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Foreword and 2004 update by the author: parallel "cartoon" models and a claim: fractal tools are inescapable in finance

THIS BOOK WAS WELL-RECEIVED. HOWEVER, AS I REGRET BUT MUST ACKNOWLEDGE, IT WAS ILL-SERVED BY EXCESS AMBITION. Thus, as demonstrated in a Scrapbook on my web homesite, many reviewers understood my point very well, but others did not at all.

Let me comment on this book, starting from scratch. The main topic is the variation of financial prices. In that context, two models were advanced near-simultaneously between 1960 and 1972: the Brownian (Gaussian independent increments) and the fractal/multifractal (strongly non Gaussian and/or strongly non independent). True and important, the Brownian originated much earlier, in a most remarkable pioneering work, Bachelier 1900. But it attracted little attention until about 1960, when it was either rediscovered or reinvented independently by several authors.

A famous English polymath J.B.S. Haldane (whom I happen to have met) observed that a scientific idea ought to be interesting even if it is not true. The Brownian is certainly both. By now, its limitations are so widely acknowledged that it has run its course and is completely discredited. I had noted its failings very early and have remained a persistent critic. Between 1962 and 1972, I put forward three models that correct those failings, first separately and then inextricably mixed together. Soon afterwards, those models became one of the streams that merged to form fractal geometry since the nineteen nineties. They are becoming broadly accepted. This book describes them and related materials that adds up to a fractal/multifractal approach to risk, ruin, and reward.

Unfortunately, the complication of this book's overall structure, the sheer mass of new material, and the concise style, all conspired to hide two very important facts. First fact: the fractal/multifractal approach is extremely parsimonious. That is, the models' input reduces to successively more general forms of fractality, that is, scale invariance – and the output is very rich. Second fact: through my work since the 1960s, the main fractal tools, namely the exponents α and H, have acquired a permanent role in finance.

Of course, my models will be superseded except as early approximations. But I shall argue that modeling could never step back to prefractal tools; it will have to move on to a better use of improved fractal tools. In particular, Section 1 will say why I see no future in the so-called variable volatility models.

Be that as it may, ever since this book's first printing in 1997, there have been pressing calls for one or another kind of alternative presentations of the same material.

A more gradual introduction to the fractal approach was widely called for. One response the book by B.B. Mandelbrot & R.L. Hudson. *The (mis)Behavior of Prices: The Fractal View of Risk, Ruin, and Reward,* New York: Basic Books, 2004, to be referenced as Mandelbrot & Hudson 2004, *Misbehavior,* or M & H 2004. Its publication has been a major reason for preparing this new Foreword and update. (Note that the style of reference is described on page 526 before the first editions bibliography. Additional references are listed on page xx below.) The tongue-in-cheek terms, *Misbehavior,* calls for an explanation. Every scientist is – or should be – motivated by the belief that the real world follows rules that are not yet known, can conceivably be identified and are not too complicated. The only rules with respect to which the real world may misbehave are oversimplified approximations conceived by earlier scientists, in this instance, Bachelier's Brownian motion. The goal of the fractal view is to represent the real-world behavior far more closely and nearly as simply.

More detailed studies *Misbehavior* contains many illustrations, several among which were specifically prepared for simultaneous use in this new foreword. But the text (as opposed to the endnotes) includes not a single mathematical formula. This policy will reinforce and widen interest in its opposite, that is, in books that cover the same ideas but do not shy away from mathematics. The original 1997 edition of the present book has indeed also led to pressing calls for more detailed developments and more graphics to illustrate the two versions of my multifractal model of price variation (a continuous time process and a cartoon) and other innovations that this volume describes far too tersely.

One overall goal of this Foreword and update is to place this work within the context of numerous technical book and articles I have authored or co-authored since 1997. Another goal is to inject the theme of fractality as a model of roughness, and motivate the bold claim (made early in this Foreword) that fractal tools will remain inevitable in finance.

Comments on the body of this book. Never meant to be an encyclopedia, nor a monograph, nor a textbook, this volume was designed as a broadly ranging resource book devoted to the innovations I brought in finance. I adopted a complex structure – juxtaposing without quite merging "something old and something new, something borrowed and something blue." More soberly restated, this book brings together reprints of old papers (1962-1972) with annotations and chapters of several different kinds that were specially written in 1996 and concerned work in progress. An unfortunate but unavoidable effect was that, among my several successive models of price variation, the first is discussed at length while the current multifractal model is discussed only briefly in the introduction and the key Chapter E6.

In this "updated reprint," the body of the book is unchanged except for a new and cleaner Figure E9.2. It illustrates more obviously the case against the lognormal distribution that is argued in Chapter E9 and deserves to receive much greater attention. The List of Chapters has been reset, and those which were new in 1997 now bear numbers preceded by a star.

Presentation of this new Author's Foreword. A few minor additions consist in references (with comments) to texts of mine published since 1997 and a few other tidbits. A major addition consist in a collection of fully captioned "cartoons" that weave together my financial models to Brownian motion and to one another. Graphics is never the last word in science, and never a substitute for analysis. True. But it is a marvelous addition no one should spurn. It is really too bad that the circumstances prevailing in 1996 made me fail to follow my own advice and the body of this book was insufficiently illustrated. The resulting graphical bareness surely contributes to the whole being sometimes described as difficult. In particular, the "cartoons" of Chapter E6 should have been made very much more explicit and better coordinated. This update is a welcome opportunity for taking a step in that direction. The Foreword is followed by photographic reprints of two short seed papers in French from the 1960s.

1. PRELIMINARY: THE INESCAPABLE NEED FOR FRACTAL TOOLS IN FINANCE

This section argues that even when the present fractal models become obsolete, fractal tools are bound to remain central to finance – the reason being that fractal geometry is the proper language of the theory of roughness in nature and culture

General questions that are often asked need not have immediate answers. How could it be that – without ever trying explicitly – fractal geometry impinges deeply on so many otherwise very different issues? How to explain that, as early as in the 1960s, my work in natural and social sciences (namely, turbulence, hydrology, and prices) could rely on closely analogous tools? In 1997, I could not answer those questions but now, at long last, I have found for those questions a short answer that runs as follows:

FRACTAL GEOMETRY MEASURES ROUGHNESS INTRINSICALLY. HENCE IT MARKS THE BEGINNING OF A QUANTITATIVE THEORY SPECIFIC TO ROUGHNESS IN ALL IT MANIFESTATIONS.

Roughness is ubiquitous in nature and culture (the latter term denoting all the works of Man, including financial markets). This is why fractality is also ubiquitous and why fractal geometry will never lack problems to deal with.

Ubiquity and antiquity were not enough. Roughness lagged very far behind steepness (of a road or a trend), heaviness, pitch, color, hotness, and the like, insofar as there was no quantitative measure of roughness until fractal geometry provided one. The steepness of a smooth incline is, of course, defined by the derivative of the height $h(x)$ along the incline. In theory, this definition implies that "normal" ratio dh/dx tend to a limit as $dx \to 0$. In practice, it suffices that dh/dx be nearly constant. But – almost by definition – rough surfaces are such that dh/dx varies all over without limit. A basic feature of the Bachelier model, and also of all three of my models is that, instead, this limit exists for the "anomalous" ratio $\log(dh)/\log(dx)$. According to the Bachelier model, this limit, α, is 1/2 at all instants in all financial data. This is both a big asset – simplicity, and a big flaw – became a limit equal to 1/2 is not available as parameter to be fitted to the data.

The key feature of the fractal/multifractal models is that the limit exists but is not found to be $\alpha = 1/2$.

In inverse historical sequences and decreasing generality, I have originated and investigated three cases. The value of α may vary in some specific way from instant to instant; this characterizes multifractality. The value of α may be the same at all time instants but different from 1/2 ; this characterizes unifractality or the HHM model as motivated below. There is also a very important intermediate case I called "mesofractality" or PLM model, as motivated below.

The derivative was specifically invented to measure quantitatively the intuitive notion of steepness and the mathematics came later. The α took the opposite path. I devised it by modifying a concept that Hölder had introduced long ago, in 1870, for mathematical reasons. Thus it began by being totally separate from intuition, and my work identified it with one of the key aspects of roughness. Examine, indeed, the various cartoons that illustrate the following sections. From one to another, the Intuitive, "eyeball," levels of roughness are immediately seen to be different. A key feature of fractal geometry is that it measures roughness by α. Moreover, the value and/or distribution of α is a directly observable parameter – not an elusive one that has to be unscrambled indirectly from many other observations. This is an aspect of parsimony.

For reasons that can perhaps be guessed but cannot be developed here, the conditions that define "true roughness" include $\alpha = 1/2$. Now recall the innumerable alternative models accepted the arguments I deployed in the 1960s against the Brownian. Most reacted by introducing "fixes" that are specifically designed to avoid the "anomalies" I pioneered, such as divergent moments and divergent dependence. Examined in this light, the alternative models that strain to avoid all anomalies automatically reset the local roughness to 1/2.

Varying volatility

An *Overview of fractals and multifractals* written in this spirit is featured in M 2002H. For emphasis, this *Overview* forms Chapter H1 that immediately follows the Preface.

2. BROWNIAN MOTION AND BEYOND: THREE SUCCESSIVE SCALING/FRACTAL MODELS OF THE VARIATION OF FINANCIAL PRICES

In this section, four models are described briefly but illustrated by a recursive fractal "cartoon" that proceeds by multiplication.

2.1. Figure 1. Brownian motion.

The standard model of the variation of financial prices has been Brownian motion conceived in Bachelier 1900. Two properties define it: There is a wide belief that they are not only convenient but explained, accounted for, by a fundamental result the: central limit theorem denoted by the letters CLT. That theorem concerns reduced *sums* of increasingly many random processes. It states that their limit behavior is "universal," that is, independent of the nature of the addends.

Of course, every theorem begins with assumptions, but those of CLT are rarely emphasized because their validity is taken for granted. First assumption: stationarity. Second assumption: the volatility, i.e. marginal variance is finite. Third assumption: finite memory. This third assumption demands to be amplified. Consider two instants of time separated by the span T. As $T \to \infty$, the dependence between price behavior at those two time instants is assumed to become negligible.

The fractal/multifractal models contradict one or several of those assumptions. This creates a paradox. My critique of Brownian motion has shown the deviations from those assumptions to be intrinsic and necessary. But, once again, it has also triggered many models that are alternative to mine and proud of preserving the central limit theorem.

Be that as it may, during the 1960s I viewed additivity as worth preserving when presenting my pre-multifractal models. This was made possible by the existence of the so-called "generalized central limit theorems." They preserve the idea of additive components but only at an extraordinarily high cost in arbitrary inputs ____. Because of the inputs' arbitrariness, Brownian motion continued to be perceived as natural while its generalizations appeared contrived. Actually, the arbitrariness was only apparent but appearances do create false impressions and must not be disregarded. Furthermore, multifractals stood outside. Their intrinsic components are not additive but multiplicative, hence they could not be presented via a generalized CLT.

Before moving on to multiplicative cartoons, it is useful to dwell on earlier cartoons that were intrinsically generated by adding contributing components – hence kept close to Brownian motion. The simplest non-random additive cartoon dates to 1906! It is the Landsberg function which M 2002H studies in Chapter H2, Section 5 and illustrates by Figure H20-2. This old and repeatedly rediscovered expression add nonrandom periodic sawtooth curves. Each tooth is a segment going UP followed by a segment going DOWN. Shuffling forsakes periodicity and chooses each tooth at random between two possibilities: UP, DOWN and DOWN, UP.

Efforts to generalize this additive cartoon have failed, creating a conceptual and pedagogical logjam. To break it, I devised a way to obtain a Brownian cartoon by multiplication. This construction was non-traditional but made the outcome more convenient and far easier to generalize, and I now think that multiplication is intrinsically the better approach.

The construction illustrated on Figure 1 yields neither Brownian motion proper nor a randomized Landsberg sum of sawteeth. But it generates a more useful "cartoon" that deepens our understanding of Brownian motion and also lends itself easily to be transformed into cartoons of either of my three successive models.

Comparing all those cartoons help underline both the clear differences and the kinship created by scaling/fractality. The relations between the different models are illustrated by the phase diagram, Figure 7.

2.2. Figure 2. The phenomenon of discontinuity and related long-tailed distribution of price changes ("Noah" Effect); the model variously denoted as “Pareto-Levy-Mandelbrot”(“PLM”), “mesofractal,” or “M 1963”

Brownian notion is continuous but financial prices are not. My M 1963 model was the first to acknowledge and face the discontinuity of prices and the corresponding long-tailedness of the distribution of price increments. This model was sketched in M 1962c – reproduced photographically in M 1997FE and now also after this Author's Foreword. It was pioneered in M 1963b{E14} and developed in half of this book.

The construction illustrated on Figure 2 preserves the principle of the fractal cartoon underlying Figure 1 but uses a different generator.

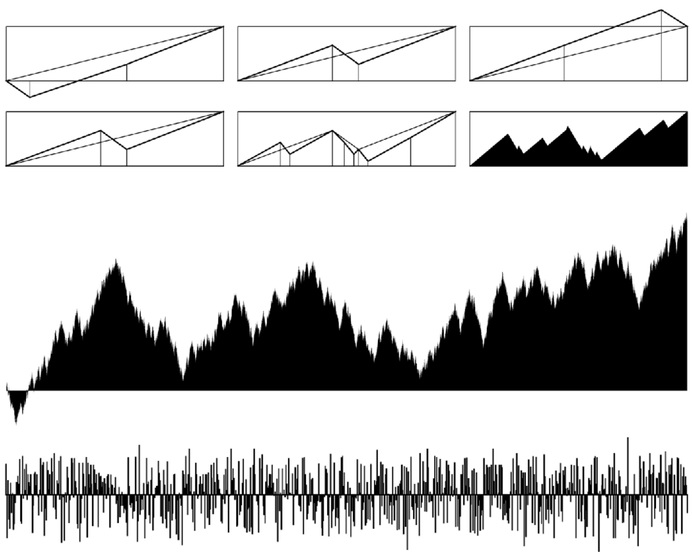

FIGURE C1-1.

An important issue: Old or new terminology. In due time, my three models were widely adopted and expanded in diverse directions by torrents of publications. This wide acceptance also had an unfortunate by-product: a multiplicity of mutually contradictory terms that could not possibly be “retrofitted” in this book. They are so varied as to convince may observers that they concern different models – which they do not.

Around 1960, the confusion began with my own hesitation. I boasted of combined inspirations from the real world and pure mathematics, namely, from the economist V. Pareto and the pure probablist P. Lévy. This led me to coin the term "Pareto-Lévy" (PL) law. Paul Samuelson replaced it by Pareto-Lévy-Mandelbrot (PLM), skillfully combining "thesis," "antithesis," and "synthesis." Instead, this book was ill-inspired in introducing the term "M 1963 model" and several later papers used "mesofractal." The confusion was compounded by authors who adopted “Lévy law” or “Lévy distribution.” All those terms have failed to take root while Samuelson's PLM has been used by others.

To avoid ambiguity and confusion and preempt inappropriate terminologies like "power-law distributions," the best seems to settle on PLM.

2.2. Figure 3. The phenomenon of long or global dependence in price variation (“Joseph Effect”) and the model variously denoted as “Holder-Hurst-Mandelbrot” (“HHM”), “unifractal,” or “M 1965.”

Everybody has always suspected *some* non-independence in price increments but its strength and infinitely long range were first faced in my "M 1965" model. It was pioneered in M 1965h{H9} – reproduced photographically in M 1997FE and now also after this Author's Foreword.

The basic unifractal model is based on Fractional Brownian motion (FBM). Reprints and specially written chapters on this account are collected in M 2002H. The recursive fractal construction illustrated on Figure 3 steps back to a cartoon using a symmetric three-segment generator, like in Figure 1 but, given H satisfying $0 < H < 1$, requires relation (log of height/log of width) $= H$ for each of the three intervals.

To the readers of the present book, what matters most is global dependence in financial prices. In, M 2002H, this topic is treated in Chapter H30, which combines excerpts from M 1969c, M 1977n, and M 1973c with diverse recent comments. The topic is subtle because, as that Chapter H30 underlines, unifractality goes far beyond FBM. On the other hand, Fractional Brownian motion is especially simple, being defined by a single long-dependence exponent H. As a result, FBM was long shunned as representing an inefficient market. By now, to the contrary, it has began to be widely studied. Most emphatically, I continue to view FBM as the basic illustration of long dependence but only a gross approximation for prices. Therefore, Chapter H30 of M 2002H adds very substantially to M 1997E and is very strongly recommended to this book's readers.

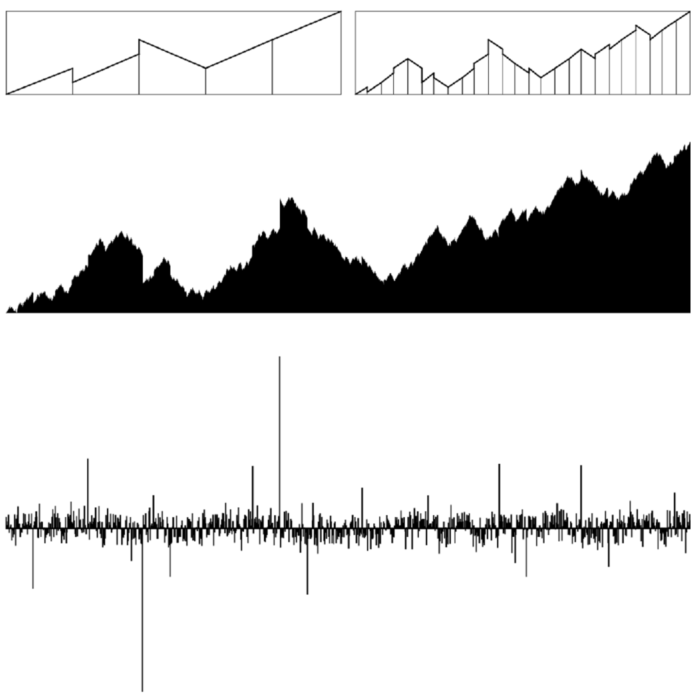

FIGURE C1-2. A shuffled recursive cartoon of a Lévy stable process with $\alpha = \log 5 / \log 3 = 1....$ In order to obtain this α and accommodate up and down discontinuities, the simplest generator combines more than three non-vertical intervals with two discontinuities that illustrate the Noah Effect. The tails and the middle of the distribution were hastily patched up by hand and the match between them leaves room for improvement in the future.

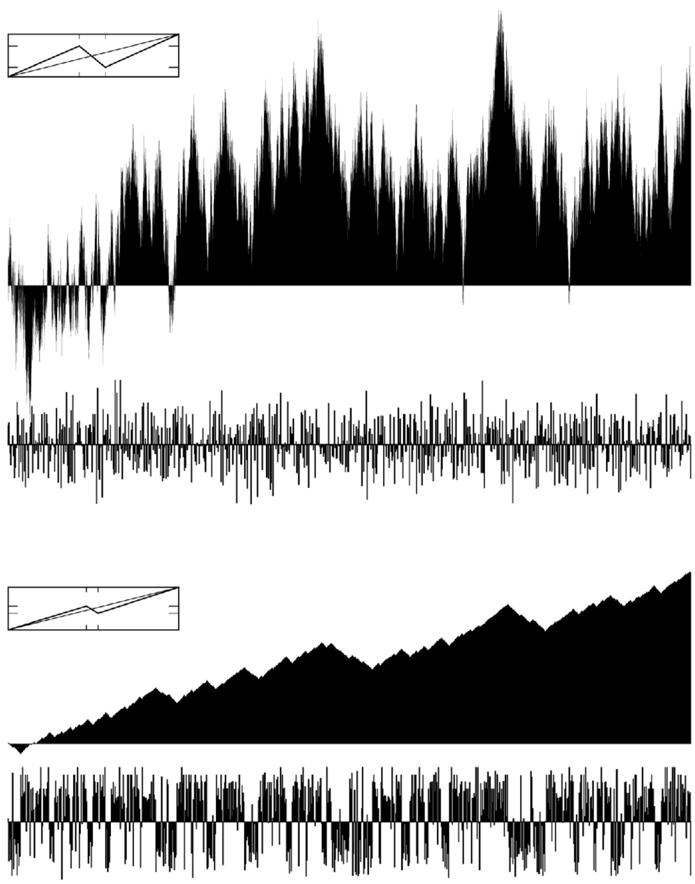

FIGURE C1-3. Shuffled recursive cartoons of fractional Brownian motions. The bottom panel – for $H = 0.75$ – is best appreciated by examining the increments: they exhibit symptoms of the Joseph Effect (seven fat and seven lean cows), long runs of mostly positive or mostly negative values. To the contrary, the upper panel – for $H = 0.3$ – is best appreciated by examining the rapid up and down flipping of the function itself.

Old or new terminology. It is also multiple and confusing but less than in the "PLM" case. Early on, just as for PLM, I credited the influence of a practical man, the hydrologist Hurst and of a mathematician, Hölder, and their names are sometimes attached to my work. This book used the notation M 1965 model and elsewhere I used "unifractal." Both failed to take root and to parallel "PLM," the best may be to settle on "HHM."

One of the main topics of M 2002H is R or R/S analysis. A slight variant of R analysis has recently received attention (again without reference to its origin) under the name "detrended range analysis." The new label does not affect the content but destroys communication.

2.4. Figure 4. The combination of the Noah and Joseph Effects and my current "multifractal model" of price variation

Throughout the 1960s my publications kept emphasizing that most observed instances of the Noah and Joseph effects do not appear separately but in combination. The great virtue of R/S analysis was that it focused on Joseph dependence and was blind to the length of the tails (the technical term is "very robust.") Nevertheless, both the PLM and HHM models could only be viewed as simplifying approximations. In different ways, each greatly improved upon Brownian motion but did not warrant being pursued separately in excessive detail, and a model embodying both effects was needed.

This model – eventually labeled multifractal – was first mentioned in the last paragraph of M 1972j{N14}. Therefore, I once called it "M 1972 model." But historical accuracy would not be sufficient. The model having been first developed – though far too tersely – in Chapter E6 of this book, it is better denoted as the "M 1972/97." To present the multifractal model, it is best to take one tack, then another. The point of departure is a Brownian or fractional Brownian cartoon and the key operation changes the generator by moving the junction points between the UP and DOWN and the DOWN and UP intervals away from each other along the horizontal. This cartoon is illustrated by Figure 4. It gave intense satisfaction but clearly defined room for improvement. The reason for satisfaction is that a construction of extreme and unexpected parsimony yields a result that is remarkably reminiscent of the actual data, as exemplified for example by Figure XX. But as the junction points move far enough they create long flat periods when almost nothing happens. This is unrealistic and opens up room for improvement.

Let me digress for the reader familiar with the multifractal formalism (which is only sketched in this book but developed in M 1999N). The flat periods correspond to $\alpha_{max} > 1$, which is the case when the junction points are so distant that one of the generator's interval is of slope < 1.

Figure 5 brings the generator's junction points closer to each other than in the Brownian case. The gain is that the flat periods vanish. The loss is that the increment diagrams are less realistic as other grounds. The result: the cartoon's are no more than cartoons and closer fit must be searched in more general multifractals.

2.5. Figure 6. "Find the fake:" side-to-side comparison of price variation according to Brownian motion and the three fractal models

Cartoon-drawing is an addictive game, and there exist shuffled recursive program in which all that is left to do is to input a generator.

An immediate idea of the resulting either is provided by Figure 6. It simply brings together the increment records already plotted in the preceding figures in this chapter. The repeats of the Brownian cartoon are omitted but it would have stood out instantly as a "fake," a far from realistic representation. The stable Lévy cartoon stands out as having long tails and no dependence. This fractional Brownian cartoons stand out as having short tails and clear-cut clustering.

To the contrary, realistic approximations of real price variation are instantly spotted among the multifractal cartoons, despite the fact that a symmetric three interval generator involves few arbitrary decisions hence is as simple as can be. It is fully defined by a single point, that when the first and second interval meet in the version UP, DOWN, UP. One point is easily graphed, as will be done in Figure 7 and the individual effects of each of those discussions are distinct and only a little training makes them easy to follow. More complex generators yield less crude cartoons – as exemplified on Figure in the body of the book. One reason is that the number of shuffled alternatives increases with the generator's number of intervals. But those more complex generators are not recommended because choosing them involves more than one point in a square. It involves a larger number of arbitrary decisions when effects interact and are difficult to follow individually.

2.6. Figure 7. A phase diagram

This Foreword, M 2001c and Chapter H1 of M 2002H share a common key feature: attention is limited to special "cartoons." A symmetric three interval generator, the position of the break between the UP and the DOWN interval is a well-defined "address" in the left half of a unit square and makes it possible to draw a "phase diagram." Figure 7 consists in a

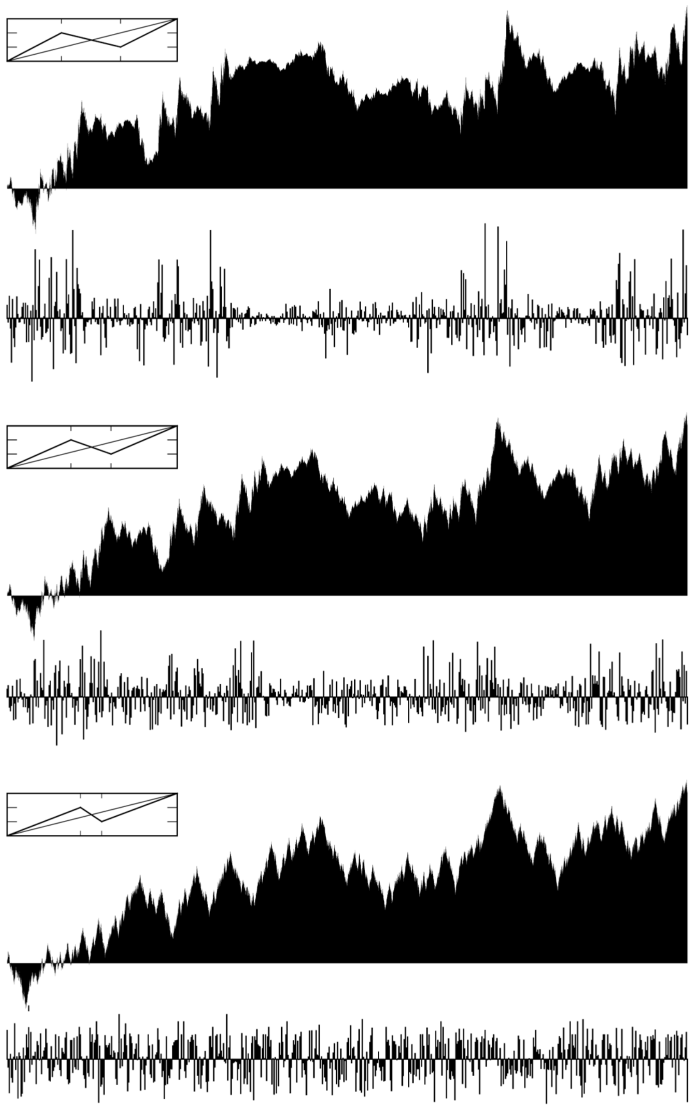

FIGURE C1-4. This figure and the next collect six shuffled recursive cartoons of multifractal functions. Both figures step back to the symmetric three-segment generator that was used in Figure 1 and is repeated here in the bottom panel. But it is subjected to a deformation different from that used in Figure 3. In the top two panels, the junction points between the UP and DOWN and DOWN and UP intervals are positioned farther away than in the bottom panel but on the same horizontal levels. This suffices to create two features that are characteristic of real price data, as will be argued in Section 2.5.

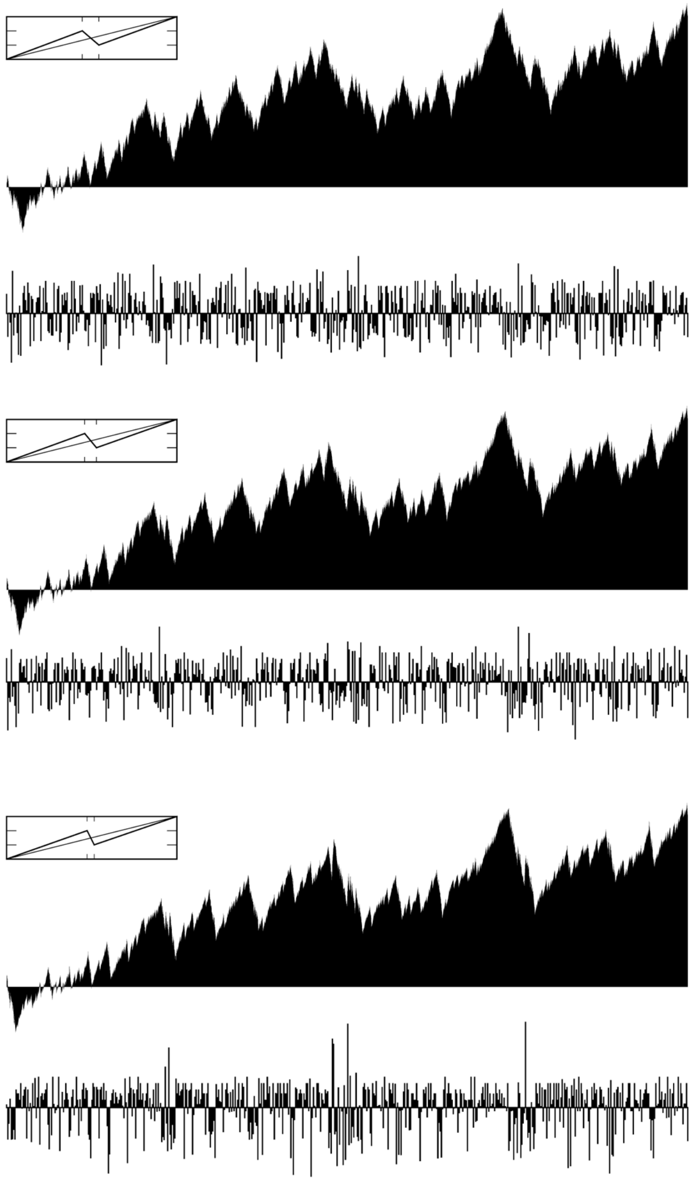

FIGURE C1-5. Compared to Figure 4, the difference is that here the positions of the generator's junction points are *not* positioned *farther* from each other than in the Brownian case but *closer*. Result: a different way of achieving large deviations and clustering, less realistic in the case of prices but bound to be of use elsewhere.

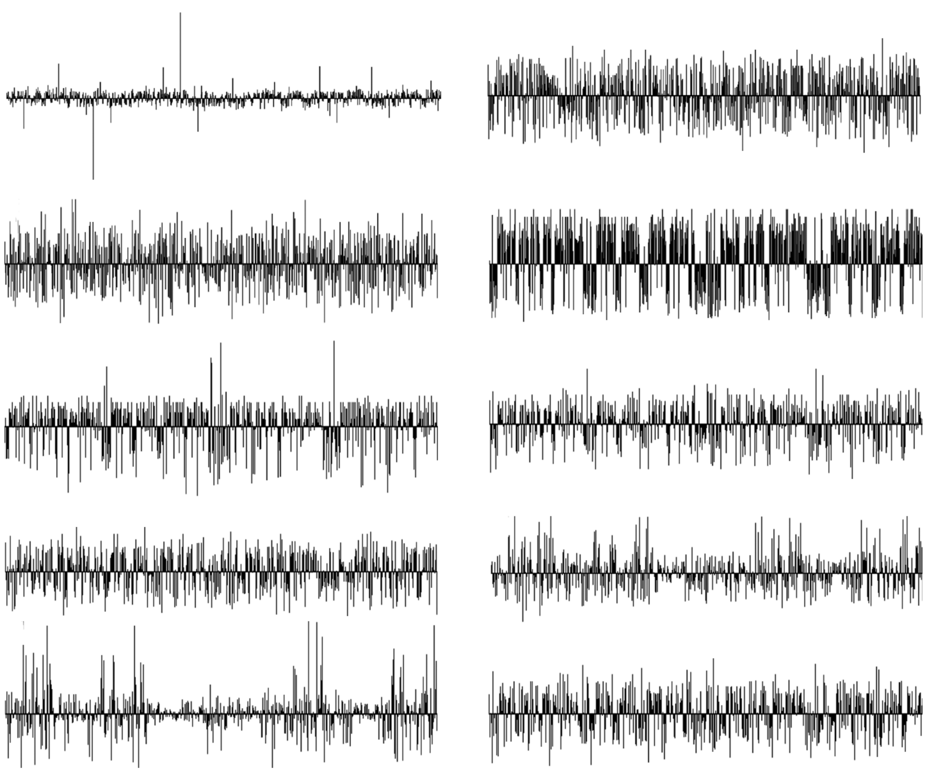

FIGURE C1-6. Side-to-side comparison of several cartoons. The Brownian static "radio hum" stands out as a "fake."

FIGURE C1-7. A phase diagram. This volume belongs to a series of my *Selecta* which will be referenced and described in Section 3.3. To those *Selecta*, the present diagram provides a broad overall index.

Since 1997, I elaborated greatly on the overly terse Chapter E6 of this book and have gone well beyond. This is witnessed by the cartoons in Section 2 and in many documents on my web "Homepage," www.math.yale.edu/mandelbrot. The *Vita* on the web includes a list of references that greatly extends this book's bibliography. The web also includes additional material of interest and the intent is to keep it up to date.

3.1. General references other than books.

Photographs of the earliest papers are available in this new foreword, M 1997FE, and my web homepage.

M 1997s B.B. Mandelbrot. Three fractal models in finance: discontinuity, concentration, risk. *Economic Notes* (Siena – currently Basil Blackwell in Oxford): **26**, 1997, 139-178. This conference text is merely an abridgement of the preface and introduction of the present book, but many readers praised it as a useful "pre-introduction."

M 1999p B.B. Mandelbrot. Renormalization and fixed points in finance, since 1962. (*Statistical Physics 20, International IUPAP Conference*, Paris, 1998. Edited by D. Iagolnitzer.) *Physica:* **A26,** 1999, 477-487. Throughout my work – even in my 1952 Ph.D. dissertation, I have made constant use of the techniques that include renormalization and fixed points. Under diverse names, many fields of science have also made use of those techniques. Are those diverse uses historically or technically related or independent of one another? This is a legitimate question and the topic of this 1999 *Physica* paper.

Most significant is the relation between my work in finance and the chapter of statistical physics called "theory of critical phenomena." That chapter's modern era began in 1965 and bloomed in 1972. The dates suffice to show that the M 1963 and M 1965 models came earlier and quite independently and the roots of the M 1972/1997 model are contemporary with that theory but also completely independent from it.

Chapter H1 of M 2002H. This chapter titled *Overview of fractals and multifractals* deserves a second mention here.

3.2. Technical references by the author, other than books, on the multifractal models in finance.

Yale Discussion Papers. B.B. Mandelbrot, L. Calvet, & A. Fisher 1997. *The Multifractal Model of Asset Returns. Large Deviations and the Distribution of Price Changes. The Multifractality of the Deutschmark/US Dollar Exchange*

Rate. Discussion Papers numbers 1164, 1165, and 1166 of the Cowles Foundation for Economics at Yale University, New Haven CT, Available on my web homepage. The second reports are largely expository, but the third – which was illustrated in advance by Figure E6.3 – continues to be a mine of important information as listed below.

Article in "Quantitative Finance." M2001a. B.B. Mandelbrot. Scaling in financial prices, I: Tails and dependence. *Quantitative Finance (IOP - currently Taylor & Francis):* **2,** 2001, 113-123. Homepage.

M2001b. B.B. Mandelbrot. Scaling in financial prices, II: Multifractals and the star equation. *Quantitative Finance:* **1,** 2001, 124-130. Homepage. This page faces a major issue concerning the tail scaling exponent α of price increments. The M 1963 model – based on independent increments and Lévy stable distribution – restricts this α to be at most 2. To the contrary, one of the most important but least known virtues of multifractal dependence is that it allows α to exceed 2. This feature was already described in Mandelbrot 1974f and is developed in M 2002r, to be referenced momentarily. (M 2001a and b are also being reprinted in *Beyond Efficiency and Equilibrium.* Edited by Doyne Farmer & John Geanakoplos, Oxford UK: The University Press.)

M2001c. B.B. Mandelbrot. Scaling in financial prices, III: Cartoon Brownian motions in multifractal time. *Quantitative Finance:* **1,** 2001, 427-440. Homepage. This paper is related to the present Foreword and similar to the *Closeup* Chapter H1 of M 2002H, which is discussed below.

M2001d. B.B. Mandelbrot. Scaling in financial prices, IV: Multifractal concentration. *Quantitative Finance:* **1,** 2001. 641-649. Homepage. The topic of this paper is a very new one. Indeed, the phenomenon of concentration discussed in Chapter E 13 of this book is limited in its scope. It relies on a classical theory of extreme values that postulates independent random variables. This theory is very popular today but fails to extend from firms to price changes. Multifractal concentration and extreme values are two very important topics that cry out for more detailed study.

M2001e. B.B. Mandelbrot. Stochastic volatility, power-laws, and long memory. *Quantitative Finance:* **1,** 2001, 558-9. Homepage. This brief paper is a concise restatement of my criticism of the many authors who claim that the defects of the Brownian model are less severe than I claim and can be "fixed" with the help of "three components" formulas.

"Handbook." M2002r. B.B. Mandelbrot. Heavy tails for independent or multifractal price increments. In *Heavy Tailed Distributions in Finance.* Edited by Zari Rachev (A volume in the series *Handbooks in Finance.*

Series editor William T. Ziemba.) Amsterdam: Elsevier. 2002. Homepage. A somewhat similar presentation addressed to a different audience appeared in *Journal of Statistical Physics*: **110**, 2003, 739-777. This paper illustrates on a very simple example the key theme of M 2001b, that the tail exponent α is restricted to be less than 2 in the case of independent price increments but can range freely in the case of multifractal dependence.

3.3. Updates concerning the "Selecta" series of books.

This book, the first in the series of my *Selecta*, announced later *Selecta* with unrealistic tentative dates that cannot be amended in the text.

M 1997FE, part of the French "Selecta." A near-simultaneous French "companion," M 1997FE is far shorter and less technical. But its treatment of "states" of randomness and variability is leisurely and detailed in ways that are not yet available in English. It complements the overly fast-paced and technical Chapter E5 of M 1997E.

M 1999N, titled "Multifractals and 1/f Noise: Wild Self-Affinity in Physics." (announced as M 1997N.) From the viewpoint of finance, the main feature of M 1997N resides in a wealth of material about multifractals. It must be mentioned that most is of an advanced nature and early papers predominate. Additional tutorial material is found in reference M2002r described in Section 3.2. A distinctive feature is that M 1999N compares two broad approaches to the study of multifractals that remain in competition. The original approach in Mandelbrot 1972j and 1974f,c defined random multifractals constructively. This approach begins with individually defined multiplicands and focuses on limits of products of such multiplicands. The cartoons of Section 2 exemplify a simple category of that original approach, technically denoted as "multinomial." Two other categories in my approach, conservative and canonical, add immensely to realism but do not allow for cartoons. An alternative approach proposed heuristically by Frisch & and Parisi 1985 and developed by Halsey et al 1986 does not involve individual multiplicands. My strong belief is that for most purposes the better approach in mine; this is documented in M 1997N. Chapter N3 of M 1999N is an extensive discussion of the concepts of renormalization and fixed points. (see also

M 2002H, titled "Gaussian Self-Affinity and Fractals: Globality, the Earth, 1/f Noise and R/S." (announced as M 1997H.) Two chapters of that book have already been mentioned: Chapter H30 on long dependence in prices, and the *Overview* Chapter H1 that treats the theme of the fractal theory of roughness. Once again, both are important and strongly recommended to the reader. Furthermore, M 2002H includes much old and new material on other topics of interest to the students of long dependence – in particular on fractional Brownian motion and *R/S*. This statistic, which is recommended to the reader, is a central tool in the study of the M 1965 model and, more generally, of global (long term) dependence.

M 2004C titled "Fractals and Chaos: the Mandelbrot Set and Beyond." (announced as 2002C) This book is largely made of reproductions of early papers. But extensive illustrations are presented here for the first time. It has no substantive, only historical relevance to finance, but it may help to quote from a review by K. Falconer in *Nature* (July 1, 2004)

"Is the Mandelbrot set just a pretty curiosity? Far from it. It is a fundamental parameter set that ...is 'universal' in that it underlies the behaviour of very large classes of more complicated nonlinear mappings, the likes of which crop up throughout modern mathematics and its applications... [This] book should be accessible to a wide readership. It provides a fascinating insight into the author's journey of seeing and discovering as the early pictures of the Mandelbrot set started to reveal a whole new world. It gives a feeling for his philosophy and approach of experimental mathematics – an approach that has changed the way we think about mathematics and science."

Comments. In an early draft, the title of M 2002C also included the words *Statistical Physics* and it included my articles on the foundations of thermodynamics, and other articles documenting that, since the 1950s, all my work, in all fields, has been rooted in a broad interpretation of the scope of Gibbs's statistical physics. This material is being put on my web but was withdrawn from M 2002C, perhaps to be reprinted elsewhere.

The promised enhancements of this book's Chapter E6, titled "Panorama." Later *Selecta* volumes were to include successive improved versions. A first enhancement was included in M 1999N but concerns only one half of Chapter E6. Instead of a *Panorama,* M 2002H incorporates the already-mentioned Chapter H1, titled *Closeup*.

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