, a fractal is a scaling geometric shape other than a straight line, a plane or the whole volume. The best current estimate is $D \sim 1.23$; it is obtained from the M(R) that is inferred indirectly, by reinterpreting the analysis by Peebles et al. of the empirical two-point correlations of the positions of the galaxies upon the sky. (These correlations are readily deduced from the above form of M(R).) By selecting suitable units of mass and length, the expectation of F can be set to be equal to 1. If so, the first significant characteristic of F is its variance. I found that the variance of F measures a property of a fractal that I dubbed "lacunarity". It is a fully quantified aspect of the loose notion of "texture". When lacunarity is high, gigantic voids are present. This is the case in my first model described below (the Seeded Heaven). When lacunarity is low, voids tend to be smaller. In my second model described below (the Parted Heaven), lacunarity can be controlled at will.

2. Historical aside.

Many other models of the distribution of galaxies have been developed over the years, going back to old amateurs like Fournier d'Albe (1907), whose work is usually called the Charlier model, to contemporary experts, like Peebles & Soneira. The Fournier-Charlier model is built to fit two features: that galaxies cluster without end, and that the mass in a sphere of radius R is not proportional to the volume $(4/3)\pi R^3$, but instead is proportional to R itself. The Fournier-Charlier model yields nothing that has not been put in. The Peebles & Soneira model is built to agree with the correlations that Peebles et al. had observed, and also to achieve the observed texture of "pseudo-filaments" or "streams". Numerous parameters are needed, and I understand that no property is predicted.

From the geometric viewpoint, the distributions involved in the above models are examples of "construction of fractals before there was a theory of fractals". The systematic study of fractals and their uses is the object of a new geometry of nature that I conceived, developed and used in many diverse theories. The comparison of the unknowingly fractal models and of my knowingly fractal one suggests it is best to do fractals with a license, because my fractal models alone yield more than what I had put into them.

3. Two tales of how the world began.

To construct cosmological models is necessarily a blasphemous activity. Moreover, to retell what happened *At the beginning* is to play at being the author of a new *Genesis*. Since *Genesis* could not be improved as literature, I prefer to praise it by imitations. Here are the main features of my two cosmological models.

THE SEEDING OF THE HEAVEN. At the beginning, the heaven was a void. And the Master of Matter, Light and Life proclaimed, Let there be matter: and matter was. It was one point. And the Master proclaimed, Let matter be seeded over the heaven, and let every small part of the heaven be just like every other small part and like every large part. And two archangels set forth hopping; wherever they alighted, they left a pinch of matter and then resumed their journey as at its beginning. And the parts of the heaven were all made just alike. And the Master was everywhere: dwelling in every pinch of matter; and the heaven looked the same from every point where the Master dwelt.

THE PARTING OF THE HEAVEN. At the beginning, the heaven was filled with matter. And the Master of Matter, Light and Life proclaimed, Let matter part away. Let it remove itself to form voids without number, and Let every small part of the heaven be just like every other small part and like every large part. And matter removed itself, and the Master was everywhere, dwelling in every place that was not in a void: and the heaven looked the same from every point where the Master dwelt.

4. Fractal implementation of the "Seeded Heaven" model.

The model asserts that, wherever the two archangels alight, they resume their journey as at its beginning. That is, each archangel's flight is a random process whose increments are statistically stationary (translation invariant) and independent. The assertion that every small part of the heaven is just like every large part means that the distribution is scaling (dilatation invariant). There is only one family of random processes with these properties: I call them Lévy flights. They depend on a single parameter one can denote by D: the probability distribution of either archangel's jumps must be $Pr(U>u) = u^{-D}$. The set of points that is generated in this fashion for given D (technically: this set's closure) is a fractal dust of fractal dimension D. The value of D can range from 0 to 2. The assertion that the Master is everywhere expresses the weakened, "conditional" form I proposed one should give to the "cosmographic" principle, which is the cosmological principle as restricted to the galaxies' distribution: The new form asserts that the distribution of the galaxies is the same everywhere, but on one condition: the origin of the frame of reference must be a point *that is itself a part of the distribution*. Thus (in opposition for example to the Fournier model) any Seeded Heaven *has no privileged center* in a statistical sense.

A first striking and satisfactory fact about the samples of a Seeded Heaven is that its galaxies look extremely clustered. Each sample does involve a privileged center, and moreover it seems to involve several distinct length sizes, the same for different observers, and apparently related to physically distinct clusters, clusters of clusters, etc... This is so despite the fact that the generating mechanism, e.g., the basic jump length distribution $Pr(U>u) = u^{-D}$, involves no length scale whatsoever. Thus, a most surprising and important fact is revealed by my model: the observed clustering need not be a property of the generative mechanism; it can be merely a property of the samples.

A subtle point is involved here: the physicist may characterize a fractal as being a shape incorporating every element of scale between a lower cutoff that may be positive or 0, and an upper cutoff that may be finite or infinite. In some fractals, the acceptable scaling are a priori restricted to a discrete geometric sequence L, bL, b^2L... but in my models the acceptable scales are unrestricted, that have a continuous and smooth distribution. However, each random sample seems to select its collection of discrete scales.

A second and third striking and satisfactory facts about the sample of a Seeded Heaven is that its galaxies seem to involve "streams" or "filaments", and leave wide open voids. Again, this is not a property of the generative mechanism but of the samples.

The significance of the streams was brought to my attention in 1974 (at the very first lecture I gave on this subject). The streams' origin is best explained by mentioning that the limit of a Lévy flight for $D = 2$ is Brownian motion, which is a continuous process with genuine filaments, and that a Lévy flight with $D<2$ can be obtained starting from Brownian motion by a process of depletion called fractal subordination. The subordinated process is totally discontinuous (it is a "fractal dust"), nevertheless it follows the trace of its Brownian "subordinand".

As to the presence of voids, it is part of the basic property that a fractal *cannot conceivably* be translationally invariant. This is even true of random sample fractals generated by a translationally invariant process. This failure of translation invariance to hold has been a source of concern and long discussion in the context of the applications of fractals to the statistical physics of condensed matter; in the present context, it means that voids are inevitable.

A fourth striking and most satisfactory feature of the Seeded Heaven model concerns the 2- and 3-point correlation functions of the galaxies' projections on the sky. A *rigorous derivation* shows them to be given by the formulas $w(\theta) = A\theta^{D-2}$ and $z(\theta_1,\theta_2,\theta_3) = Q [w(\theta_1)w(\theta_2) + w(\theta_2)w(\theta_3) + w(\theta_3)w(\theta_1)]$, with $Q = 2/3$. These formulas happen to be precisely identical to the formulas that have been obtained by Peebles and his associates on the basis of pure curve-fitting.

A fifth and not too happy feature of the Seeded Heaven model is that the 4-point correlation it yields is in clear disagreement with Peebles' experimental data.

A sixth fact about the samples of a Seeded Heaven is very unsatisfactory: the voids they involve are of absurdly excessive size. For example, after removal of nearby stars, the biggest void in a sample may seem to cover a quarter of the sky. Recalling that the Latin for hole is *lacuna,* the seeded skies can be said to be of "excessive lacunarity". Lacunarity is an aspect of the loose notion of "texture". Thus we see that the correlation functions, hence the fractal dimension D, fail to tell us anything about lacunarity. It is disappointing that the same correlations should be compatible with a variety of textures.

Many cosmological models require the distribution of matter to be asymptotically homogeneous. In my Seeded Heaven model, asymptotic homogeneity can only be achieved by a multitude of simultaneous decentralized Geneses: by resorting to the juxtaposition of separate and independent seeded skies of finite size. Fortunately, this brutal expedient is only a defect of my first fractal model, and is readily remedied in the Parted Heaven model. (This is an example of a general phenomenon I have experienced in developing the new fractal geometry of nature: there is a temptation to resort rapidly to finite outer cutoffs, but one gains by resisting it.)

THE HEAVENLY STAIRCASE. The reader familiar with the Cantorian Devil's Staircase and its Lévy variant may want to combine the trails that the two archangels left behind while seeding. Then, for each point of their trail, mark along a fourth coordinate axis the total mass seeded between this point and the trail's origin. Plot the mass seeded by Gabriel in the positive direction, and the mass seeded by Michael in the negative direction. The resulting four-dimensional curve is a generalization of the Lévy staircase, wherein the one-dimensional abscissa is replaced by a three-dimensional one. This curve is far more "devilish" than either the Cantor or the Lévy Devil's Staircase: for example, risers go in all possible directions and every step is infinitely narrow, whichever its length. Since it links any two points where the Master dwells in heaven, it must be called a *Heavenly Staircase.*

5. Fractal implementation of the "Parted Heaven" model.

The model asserts that big "holes" are scattered at random over space, and that matter is prohibited from being located within them. I call these holes *tremas* (from the Greek word for hole). Their being scaling requires their volumes' distribution to satisfy $\Pr(V>v)$ is $\propto v^{-1}$; we shall write $\Pr(V>v) = (1-D/3)v^{-1}$. It is obvious that the tremas overlap heavily. And the points that lie outside of every trema can be shown to add up to a fractal of dimension D. This immediately implies that $M(R) = FR^D$ and that the 2-point correlation takes the desirable from $w(\theta) = A\theta^{D-2}$.

In earliest variant of this model, the tremas were spheres, and the qualitative shape properties of the remainder set are best understood by decreasing D gradually from an initial value of 3. Initially, the tremas overlap little, and the remainder is connected. Later, the tremas overlap into increasingly large voids, while the remainder set shrinks gradually. Finally, for a certain "critical" value of D, a "critical percolation" phenomenon is observed: the remainder dissolves from being a number of separate connected filaments into being a totally disconnected dust. Yet, as one should have expected, it continues to be true for a whole range of values of D that this dust seems to follow along the pre-dissolution filaments: this argument explains why one should expect the remainder fractal to exhibit pseudo-filaments. Thus, the empirical pseudo-filaments characteristic of galaxy distributions can be modeled by remainder fractals with a value of D just below the critical percolation value.

While these predictions were most gratifying, the original spherical trema model remained unsatisfactory because of the excessive visual lacunarity of the remainder set. The only way to erase it was to assume that the trema radii are bounded by a finite cutoff. Though less brutal than the expedient necessary to homogenize a Seeded Heaven (simultaneous decentralized Geneses), this approach was unsatisfactory. Again, it turned out to be premature — and possibly unnecessary. Indeed, I soon observed that, while allowing other trema shapes did not change the dimension hence preserved $M(R) = FR^D$ and $w(\theta)$, other features of the model were changed very significantly.

To begin with a qualitative change, the critical percolation D, i.e. the value for which filaments dissolve into a dust, can be modified. Hence, for a fixed D, the set's propensity to form pseudo-filaments can be changed. (For example, let the spherical trema be replaced by a thin shell of the same volume contained between two spheres; the propensity of the filaments to be cut will increase.)

To continue with quantitative changes, the 3-point correlation $z(\theta_1,\theta_2,\theta_3)$ is changed, insofar as the constant prefactor Q = 2/3 is replaced by a function that depends symmetrically upon θ_1, θ_2, θ_3; this function varies little, nevertheless its presence improves the fit upon the Seeded Heaven formula in which Q = 2/3 (as in Peebles's curve-fitted correlation).

Next, the theoretical 4-point correlation becomes compatible with the Peebles data.

More important, the trema shape has a direct effect upon the distribution of the prefactor F in the formula $M(R) = FR^D$. Let us elaborate upon this prefactor. In ordinary geometry, it is a numerical function of D (π when D = 2 and $4\pi/3$ when D = 3). In any fractal model, it is a random variable. In the Seeded Heaven model, its distribution was a fixed function of D and could not be acted upon, but in the Parted Heaven model, its distribution depends upon the shape of the tremas *and can be varied over a very broad range.* The variance of F comes from combining some cases when our sphere of radius R sits inside a big tight clump, some cases when its center sits near the boundary of a big clump population for each R some cases where one sphere of radius include only a small clump surrounded by void on all sides. The more nearly homogeneous a fractal, the smaller will be the variance of this F. Thus the variance of F is a good measure of a fractal's lacunarity, i.e. of its lacunarity.

6. Bibliography.

This handout being meant for specialists in astronomy, this bibliography will cover *neither* the old experimental evidence (Carpenter, de Vaucouleurs, Joĕveer, et al.), nor the recent evidence. All the references concern the fractal aspect.

My cosmological models are both investigated in Chapters 9 and 32 to 35 of

¤ B. B. Mandelbrot, *The Fractal Geometry of Nature,* San Francisco, W. H. Freeman, 1982.

The second (January '83) and third (May '83) printings are updated and augmented, and some blemishes in the text have been corrected.

The "Seeded Heaven" model was originally sketched in

¤ B. B. Mandelbrot *Les objets fractals*, Paris: Flammarion, 1975.

¤ B. B. Mandelbrot *Fractals; Form, Chance and Dimension*, San Francisco, W. H. Freeman, 1977.

The derivation of the correlations in the "Seeded Heaven" model is sketched in

¤ B. B. Mandelbrot, Comptes Rendus (Paris) **280**A, 1551-1554 (1975)

and it is given in fuller detail in

¤ P. J. E. Peebles, *The Large-Scale Structure of the Universe,* Princeton University Press, 1980 (Section 62).

A full treatment of both models is due to appear in a book on fractals edited by B. B. Mandelbrot, to combine various unpublished memoranda with some of the papers read (in July, 1982) at the (first) Fractals Workshop held in Courchevel (France).