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FRACTIONAL BROWNIAN MOTIONS, FRACTIONAL NOISES AND APPLICATIONS*

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1. Introduction. By "fractional Brownian motions" (fBm's), we propose to designate a family of Gaussian random functions defined as follows:[1] $B(t)$ being ordinary Brownian motion, and H a parameter satisfying $0 < H < 1$, fBm of exponent H is a moving average of $dB(t)$, in which past increments of $B(t)$ are weighted by the kernel $(t - s)^{H-1/2}$. We believe fBm's do provide useful models for a host of natural time series and wish therefore to present their curious properties to scientists, engineers and statisticians.

The basic feature of fBm's is that the "span of interdependence" between their increments can be said to be infinite. By way of contrast, the study of random functions has been overwhelmingly devoted to sequences of independent random variables, to Markov processes, and to other random functions having the property that sufficiently distant samples of these functions are independent, or nearly so. Empirical studies of random chance phenomena often suggest, on the contrary, a strong interdependence between distant samples. One class of examples arose in economics. It is known that economic time series "typically" exhibit cycles of all orders of magnitude, the slowest cycles having periods of duration comparable to the total sample size. The sample spectra of such series show no sharp "pure period" but a spectral density with a sharp peak near frequencies close to the inverse of the sample size [1], [4]. Another class of examples arose in the study of fluctuations in solids. Many such fluctuations are called "1:f noises," because their sample spectral density takes the form λ^{1-2H}, with λ the frequency, $\frac{1}{2} < H < 1$ and H frequently close to 1. Since, however, values of H far from 1 are also frequently observed, the term "1:f noise" is inaccurate. It is also unwieldy. With some trepidation due to the availability of several alternative expressions, we take this opportunity to propose that "1:f noises" be relabeled as *fractional noises* (see [13]). A third class of phenomena with extremely long interdependence is encountered in hydrology, where Hurst [6] found the range (to be defined below) of cumulated water flows to

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[1] Some results of this paper were sketched in [11]. The present paper is based upon (and supersedes) the following privately circulated IBM reports by B. Mandelbrot: *Self-similar random processes and the range*, April 13, 1964, and *Self-similar random processes: extrapolation, interpolation, and decay of perturbations*, May 1, 1964.

The reader may wonder why we selected a parameter H in $(0, 1)$ in preference to a parameter $H' = H - \frac{1}{2}$ in $(-\frac{1}{2}, \frac{1}{2})$. Many formulas would be simplified using H' but the statements of Corollaries 3.4 and 3.6 would be made more complicated. This would be bad because T^H laws were the rationale behind fBm's.

vary proportionately to t^H with $\frac{1}{2} < H < 1$. This fact will be seen in the sequel to be intimately related to the presence of an infinite span of interdependence between successive water flows. Hurst's law is likely to acquire significant practical importance in the design of water systems.[2]

These and related empirical findings suggest two tasks to the probabilist: (i) to press the development of the general theory to embrace the new phenomena, and (ii) to single out and study in detail many specific simple families of random functions that could in some way be expected to be "typical" of what happens in the absence of asymptotic independence. The present paper contributes to this second task. Since our purpose is *not* to contribute to the development of analytical techniques of probability, we selected fBm so as to be able to derive the results of practical interest with a minimum of mathematical difficulty. Extensive use has been made of the concept of "self-similarity," a form of invariance with respect to changes of time scale. A few self-similar processes other than fBm's will be considered in passing. From the purely mathematical viewpoint, our work has turned out to be largely expository since we discovered (while writing our paper) that fBm's have already been considered (implicitly) by Kolmogorov [7] and others [5], [8], [22, p. 122], [24, p. 262]. These references contain a wealth of material to which the applications we listed should draw general interest.

2. The definition of fractional Brownian motion. As usual, t designates time, $-\infty < t < \infty$, and ω designates the set of all the values of a random function. (This ω belongs to a sample space Ω.) The ordinary Brownian motion, $B(t, \omega)$, of Bachelier, Wiener and Lévy is a real random function with independent Gaussian increments such that $B(t_2, \omega) - B(t_1, \omega)$ has mean zero and variance $|t_2 - t_1|$, and such that $B(t_2, \omega) - B(t_1, \omega)$ is independent of $B(t_4, \omega) - B(t_3, \omega)$ if the intervals (t_1, t_2) and (t_3, t_4) do not overlap. The fact that the standard deviation of the increment $B(t + T, \omega) - B(t, \omega)$, with $T > 0$, is equal to $T^{1/2}$ is often referred to as the "$T^{1/2}$ law."

DEFINITION 2.1. Let H be such that $0 < H < 1$, and let b_0 be an arbitrary real number. We call the following random function $B_H(t, \omega)$, *reduced fractional Brownian motion* with parameter H and starting value b_0 at time 0. For $t > 0$, $B_H(t, \omega)$ is defined by

$$B_H(0, \omega) = b_0,$$

$$B_H(t, \omega) - B_H(0, \omega)$$
$$= \frac{1}{\Gamma(H + \frac{1}{2})} \left\{ \int_{-\infty}^{0} [(t - s)^{H-1/2} - (-s)^{H-1/2}]\, dB(s, \omega) + \int_0^t (t - s)^{H-1/2}\, dB(s, \omega) \right\} \tag{2.1}$$

[2] *Footnote added in proof.* Papers [15], [16] and [17], written after the present work was submitted, carry out in considerable detail the application of fBm's to hydrology, as first suggested in [11]. In particular, the mathematical appendix to [16] contains a number of complements to the present work. The body of [16] gives a number of graphical illustrations we consider striking.

(and similarly for $t < 0$). The integration is taken in the pointwise sense (as well as the mean square sense) by using the usual methods involving integration by parts. Note that if $b_0 = 0$, $B_{1/2}(t, \omega) = B(t, \omega)$. For other values of H, $B_H(t, \omega)$ is called a *fractional derivative or integral*[3] of $B(t, \omega)$ in the sense of Weyl [21]. FBm's really divide into three very different families corresponding, respectively, to $0 < H < \frac{1}{2}$, $\frac{1}{2} < H < 1$, and $H = \frac{1}{2}$.

Paul Lévy [9, p. 357] briefly commented on a similar but better known moving average of $B(t, \omega)$, namely, the Holmgren-Riemann-Liouville fractional integral:

$$B_H{}^0(t, \omega) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t - s)^{H-1/2} \, dB(s, \omega),$$

where H may be any positive number. This integral puts too great an importance on the origin for many applications, which is why Weyl's integral was introduced (see comments in Zygmund [25, §XII.8]).

If $B(t, \omega)$ is replaced by a complex-valued Brownian motion, the integral that now defines B_H will yield the complex fractional Brownian motion.

3. Self-similarity properties.

DEFINITION 3.1. The notation $\{X(t, \omega)\} \triangleq \{Y(t, \omega)\}$ will mean that the two random functions $X(t, \omega)$ and $Y(t, \omega)$ have the same finite joint distribution functions (a fortiori, the same state space).

DEFINITION 3.2. The increments of a random function $\{X(t, \omega); -\infty < t < \infty\}$ will be said to be *self-similar*[4] (*s-s*) with parameter H ($H \geqq 0$) if for any $h > 0$ and any t_0,

$$\{X(t_0 + \tau, \omega) - X(t_0, \omega)\} \triangleq \{h^{-H}[X(t_0 + h\tau, \omega) - X(t_0, \omega)]\}. \tag{3.1}$$

The following obvious theorem motivated the introduction of fBm.

THEOREM 3.3. *The increments of* fBm, $B_H(t, \omega)$, *are stationary and s-s with parameter H.*

[3] The introduction of $\Gamma(H + \frac{1}{2})$ as denominator has the following motivation: it insures that, when $H - \frac{1}{2}$ is an integer, a fractional integral becomes an ordinary repeated integral. Note also that the definition of B_H is made more symmetric by writing it as the following convergent difference of divergent integrals:

$$B_H(t_2, \omega) - B_H(t_1, \omega) = \frac{1}{\Gamma(H + \frac{1}{2})} \left\{ \int_{-\infty}^{t_2} (t_2 - s)^{H-1/2} \, dB(s, \omega) - \int_{-\infty}^{t_1} (t_1 - s)^{H-1/2} \, dB(s, \omega) \right\}.$$

[4] For every t_0, this definition means that, when t is restricted to $t \geqq t_0$, $X(t, \omega) - X(t_0, \omega)$ is a "semistable stochastic process" in the sense of Lamperti [8]. Semistability is weaker than the property of s-s increments. For example, Lévy's Riemann-Liouville fractional integral of $B(t, \omega)$ is semistable for all $H > 0$.

If $X(t, \omega)$ is semistable with parameter H and has stationary increments, then $X(t, \omega)$ is the restriction to $t \geqq 0$ of a process with s-s increments with parameter H.

Definition 3.2 could apparently be generalized by replacing the h^{-H} in (3.1) by $A(h)$. However, $A(h)$ must satisfy $A(h'h'') = A(h')A(h'')$. If $A(h)$ is measurable, or satisfies some such condition, one must have $A(h) = h^{-H}$ as postulated.

COROLLARY 3.4. *A T^H law for the standard deviation of B_H can be stated as follows:*

$$E[B_H(t+T,\omega) - B_H(t,\omega)]^2 = T^{2H}V_H, \tag{3.2}$$

where

$$V_H = [\Gamma(H+1/2)]^{-2}\left\{\int_{-\infty}^{0}[(1-s)^{H-1/2} - (-s)^{H-1/2}]^2\,ds + \frac{1}{2H}\right\}.$$

A quantity which is very important in many applications (see below) is the sequential range.

DEFINITION 3.5. Let $X(t, \omega)$ be a real-valued random function. Its *sequential range* is defined to be

$$M(t, T, \omega) = \sup_{t \leqq s \leqq t+T}[X(s,\omega) - X(t,\omega)] - \inf_{t \leqq s \leqq t+T}[X(s,\omega) - X(t,\omega)].$$

Also define $M(T, \omega)$ as $M(0, T, \omega)$. If $X(t, \omega)$ has continuous sample paths (as does B_H by Proposition 4.1) and t and T are finite, one can of course replace sup by max and inf by min.

COROLLARY 3.6. *A T^H law for the sequential range of a process of s-s increments can be stated as follows: if $X(t, \omega)$ has s-s increments with parameter H, then*

$$M(T,\omega) \triangleq T^H M(1,\omega).$$

If, for example, $X(t, \omega) = B(t, \omega)$ so that $H = \frac{1}{2}$, then $T^{-1/2}M(t, T, \omega)$ has a distribution independent of both t and T (which has been calculated by Feller [2]).

3.1. Some partial converses.

PROPOSITION 3.7. *If $X(t, \omega)$ has s-s and stationary increments and is mean square continuous, then $0 \leqq H < 1$.*

Proof. By Minkowski's inequality, for any τ_1 and $\tau_2 > 0$,

$$\{E[X(t+\tau_1+\tau_2) - X(t)]^2\}^{1/2} \leqq \{E[X(t+\tau_1+\tau_2) - X(t+\tau_1)]^2\}^{1/2} + \{E[X(t+\tau_1) - X(t)]^2\}^{1/2}.$$

By hypothesis, there is a constant V such that

$$E[X(t+\tau,\omega) - X(t,\omega)]^2 = V\tau^{2H}.$$

Therefore,

$$V^{1/2}[\tau_1 + \tau_2]^H \leqq V^{1/2}[\tau_1{}^H + \tau_2{}^H],$$

which implies $H < 1$. Mean square continuity requires $H \geqq 0$.

PROPOSITION 3.8. *If $X(t, \omega)$ is a nonconstant Gaussian random function satisfying the conditions of Proposition 3.7, then it must be fBm.*

Proof. A Gaussian process is determined by its covariance and mean properties.

3.2. Digression concerning some non-Gaussian self-similar processes. $X(t, \omega)$ may satisfy the conditions of Proposition 3.7 without being Gaussian. This is indicated by an example given by Rosenblatt [19, pp. 434–435].

If the requirement of continuity is abandoned, many other interesting self-similar processes suggest themselves. One may for example replace $B(t)$ by a non-Gaussian process whose increments are stable in the sense of Paul Lévy. Such increments necessarily have an infinite variance. "Fractional Lévy-stable random functions" have moreover an infinite span of interdependence.

3.3. Digression concerning data analysis: Hurst's empirical results concerning $M(T, \omega)$. Our original motivation in singling out fBm came from some empirical results concerning M due to Hurst [6]. This author studied the records of water flows through the Nile and through other rivers, the price of wheat and other physical series such as rainfall, temperatures, pressures, thickness of tree rings, thickness of varves (stratified mudbeds) and sunspot numbers.

His empirical conclusion is, in the first approximation, that the range is proportional to T^H, where $\frac{1}{2} < H < 1$. This was a source of great surprise for statisticians because models such as

$$X(t, \omega) = \int_0^t Y(s, \omega)\, ds,$$

where $Y(s, \omega)$ is stationary with summable covariance function, have a sequential range asymptotically proportional to $\sqrt{T}$. Thus, as may be seen in the discussions of his papers, Hurst's findings led some commentators to conclude that the river flows cannot be represented by stationary stochastic processes. As is shown in the next section, the existence of fBm with $\frac{1}{2} < H < 1$ indicates that this conclusion is not necessarily correct. We shall, however, have to return to Hurst's evidence because his empirical evaluation actually deals with the sequential range after removal of the sample mean (see §5.10).

4. Continuity and differentiation. Since (3.2) tends to zero with τ, $B_H(t, \omega)$ is mean square continuous. This, however, does not tell us anything about the sample paths.

PROPOSITION 4.1. $B_H(t, \omega)$, $0 < H < 1$, *has almost all sample paths continuous* (t *in any compact set*).

Proof. If $H > \frac{1}{2}$, the statement follows immediately from (3.2) and a theorem of Kolmogorov's (see Loève [10, p. 519]). In any case we can choose k such that $0 < k < H$ and note that (dropping the ω's in the notation)

$$\begin{aligned}\Gamma(H + \tfrac{1}{2})^{1/k} E\,|B_H(t+\tau) - B_H(t)|^{1/k} \\ &= \Gamma(H + \tfrac{1}{2})^{1/k} E\,|B_H(\tau) - B_H(0)|^{1/k} \\ &= E\left|\int_{-\infty}^{\tau} [(\tau - s)^{H-1/2} - N(s)(-s)^{H-1/2}]\, dB(s)\right|^{1/k},\end{aligned}$$

where $N(s) = 1$ if $s \leqq 0$ and zero if $s > 0$. Making a change of variable, the above becomes

$$|t|^{H/k} E\left|\int_{-\infty}^{1} [(1 - s)^{H-1/2} - N(s)(-s)^{H-1/2}]\, dB(s)\right|^{1/k} = |t|^{H/k} V(H, k)$$

and we again apply Kolmogorov's theorem.

The process $B_H(t, \omega)$ is not mean square differentiable (this follows by an obvious modification of the next proposition), and it almost surely does not have differentiable sample paths.

PROPOSITION 4.2. *$B_H(t, \omega)$ is almost surely not differentiable; in fact,*

$$\limsup_{t \to t_0} \left| \frac{B_H(t, \omega) - B_H(t_0, \omega)}{t - t_0} \right| = \infty$$

with probability one.

Proof. By (3.1) (take $B_H(0) = 0$),

$$\frac{B_H(t, \omega) - B_H(t_0, \omega)}{t - t_0} \triangleq (t - t_0)^{H-1} \{B_H(t_0 + 1, \omega) - B(t_0, \omega)\}$$
$$\triangleq (t - t_0)^{H-1} B_H(1, \omega).$$

Define the events

$$A(t, \omega) = \left\{ \sup_{0 \leqq s \leqq t} \left| \frac{B_H(s, \omega)}{s} \right| > d \right\}.$$

For any sequence $t_n \downarrow 0$ we have

$$A(t_n, \omega) \supset A(t_{n+1}, \omega);$$

thus,

$$P\{\lim_{n \to \infty} A(t_n)\} = \lim_{n \to \infty} P\{A(t_n)\}$$

and

$$P\{A(t_n)\} \geqq P\left\{ \left| \frac{B_H(t_n)}{t_n} \right| > d \right\}$$
$$= P\{ | B_H(1) | > t_n^{1-H} d\},$$

which tends to one as $n \to \infty$.

Note that the proof goes through under the assumption of self-similarity.

4.1. Fractional Gaussian noises and approximations thereto. It is inconvenient that fBm does not have a derivative. This difficulty is also encountered, as is well known, in the case of ordinary Brownian motion. Many methods, not always rigorous, have been evolved to give meaning to the concept of the "derivative of Brownian motion," the constructs so obtained being called "white Gaussian noises." Analogous approaches can be followed with the fractional Brownian motions and they lead to what may be called "fractional Gaussian noises."

The most elementary method of circumventing the fBm's lack of derivative is to smooth B_H and introduce the random function

$$B_H(t, \omega; \delta) = \delta^{-1} \int_t^{t+\delta} B_H(s, \omega)\, ds$$

(4.1)

$$= \int_{-\infty}^{\infty} B_H(s, \omega) \varphi_1(t - s)\, ds, \qquad \delta > 0,$$

where

$$\varphi_1(t) = \begin{cases} \delta^{-1} & \text{if } 0 \leqq t \leqq \delta, \\ 0 & \text{otherwise.} \end{cases}$$

The function $B_H(t, \omega; \delta)$ does have a stationary derivative, namely,

$$\begin{aligned} B_H{}'(t, \omega; \delta) &= \delta^{-1}[B_H(t+\delta, \omega) - B_H(t, \omega)] \\ &= -\int_{-\infty}^{\infty} B_H(s, \omega)\, d\varphi_1(t-s), \end{aligned} \tag{4.2}$$

which is almost surely continuous, but surely nondifferentiable.

For δ small enough, $B_H(t, \omega)$ and $B_H(t, \omega; \delta)$ are indistinguishable for many "practical purposes," i.e., excluding the high frequency effects to which the nondifferentiability of $B_H(t, \omega)$ is due (see §7).

One can, thus, proceed step by step, replacing φ_1 by ever smoother kernels. Finally, one could use an infinitely differentiable kernel φ, which vanishes outside some finite interval and integrates to one. Then the kth derivative of

$$\int_{-\infty}^{\infty} B_H(s, \omega)\varphi(t-s)\, ds$$

is

$$(-1)^k \int_{-\infty}^{\infty} B_H(s, \omega)\varphi^{(k)}(t-s)\, ds,$$

which is continuous and stationary for all positive integers k. Following up this approach, one can interpret $B_H{}'$ as being not a random function but a "generalized random function" in the sense of Schwartz distributions[5] (see Gelfand and Vilenkin [3]). For practical purposes, it may be desirable to avoid Schwartz distributions, and we shall be concerned with determining whether finite differences of B_H are reasonable approximations of $B_H{}'$.

4.2. Digression concerning some non-Gaussian fractional noises. The non-Gaussian fractional functions of §3.2 are also, in most cases, nondifferentiable. But ways may exist of defining a generalized differential, or of defining a differential after smoothing. Such constructs, when possible, may be called "fractional non-Gaussian noises." There is no doubt that such noises are required to model some of the phenomena listed in the Introduction.

5. Some correlations and their applications to the extrapolation and interpolation of $B_H(t, \omega)$. We pause to examine certain interesting properties which fractional Brownian motion has with regard to extrapolation and interpolation. This will give the reader more feeling for such processes and will help identify problems for which fBm is a good model.

[5] A very different generalization of processes directed towards low-frequency rather than high-frequency effects is proposed in [14].

5.1. The correlation between two increments of $B_H(t, \omega)$. Let T, T_1, and T_2 be fixed and nonnegative. Then (dropping the ω in the notation) compute the correlation between the increments of $B_H(t)$ over the following time intervals: $T/2$ to T_1 and $-T/2$ to T_2. One has

$$\begin{aligned}
&2E\{[B_H(\tfrac{1}{2}T + T_1) - B_H(\tfrac{1}{2}T)][B_H(-\tfrac{1}{2}T) - B_H(-\tfrac{1}{2}T - T_2)]\} \\
&\quad = E[B_H(\tfrac{1}{2}T + T_1) - B_H(-\tfrac{1}{2}T - T_2)]^2 + E[B_H(\tfrac{1}{2}T) - B_H(-\tfrac{1}{2}T)]^2 \\
&\qquad - E[B_H(\tfrac{1}{2}T + T_1) - B_H(-\tfrac{1}{2}T)]^2 - E[B_H(\tfrac{1}{2}T) - B_H(-\tfrac{1}{2}T - T_2)]^2.
\end{aligned}$$

Thus, the desired correlation is

$$(5.1)\qquad \frac{1}{2}\,\frac{(T + T_1 + T_2)^{2H} + T^{2H} - (T + T_1)^{2H} - (T + T_2)^{2H}}{T_1{}^H T_2{}^H}\,.$$

If $T > 0$, we can write $S_1 = T_1/T$ and $S_2 = T_2/T$ and we see that the correlation is only a function of the reduced variables S_1 and S_2 (as expected from self-similarity):

$$(5.2)\quad C(S_1, S_2) = \frac{1}{2}\,\frac{(1 + S_1 + S_2)^{2H} + 1 - (1 + S_1)^{2H} - (1 + S_2)^{2H}}{(S_1S_2)^H}\,.$$

For all S_1 and S_2, this correlation is positive if $\frac{1}{2} < H < 1$ and negative if $0 < H < \frac{1}{2}$. This is the first example of a series of distinctions based on the sign of $H - \frac{1}{2}$.

5.2. Strong mixing. Now consider the least upper bound of the absolute value of the correlation (5.1) over various sets of values of T, T_1 and T_2. Fixing T_1 and T_2, we see that this absolute value attains a maximum for $T = 0$. Then varying T_1/T_2, we see that for $T_1 = T_2$ it attains a maximum equal to $|2^{2H-1} - 1|$. If T is fixed and >0, $|2^{2H-1} - 1|$ is not an attainable maximum but remains a least upper bound (corresponding to $T_1 = T_2 = \infty$).

This leads us to Rosenblatt's [19] condition of strong mixing, a form of asymptotic independence. In the Gaussian case, Kolmogorov and Rozanov showed that strong mixing requires that a certain maximal correlation coefficient tends to zero as the distance between the two time points tends to infinity. By self-similarity and (5.1), that coefficient is bounded below by $|2^{2H-1} - 1| > 0$ in the case of fBm. Therefore, strong mixing does not hold for the increments of fBm, except in the classical Brownian case $H = \frac{1}{2}$.

Strong mixing was originally introduced as one of several conditions that a random process must satisfy in order that the central limit theorem be applicable. This question does not arise here, since the increments of fBm constitute a Gaussian process and satisfy the central limit theorem trivially. The practical importance of strong mixing is therefore to be found elsewhere. To say that the increments of a fBm are not strongly mixing happens to be a convenient way of expressing the idea that the span of interdependence between such increments is infinite (see end of §6.3).

5.3. Extrapolation and interpolation of $B_H(t, \omega)$ from its values $B_H(0, \omega) = 0$ and $B_H(T, \omega)$ with $T > 0$ to its values for $-\infty < t < \infty$. Recall that if G_1 and G_2 are two dependent Gaussian random variables with zero mean, then

$$\frac{E[G_1 \mid G_2]}{G_2} = \frac{E[G_1 G_2]}{E[G_2^2]}. \tag{5.3}$$

Thus, by setting $B_H(0) = 0$,

$$\begin{aligned}\frac{E[B_H(t) \mid B_H(T)]}{B_H(T)} &= \frac{E[B_H(t) B_H(T)]}{E[B_H^2(T)]} \\ &= \frac{EB_H^2(t) + EB_H^2(T) - E[B_H(t) - B_H(T)]^2}{2E[B_H^2(T)]}.\end{aligned} \tag{5.4}$$

This yields the interpolatory-extrapolatory formula

$$\frac{E[B_H(t) \mid B_H(T)]}{B_H(T)} = \frac{t^{2H} + T^{2H} - |t - T|^{2H}}{2T^{2H}}. \tag{5.5}$$

By defining the "reduced" variable $s = t/T$, (5.5) becomes

$$\begin{aligned}\frac{E[B_H(sT) \mid B_H(T)]}{B_H(T)} &= \frac{1}{2}[s^{2H} + 1 - |s - 1|^{2H}] \\ &\equiv Q_H(s), \qquad \text{by definition}\end{aligned}$$

(see Fig. 1).

In the case of Brownian motion $H = \frac{1}{2}$, $Q_H(s)$ is represented by a kinked curve made up of sections of straight lines. However, if $\frac{1}{2} < H < 1$, $Q_H(s)$ has a continuous derivative $Q_H'(s)$ which satisfies the following:

$$0 < Q_H'(0), \qquad Q_H'(1) < 1 \quad \text{and} \quad Q_H'(\tfrac{1}{2}) > 1.$$

Finally, for $0 < H < \frac{1}{2}$, $Q_H(s)$ is differentiable except at $s = 0$ and $s = 1$, where it has a cusp.

5.4. Extrapolation for large $|t|$. For the Brownian case $H = \frac{1}{2}$, we have that for all $t > T$,

$$E[B(t, \omega) \mid B(T, \omega)] = B(T, \omega).$$

Thus, the best forecast is that $B(t, \omega)$ will not change. For $\frac{1}{2} < H < 1$, on the contrary,

$$Q_H(s) \sim H|s|^{2H-1} \quad \text{for } s \text{ large},$$

and the extrapolation involves a *nonlinear "pseudo-trend"* that diverges to

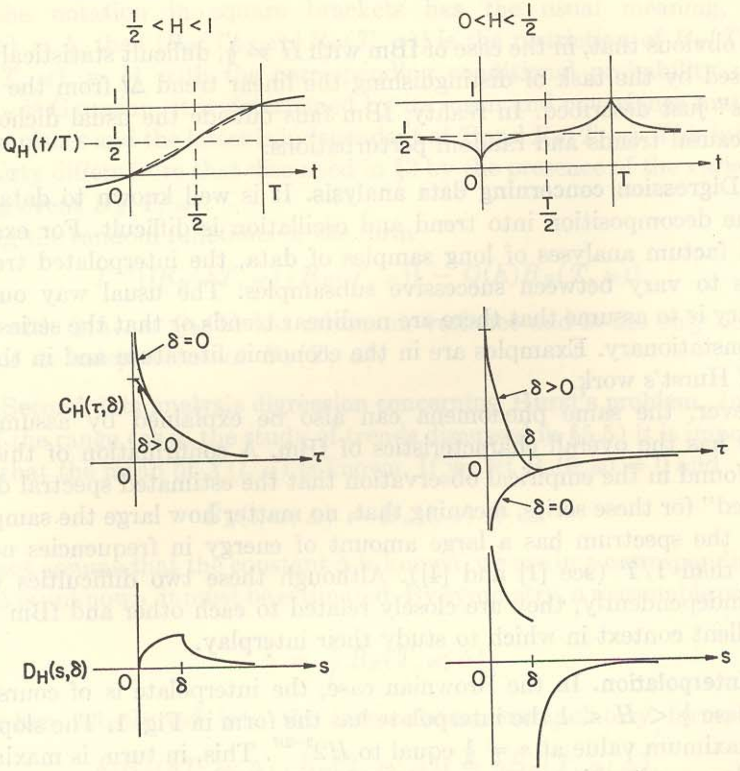

FIG. 1. *Freehand graphs of the shape of several important functions used in the text. The function Q_H occurs in the interpolation and extrapolation of B_H (§5.3). The function $C_H(\tau, \delta)$ is covariance of the process of finite differences $B_H(t + \delta, \omega) - B_H(t, \omega)$, where t is a continuous time (§6). The function $D_H(s, \delta)$ occurs in §6.1. The differences between the two cases $0 < H < \frac{1}{2}$ and $\frac{1}{2} < H < 1$ are striking.*

infinity. In the remaining case, $0 < H < \frac{1}{2}$,

$$Q_H(s) \sim \tfrac{1}{2} \quad \text{for } s \text{ large},$$

and the extrapolation has a *nonlinear "pseudo-trend"* that converges to

$$\tfrac{1}{2}[B_H(0, \omega) + B_H(T, \omega)].$$

5.5. Extrapolation for large $|t|$ when $E[B_H(t, \omega)] \neq 0$. The problem of "variable trends." In analyzing time series $X(t, \omega)$ without "seasonal effects," it is customary to search for a decomposition into a "linear trend component" and an "oscillatory component." The former usually is an estimate of

$$E[X(t + \tau, \omega) - X(t, \omega)],$$

and it is interpreted as due to major "causal" changes in the mechanism generating $X(t, \omega)$. The latter, on the contrary, is taken to be an "uncontrollable" stationary process, hopefully free of low-frequency components.

It is obvious that, in the case of fBm with $H \neq \frac{1}{2}$, difficult statistical problems are raised by the task of distinguishing the linear trend Δt from the nonlinear "trends" just described. In reality, fBm falls outside the usual dichotomy between causal trends and random perturbations.

5.6. Digression concerning data analysis. It is well known to data analysts that the decomposition into trend and oscillation is difficult. For example, in ex-post factum analyses of long samples of data, the interpolated trend often appears to vary between successive subsamples. The usual way out of this quandary is to assume that there are nonlinear trends or that the series is otherwise nonstationary. Examples are in the economic literature and in the discussions of Hurst's work.

However, the same phenomena can also be explained by assuming that $X(t, \omega)$ has the overall characteristics of fBm. A confirmation of this conjecture is found in the empirical observation that the estimated spectral density is very "red" for these series, meaning that, no matter how large the sample duration T, the spectrum has a large amount of energy in frequencies not much greater than $1/T$ (see [1] and [4]). Although these two difficulties were observed independently, they are closely related to each other and fBm provides an excellent context in which to study their interplay.

5.7. Interpolation. In the Brownian case, the interpolate is of course linear. In the case $\frac{1}{2} < H < 1$ the interpolate has the form in Fig. 1. The slope $Q_H'(s)$ has a maximum value at $s = \frac{1}{2}$ equal to $H2^{2-2H}$. This, in turn, is maximum for $H = \frac{1}{2}\log_2 e$, where it turns out to be 1.06. Thus, $Q_H(s)$ for $0 < s < 1$ is quite close to linear if $\frac{1}{2} < H < 1$. Lastly, if $0 < H < \frac{1}{2}$, the interpolate has an S-shape which is inverted with respect to that of the previous case (see Fig. 1).

5.8. The variance of $B_H(t, \omega)$ conditioned by $B_H(0, \omega)$ and $B_H(T, \omega)$. The conditional expectation is the interpolate and extrapolate having smallest variance. The usual formulas for the Gaussian case tell us that given $B_H(0, \omega)$ and $B_H(T, \omega)$ the variance of $B_H(sT, \omega)$ is

$$V_H(Ts)^{2H}\left\{1 - [1 + s^{2H} - |1 - s|^{2H}]^2 \frac{1}{4s^{2H}}\right\}.$$

For s large this tends to $V_H(Ts)^{2H}$. Thus, the standard deviation, σ, is asymptotically proportional to s^H. Moreover, as $s \to \infty$,

$$\frac{\sigma}{E[B(sT, \omega) \mid B(T, \omega)]} \sim \begin{cases} s^{1-H} & \text{if } \frac{1}{2} < H < 1, \\ s^H & \text{if } 0 < H < \frac{1}{2}. \end{cases}$$

Note that this ratio always increases without bound as s increases.

5.9. Conditional s-s property. While on the subject of conditional random variables we might mention a property which we call *conditional self-similarity.* This concept plays an important role in the theory developed in [14]. Let us look at the random function

$$U_H(h, \omega; T, B_H(T, \omega)) = T^{-H}\{[B_H(Th, \omega) \mid B_H(T, \omega)] - Q_H(h)B_H(T, \omega)\},$$

where the notation in square brackets has the usual meaning, e.g., if $B_H(T, \omega) = b$, then $[B_H(Th, \omega) \mid B_H(T, \omega)]$ is the restriction of $B_H(Th, \omega)$ to $\{\omega \mid B_H(T, \omega) = b\}$ with the corresponding conditional probability measure. Since U_H is Gaussian, it is determined by its mean and covariance matrix. The former vanishes and the latter is independent of T and $B_H(T, \omega)$. This interesting s-s property differs from that discussed in §3 by the presence of the variable conditioning event $B_H(T, \omega)$.

Among the random functions of the form

$$T^{-H}\{[B_H(hT, \omega) \mid B_H(T, \omega)] - Q(h)B_H(T, \omega)\},$$

the one with $Q(h) = Q_H(h)$ has minimum variance and is the only one where the value is independent of $B_H(T, \omega)$.

5.10. Second data analysis digression concerning Hurst's problem. In Hurst's study of the range (as in the study of trends discussed in §5.5) it is impossible to assume that the mean of $X(t, \omega)$ is known. If we let $B_H(0, \omega) = 0$ and

$$\tilde{B}_H(t, \omega; \Delta) \equiv B_H(t, \omega) + t\Delta$$

and do not assume that the constant Δ is known, we are in a corresponding situation. If Δ is unknown, it must be estimated. By symmetry, a reasonable estimate is

$$\hat{\Delta} = \frac{1}{T} \tilde{B}_H(T, \omega; \Delta),$$

which when substituted into the interpolatory-extrapolatory formula yields

$$\hat{E}[\tilde{B}_H(hT, \omega; \Delta) \mid \tilde{B}_H(T, \omega; \Delta)] = h\tilde{B}_H(T, \omega; \Delta).$$

In as far as the range is concerned, one is led to consider

$$M_H^*(T, s, \omega) = \max_{0 \leq h \leq s} [\tilde{B}_H(hT, \omega; \Delta) - h\tilde{B}_H(T, \omega; \Delta)]$$
$$- \min_{0 \leq h \leq s} [\tilde{B}_H(hT, \omega; \Delta) - h\tilde{B}_H(T, \omega; \Delta)],$$

which can readily be seen to be independent in distribution of T and to satisfy the s^H law. This helps further explain the empirical finding of Hurst.

The results of §5 could, of course, be easily generalized to cases where the process is given at more than two points. The formulas become much more complicated, but it is worth noting that they again are functions of certain "reduced" variables.

6. The derivative of the smoothed process $B_H(t, \omega; \delta)$. The derivative process, $B_H'(t, \omega; \delta)$, is itself interesting as a stochastic model. Being stationary, it has a covariance of the form

$$C_H(\tau; \delta) = E[B_H'(t, \omega; \delta) \mid B_H'(t + \tau, \omega; \delta)].$$

Without loss of generality assume $B_H(0, \omega) = 0$; then

$$C_H(\tau; \delta) = \frac{1}{2} V_H \delta^{2H-2} \left[\left(\frac{|\tau|}{\delta} + 1 \right)^{2H} - 2 \left| \frac{\tau}{\delta} \right|^{2H} + \left| \frac{|\tau|}{\delta} - 1 \right|^{2H} \right].$$

If $\tau \gg \delta$,

$$C_H(\tau, \delta) \sim V_H H(2H - 1) | \tau |^{2H-2}.$$

This has the same sign as $H - \frac{1}{2}$. It tends to zero as $| \tau | \to \infty$, which (by a theorem of Maruyama [18]) means that $B_H'(t, \omega; \delta)$ is weakly mixing and ergodic. However, from our remarks in §5, $B_H'(t, \omega; \delta)$ is not strong mixing (this follows from the representation (4.2)).

For $\tau = 0$,

$$C_H(0, \delta) = V_H \delta^{2H-2};$$

for small values of $| \tau |/\delta$,

$$C_H(0; \delta) - C_H(\tau; \delta) \sim V_H \delta^{-2} \tau^{2H}.$$

If $\frac{1}{2} < H < 1$, $C_H(\tau; \delta)$ is positive and finite for all τ, and one has

$$\int_0^\infty C_H(s, \delta)\, ds = \infty.$$

If $0 < H < \frac{1}{2}$, $C_H(\tau; \delta)$ changes sign once from positive to negative, at a value of τ proportional to δ, and one has

$$\int_0^\infty C_H(s, \delta)\, ds = 0.$$

6.1. Extrapolation of $B_H'(t, \omega; \delta)$ and $B_H(t, \omega)$. Given $\{B(s, \omega), -\infty < s < t\}$ the least squares estimate of $B_H(t + \tau, \omega)$ is, with $N(s)$ defined p. 426,

$$\hat{B}_H(t + \tau, \omega) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{-\infty}^t [(t + \tau - s)^{H-1/2} - (-1)^{H-1/2} N(s)\, dB(s, \omega)].$$

If $\tau > 0$, $\hat{B}_H$ is infinitely differentiable (mean square or a.e.) in τ. Thus,

$$\frac{d\hat{B}_H(t + \tau, \omega)}{d\tau} = \frac{H - \frac{1}{2}}{\Gamma(H + \frac{1}{2})} \int_{-\infty}^t (t + \tau - s)^{H-3/2}\, dB(s, \omega).$$

Define the decay kernel $D_H(t, \delta)$ as

$$D_H(t; \delta) = \begin{cases} [\delta\Gamma(H + \frac{1}{2})]^{-1} t^{H-1/2} & \text{for } t \leqq \delta, \\ [\delta\Gamma(H + \frac{1}{2})]^{-1} [t^{H-1/2} - (t - \delta)^{H-1/2}] & \text{for } t > \delta. \end{cases}$$

Then

$$B_H'(t, \omega; \delta) = \int_{-\infty}^t D_H(t - s; \delta)\, dB(s + \delta, \omega),$$

which is a one-sided moving average. It follows that the least squares predictor of $B_H'(t + \tau, \omega; \delta)$ given $\{B(s, \omega), -\infty < s \leqq t\}$ is

$$\hat{B}_H'(t + \tau, \omega; \delta) = \int_{-\infty}^{t-\delta} D_H(t + \tau - s; \delta)\, dB(s + \delta, \omega),$$

which tends to $\hat{B}_H'(t + \tau, \omega)$ as $\delta \to 0$.

A fundamental relation between the "dynamic" law of relaxation of perturbations D and the "static" law of the distribution of the spontaneous fluctuations as expressed by the covariance $C_H(t;\delta)$ is the well-known formula:

$$C_H(t;\delta) = \int_0^\infty D_H(s,\delta)D_H(s+t,\delta)\,ds.$$

6.2. Digression concerning data analysis. A primary reason for the practical importance of fractional Brownian motion as a model arises from the fact that power function decay laws have often been observed by experimentalists. It seems, in fact, likely that they will be useful even in cases where at present the exponential law $D(s) \sim e^{-s/a}$ is used but has been adopted only because of its tractability, and because the span of observable events is too short to conclude reliably otherwise. The exponential decay law arises in the classical case when $X(t,\omega)$ is a stationary Markov-Gauss process. Then, the "age of perturbation" is not important, since for any $0 < t_0 < t$ the percentage attenuation between 0 and t can be obtained as the product of two independent decays—between 0 and t_0, and between t_0 and t. Things are very different in the case of fractional Brownian motion, when the age is critically important in assessing future behavior. In economics, for example, an age-dependent law like $s^{H-3/2}$ seems preferable to the exponential, both as a law of depreciation or as a way of expressing the attenuation of the effects of way past "causes." (We say $s^{H-3/2}$ and not $s^{H-1/2}$, because we think of the "derivative" of $B_H(t,\omega)$.)

6.3. Some conditional expectation least squares predictors. Given $B_H(0,\omega) = 0$ and $B_H'(0,\omega;\delta)$, it is illuminating to resume in terms of $C_H(t,\delta)$ certain of the extrapolation problems discussed in §5. We clearly have

$$\frac{E[B_H'(s,\omega;\delta) \mid B_H'(0,\omega;\delta)]}{B_H'(0,\omega;\delta)} = \frac{C_H(s;\delta)}{C_H(0;\delta)}.$$

Integrating from 0 to t we obtain

$$\frac{E[B_H(t,\omega;\delta) - B_H(0,\omega;\delta) \mid B_H'(0,\omega;\delta)]}{B_H'(0,\omega;\delta)} = \frac{\int_0^t C_H(s,\delta)\,ds}{C_H(0;\delta)}.$$

Consider, then, the limit for $t \to \infty$ of the expectation written on the left-hand side. This limit is infinite when $\frac{1}{2} < H < 1$, and it vanishes when $0 < H < \frac{1}{2}$. It is interesting in this light to examine briefly a measure of the span of memory of a process, proposed by G. I. Taylor [20, p. 425], namely, the integral of the covariance function. If $\frac{1}{2} < H < 1$, this measure correctly asserts that the memory of the process is infinite; if $0 < H < \frac{1}{2}$, however, Taylor's measure asserts that the memory vanishes, while in fact (as we saw in discussing strong mixing) it is infinite.

7. The spectra. A very interesting frequency representation of the increments of fractional Brownian motion was obtained by Hunt [5, p. 67]:

$$B_H(t_2,\omega) - B_H(t_1,\omega) = V_H^* \int_0^\infty (e^{2\pi i\lambda t_2} - e^{2\pi i\lambda t_1})\lambda^{-H-1/2}\,dB(\lambda,\omega),$$

where V_H^* is a constant. This suggests that $B_H(t, \omega)$ has a "spectral density" proportional to λ^{-2H-1}. Spectral densities of nonstationary random functions are, however, difficult to interpret. It is tempting to differentiate B_H and claim that B_H' has a spectral density proportional to λ^{1-2H}. If $\frac{1}{2} < H < 1$, this formal density is such that it becomes infinite for $\lambda = 0$.

Spectral densities proportional to λ^{1-2H} near $\lambda = 0$, where $\frac{1}{2} < H < 1$, are very important in electronics [13].[6] The proportionality of the spectral density to λ^{1-2H} also suggests that there is infinite energy at high frequencies. Both the derivative B' and its spectrum can be interpreted via Schwartz distributions. These are not needed, however, to examine the spectrum of $B_H'(t, \omega; \delta)$.

The spectral density of $B_H'(t, \omega; \delta)$ is

$$\begin{aligned} G_H'(\lambda; \delta) &= 4 \int_0^\infty C_H(s; \delta) \cos (2\pi\lambda s)\, ds \\ &= 2V_H\delta^{-2} \int_0^\infty [(s + \delta)^{2H} - 2s^{2H} + |s - \delta|^{2H}] \cos (2\pi\lambda s)\, ds. \end{aligned}$$

A sort of self-similarity property of B_H' is expressed by the fact that one can define a function G^* by writing

$$G_H'(\lambda; \delta) = 2V_H\delta^{2H-1}G_H^*(\delta\lambda).$$

For small values of $\lambda\delta$ one has

$$G_H^*(\delta\lambda) \sim K_H(2\pi\delta\lambda)^{1-2H},$$

with

$$K_H = \frac{\pi H(2H - 1)}{\Gamma(2 - 2H)} [\cos \pi(H - 1)]^{-1} > 0.$$

Thus, $G_H'(\lambda; \delta)$ behaves like $2K_HV_H(2\pi\lambda)^{1-2H}$. For fixed $\lambda > 0$, $\lim_{\delta\to0} G_H'(\lambda, \delta)$ is positive and finite and equal to the formal spectral density of B_H'. In other words, changes in the value of δ involve detail whose energy is primarily at high frequencies.

REFERENCES

[1] I. Adelman, *Long cycles—fact or artifact?*, Amer. Economic Rev., 60 (1965), pp. 444–463.

[2] W. Feller, *The asymptotic distribution of the range of sums of independent random variables*, Ann. Math. Statist., 22 (1951), pp. 427–432.

[3] I. M. Gelfand and N. Ya. Vilenkin, *Generalized Functions*, vol. 4, Academic Press, New York, 1964.

[4] C. W. J. Granger, *The typical spectral shape of an economic variable*, Econometrica, 34 (1966), pp. 150–161.

[5] G. A. Hunt, *Random Fourier transforms*, Trans. Amer. Math. Soc., 71 (1951), pp. 38–69.

[6] A rough examination of the data also suggests that some empirical spectral densities are proportional to λ^{1-2H}, where $1 < H < \frac{3}{2}$. Such formal densities cannot be observed in samples from stationary second order processes, and the concept of sporadic process was introduced in [14] as a model of the corresponding phenomena.

[6] H. E. Hurst, R. P. Black and Y. M. Sinaika, *Long Term Storage in Reservoirs. An Experimental Study*, Constable, London, 1965.

[7] A. N. Kolmogorov, *Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum*, C. R. (Doklady) Acad. Sci. URSS (N.S.), 26 (1940), pp. 115–118.

[8] J. Lamperti, *Semi-stable stochastic process*, Trans. Amer. Math. Soc., 104 (1962), pp. 62–78.

[9] P. Lévy, *Random functions: General theory with special reference to Laplacian random functions*, Univ. California Publ. Statist., 1 (1953), pp. 331–390.

[10] M. Loève, *Probability Theory*, 2nd ed., Van Nostrand, Princeton, 1960.

[11] B. Mandelbrot, *Une classe de processus homothétiques à soi*; *application à la loi climatologique de H. E. Hurst*, C. R. Acad. Sci. Paris, 260 (1965), pp. 3274–3277.

[12] ——— , *Self-similar error-clusters in communication systems and the concept of conditional stationarity*, IEEE Trans. Comm. Tech., COM-13 (1965), pp. 71–90.

[13] ——— , *Noises with an l/f spectrum, a bridge between direct current and white noise*, IEEE Trans. Information Theory, IT-13 (1967), pp. 289–298.

[14] ——— , *Sporadic random functions and conditional spectral analysis; self-similar examples and limit*, Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability, vol. III, University of California Press, Berkeley, 1967, pp. 155–179.

[15] B. Mandelbrot and J. R. Wallis, *Noah, Joseph and operational hydrology*, Water Resources Research, to appear.

[16] ———, *Computer experiments with fractional Gaussian noise*, Ibid., to appear.

[17] ———, *Some long run properties of geophysical records*, Ibid., to appear.

[18] G. Maruyama, *The harmonic analysis of stationary stochastic processes*, Mem. Fac. Sci. Kyushu Univ. Ser. A, 4 (1949), pp. 45–106.

[19] M. Rosenblatt, *Independence and dependence*, Proc. Fourth Berkeley Symposium on Mathematical Statistics and Probability, vol. II, University of California Press, Berkeley, 1961, pp. 431–443.

[20] G. I. Taylor, *Statistical theory of turbulence*, Proc. Roy. Soc. Ser. A, 151 (1935), pp. 421–478.

[21] H. Weyl, *Bemerkungen zum Begriff der Differential-Quotenten gebrochener Ordnung*, Vierteljschr. Naturforsch. Ges. Zürich, 62 (1967), pp. 296–302.

[22] A. M. Yaglom, *Correlation theory of processes with random stationary nth increments*, Amer. Math. Soc. Transl. (2), 8 (1958), pp. 87–141.

[23] ———, *Stationary Gaussian processes satisfying the strong mixing condition and best predictable functionals*, Bernoulli-Bayes-Laplace Anniversary Volume, L. M. LeCam and J. Neyman, eds., Springer-Verlag, New York, 1965, pp. 241–252.

[24] ———, *Outline of some topics in linear extrapolation of stationary random processes*, Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability, vol. II, University of California Press, Berkeley, 1967, pp. 259–278.

[25] A. Zygmund, *Trigonometrical Series*, 2nd ed., Cambridge University Press, London, 1959.