Stable Fractal Sums of Pulses: the General Case. *†

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Abstract

The paper introduces new classes of stable self-affine stochastic processes with stationary increments and global (long-range) dependence. They are *fractal* sums of pulses (FSP) for which the pulses are such that the time of occurrence, duration and amplitude are independent random variables. It is shown that different pulse shapes give rise to different FSPs. These FSPs differ from the known self-affine processes with stationary increments. Path properties of FSPs are described as well as partial domains of attraction.

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1 Introduction

The processes obtained in this paper are Lévy stable with the scaling exponent α, $0 < \alpha < 2$, and have stationary increments. That is, $\{X(t+b) - X(b),\ t \geq 0\} \stackrel{d}{=} \{X(t) - X(0),\ t \geq 0\}$ for all $b > 0$, where "$\stackrel{d}{=}$" denotes equality of finite-dimensional distributions. They are also self-affine, therefore, fractal. That is, there exists an "affinity exponent" $H > 0$, such that $\{X(at),\ t \geq 0\} \stackrel{d}{=} \{a^H X(t),\ t \geq 0\}$ for all $a > 0$. Our processes are *stable fractal sums of pulses* (SFSP), because they are sums of an infinite number of pulses occurring according to some Poisson random measure. The value of $X(t) - X(0)$ is defined as the sum of the changes in the pulse amplitude between times 0 and t. The pulses' shape is determined by a prescribed "template" f, which is a function supported in the interval $[0, 1]$. To obtain a pulse, the template is translated by $\tau \in R$, "stretched" vertically by $\lambda \neq 0$ and "stretched" horizontally by $w > 0$. Hence, τ, λ and w correspond to the time of pulse birth, vertical scaling and width (duration) of a pulse, respectively. These quantities are random and determined by the scaling Poisson intensity shown in equation (2.1).

For an SFSP to be well-defined (in the sense of a.s., possibly conditional, convergence), Section 2 gives sufficient conditions on the template f that are analogous to ones considered in Cioczek-Georges and Mandelbrot (1994a) and (1994b) for the micropulse construction of fractional Brownian motion (FBM). Those conditions allow for great diversity of pulse shapes and yield $0 < H < \max(1, 1/\alpha)$. It is known that for α-stable processes with stationary increments, the affinity exponent H must satisfy $0 < H \leq \max(1, 1/\alpha)$. Moreover, for $\alpha < 1$ there is a unique process with $H = 1/\alpha$; it is Lévy stable motion (LSM), which has independent increments. Hence, the SFSPs yield the maximal range of H for $\alpha < 1$, and "almost" yield the maximal range for $\alpha \geq 1$. The specific value of H for a given SFSP depends on the distribution of the pulse width.

Section 3 proves that the processes obtained in this paper are new, that is, differ in the sense of finite-dimensional distributions from previously described α-stable, H-self-affine processes with stationary increments. Section 3 also examines sample path properties. For $H > 1/\alpha$, there exists a sample continuous version of a process as for all self-affine processes with stationary increments in this case; for $H \leq 1/\alpha$, the sample paths are highly irregular, namely, nowhere bounded with probability one.

Section 4 deals with partial domains of attraction. We describe a large class of sums of pulses processes which are attracted to a given SFSP. The novelty is that those pulses are not self-affine but governed by a Poisson measure with an intensity that differs from the scaling intensity by a slowly varying factor.

A different broad class of processes related to the SFSPs is the *partly random fractal sums of pulses* (PFSP). They are investigated in Mandelbrot (1995a) and (1995b), together with the new notion of "lateral attraction" for processes. They are not Lévy stable themselves, and are in the domain of lateral attraction of either FBM or LSM.

2 Definition and Existence of a Class of SFSPs

In the pulse address space $A = R_0 \times R \times R_+$, where $R_0 = R \backslash \{0\}$ and $R_+ = (0, \infty)$, each pulse is represented by a point with coordinates λ, τ and w. Once again, they correspond, respectively, to the vertical scaling, time of birth and width of a pulse. Fix a template function f, where f is supported in $[0,1]$ and determines the shape of pulses we use in the construction. Then, at time t, the amplitude of an f-shaped pulse, starting at τ and ending at $\tau + w$, equals

$$\lambda f\left(\frac{t-\tau}{w}\right).$$

The number of pulses with a given set of coordinates is determined by a Poisson random measure. Let $\mathcal{A} \equiv \mathcal{B}(A)$ be the Borel σ-field on A. We consider a Poisson random measure N on $(A, \mathcal{A})$ with mean n given by

$$n(d\lambda, d\tau, dw) = \begin{cases} c'\lambda^{-\alpha-1}w^{-\theta-1}d\lambda d\tau dw & \text{if } \lambda > 0, \\ c''|\lambda|^{-\alpha-1}w^{-\theta-1}d\lambda d\tau dw & \text{if } \lambda < 0, \end{cases} \tag{2.1}$$

for $\tau \in R$, $w \in R_+$, and some $0 < \alpha < 2$, $min(1-\alpha, 0) < \theta < 1$, $c', c'' \geq 0$, $c' + c'' > 0$. When $\alpha = 1$, let us simply assume, for reasons which will become apparent later, $c' = c''$.

Adding up pulses correspond to the integration of pulse amplitude with respect to the Poisson measure. Hence, the sum of f-shaped pulses described in the Introduction is defined by

$$\int_A \lambda \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right] N(d\lambda, d\tau, dw). \tag{2.2}$$

The question arises whether the above integral is well-defined, i.e. in what sense it is finite a.s. To answer this, we use an equivalence between integrals with respect to Poisson measures and stable integrals.

A stable integral $\int_E g(x)M(dx)$ is a stable random variable with the characteristic function

$E\exp(it\int_E g(x)M(dx)) =$

$$(2.3)\begin{cases} \exp\{-\int_E |tg(x)|^\alpha(1-i\beta(x)\mathrm{sgn}(tg(x))\tan\frac{\pi\alpha}{2})m(dx)\}, & \text{if } \alpha\neq 1,\ 0<\alpha<2, \\ \exp\{-\int_E |tg(x)|(1+i\frac{2}{\pi}\beta(x)\mathrm{sgn}(tg(x))\ln|tg(x)|)m(dx)\} & \text{if } \alpha=1, \end{cases}$$

where $(E,\mathcal{E},m)$ is a σ-finite measure space, $\beta\colon E\to[-1,1]$ is a measurable function, called skewness intensity, and a function g is *integrable*, i.e. satisfies $\int_E |g(x)|^\alpha m(dx) < \infty$ for $0<\alpha<2$, and additionally, $\int_E |g(x)(\ln|g(x)|)\beta(x)|m(dx) < \infty$ for $\alpha = 1$. M is an α-stable measure with the control measure m and skewness intensity β. It can be defined by $M(A) := \int_E I[A]M(dx)$ for all m-finite sets $A\in\mathcal{E}$. For details as well as alternative definitions of stable measures and integrals see Samorodnitsky and Taqqu (1994)). Theorem 3.12.1 of this monograph relates integrals with respect to a Poisson random measure to those with respect to a stable measure.

Bearing in mind the possible Poisson integral representation of a pulse process it is more convenient for us to deal with an equivalent in distribution (up to a constant C_α defined in Theorem 3.12.1 of Samorodnitsky and Taqqu (1994)) stable integral:

$$X(t) := \int_E \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right] M(d\tau, dw), \tag{2.4}$$

where M is an α-stable random measure on $(E,\mathcal{E}) := (R\times R_+, \mathcal{B}(R\times R_+))$ with constant skewness $\beta = (c'-c'')/(c'+c'')$ and control measure m given by

$$m(d\tau, dw) = \frac{c'+c''}{2} w^{-\theta-1} d\tau dw. \tag{2.5}$$

It is clear that the integral in (2.2) converges a.s. iff the corresponding stable integral in (2.4) is well-defined, i.e. when its integrand is integrable in the sense described below (2.3). However, even if this Poisson integral exists, its convergence is conditional in the case $\alpha\geq 1$, as in the following:

THEOREM 2.1 *If*

$$\int_E \left|f\Big(\frac{1-\tau}{w}\Big) - f\Big(\frac{-\tau}{w}\Big)\right|^\alpha w^{-\theta-1} d\tau dw < \infty, \tag{2.6}$$

then the process $\{X(t),\ t\geq 0\}$ *given by* (2.4) *is well-defined and its finite-dimensional distributions equal those of* $\{C_\alpha X'(t),\ t\geq 0\}$, *where*

$$X'(t) = \begin{cases} \int_A \lambda\left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right] N(d\lambda, d\tau, dw) & \text{if } \alpha < 1, \\ \lim_{\epsilon\to 0}\sum_{i=-\infty}^{\infty}\int_{(-\epsilon,\epsilon)^c}\int_{E_i} \lambda\left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right] N(d\lambda, d\tau, dw) & \text{if } \alpha\geq 1, \end{cases}$$

and $E_i = \{u : 2^{-i} < |f((t-\tau)/w) - f(-\tau/w)| < 2^{-i+1}\}$, $i = -\infty, \ldots, +\infty$.

Both processes have α*-stable stationary increments and are self-affine with the exponent* $H = (1-\theta)/\alpha$. *In particular, the scale parameter of* $X(t)$ *is proportional to* $t^{(1-\theta)/\alpha}$.

PROOF: Since $\int_E |f((t-\tau)/w) - f(-\tau/w)|^\alpha m(d\tau, dw) = t^{1-\theta} \int_E |f((1-\tau)/w) - f(-\tau/w)|^\alpha$ $m(d\tau, dw)$, the first statement is a straightforward corollary from Theorem 3.12.1 of Samorodnitsky and Taqqu (1994). (When $\alpha = 1$, the additional condition for the existence of a stable integral involving a logarithmic term is always satisfied under the assumption $c' = c''$, i.e. $\beta = 0$.) Note also that the compensating constants for the Poisson integral can be taken 0 for $\alpha \geq 1$ since $\int_{E_i} [f((t-\tau)/w) - f(-\tau/w)] m(d\tau, dw) = 0$ in this case and $\beta = 0$ for $\alpha = 1$.

Because stable integrals are linear functionals of their integrands, the finite-dimensional distributions of both processes are also α-stable. Their multidimensional characteristic functions are again of the form (2.3) with $g(x)$ replaced by a linear combination of the respective integrands. Using this fact and an appropriate change of variables it is easy to establish self-affinity as well as stationarity of increments for $\{X(t),\ t \geq 0\}$, and hence for $\{X'(t),\ t \geq 0\}$. This is where the assumption $\beta = 0$ for $\alpha = 1$ plays the crucial role.

The exact value of the scale parameter raised to power α for $X(t)$ equals again $\int_E |f((t-\tau)/w) - f(-\tau/w)|^\alpha m(d\tau, dw)$ which is clearly proportional to $t^{1-\theta}$. ■

The next proposition specifies conditions on f which imply (2.6).

PROPOSITION 2.1 *(i) Let a function* f*, with the support in* $[0,1]$*, be Hölder continuous in* $[0,1]$ *with an exponent* $\eta > 0$*, i.e.,*

$$|f(x) - f(y)| \leq M|x-y|^\eta$$

for some $M > 0$ *and any* $x, y \in [0,1]$. *Then* (2.6) *holds for* $\max(0, 1-\alpha\eta) < \theta < 1$. *If in addition* $f(0) = f(1) = 0$, *then* (2.6) *holds for* $1 - \alpha\eta < \theta < 1$.

(ii) Let f *be a step function, i.e. there exists* $a_i \in R$, $i = 1, 2, \ldots, k \in N$, *such that*

$$f(x) = \sum_{i=1}^{k} a_i I[s_{i-1} < x \leq s_i],$$

where $0 = s_0 < s_1 < \ldots < s_k = 1$. *Then* (2.6) *holds for* $0 < \theta < 1$.

PROOF: In the proof of the first part of (i) use the boundedness of f to evaluate the integral in (2.6) over the regions $\{(\tau, w) : 0 < (1-\tau)/w < 1,\ -\tau/w < 0\}$ and $\{(\tau, w) :$

$1 < (1-\tau)/w,\ 0 < -\tau/w < 1\}$, and use Hölder continuity in the region $\{(\tau, w) :\ 0 < (1-\tau)/w < 1,\ 0 < -\tau/w < 1\}$. In the proof of the second part of (ii), note that $f((t-\tau)/w) = f((t-\tau)/w) - f(0)$ and $-f(-\tau/w) = f(1) - f(-\tau/w)$ respectively in the above first two regions, and then use Hölder continuity. To prove (ii), notice that in the region $\{(\tau, w) :\ 0 < (1-\tau)/w < 1,\ 0 < -\tau/w < 1\}$, when w is large enough, i.e. such that $1 < \min_i(s_i - s_{i-1})w$, the integrand (for fixed w) is non-zero in at most k intervals of the length 1. Hence, for $L = 1/\min_i(s_i - s_{i-1})$,

$$\int_L^\infty \int_{1-w}^0 \left|f\left(\frac{1-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right|^\alpha w^{-\theta-1} d\tau dw \leq \int_L^\infty k \max_i |a_i - a_{i-1}|^\alpha w^{-\theta-1} dw < \infty.$$

In other regions again use the boundedness of f. ∎

COROLLARY 2.1 *If a function f, with the support in $[0,1]$, is piecewise Hölder continuous in $[0,1]$ with an exponent $\eta > 0$, i.e. it has a finite number of jumps and is Hölder continuous in the intervals between jumps, then* (2.6) *holds for* $\max(0, 1-\alpha\eta) < \theta < 1$.

REMARK 2.1 Proposition 2.1 and Corollary 2.1 show that we are able to construct SFSP processes with any H and α satisfying $0 < H < \max(1, 1/\alpha)$.

REMARK 2.2 For fixed α and Hölder exponent η of f continuous on $(0,1)$, the self-affinity constant H can be any number, depending on θ, from the interval $(0, \eta)$ for $\{X(t),\ t \geq 0\}$ obtained as a sum of continuous (on R) pulses, and from the interval $(0, \min(\eta, 1/\alpha))$ for $\{X(t),\ t \geq 0\}$ obtained from the pulses with discontinuities at the end points. Hence, the parameter H covers different regions for $\alpha > 1$ depending on the pulse shape, and continuous templates allow for bigger range of H, i.e. also for $H \geq 1/\alpha$ (ultimately $0 < H < 1$). On the other hand, for $\alpha \leq 1$ these are the simplest step functions which may produce H greater than 1 (in this case $0 < H < 1/\alpha$). (For detailed treatment of rectangular (up-and-down) sums of pulses see Cioczek-Georges *et al* (1995).)

3 Properties and Examples of Stable SFSPs

The main topic of this Section is comparison of SFSPs with other known stable processes. We prove that SFSPs are indeed new processes and we also determine their path behavior. Examples at the end of the Section include two specific triangular shape templates which inspired current investigations.

Clearly, $\{X(t),\ t \geq 0\}$ constitutes a different stable process for different values of stable exponent α, or intensity β, or self-affine constant H. The question we must answer is whether for fixed admissible parameters α, c', c'', and θ, hence also fixed β, H, and the measure m, the process $\{X(t),\ t \geq 0\}$ is indeed different from other known stable processes with the same values of α, β, and H, i.e. whether or not their finite dimensional distributions are the same. To address this problem we look at two-dimensional characteristic functions of $(X(s), X(t))$, $0 < s < t$, more precisely, at so called spectral measures Γ, determined by these vectors.

In general, for an α-stable vector (X, Y) we have:

$$
E\exp(i(\xi_1 X + \xi_2 Y)) =
\begin{cases}
\exp\{-\int_{S_2} |\xi_1 s_1 + \xi_2 s_2|^\alpha (1 - i \text{ sgn } (\xi_1 s_1 + \xi_2 s_2) \tan \frac{\pi\alpha}{2})\Gamma(ds) \\ \quad + i(\xi_1\mu_1 + \xi_2\mu_2)\} & \text{if } \alpha \neq 1, \\
\exp\{-\int_{s_2} |\xi_1 s_1 + \xi_2 s_2| (1 + i\frac{2}{\pi} \text{ sgn } (\xi_1 s_2 + \xi_2 s_2) \ln |\xi_1 s_1 + \xi_2 s_2|)\Gamma(ds) \\ \quad + i(\xi_1\mu_1 + \xi_2\mu_2)\} & \text{if } \alpha = 1,
\end{cases}
$$

where the unique finite measure Γ (called the spectral measure) is defined on Borel sets of the unit circle S_2 and $\mu_1, \mu_2 \in R$ are the location parameters. In our case, since Γ is symmetric for $\alpha = 1$, the logarithmic term vanishes. Also $\mu_1 = \mu_2 = 0$. To determine Γ for $(X(s), X(t))$ it is necessary to use a special change of variables transforming the form of the characteristic function of $\int_E (\xi_1(f((t-\tau)/w) - f(-\tau/w)) + \xi_2(f((s-\tau)/w) - f(-\tau/w)))M(d\tau, dw)$ as in (2.3) into the form involving the spectral measure. For details look at Section 3.2 (in particular, formula (3.2.4)) in Samorodnitsky and Taqqu (1994). Here, we state only that subsets of S_2 where Γ is concentrated are determined by different values of the ratio

$$
\frac{f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)}{f\left(\frac{s-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)}.
$$

This ratio equals s_2/s_1 for some point $(s_1, s_2) \in S_2$. Note also that point (s_1, s_2) (and, if $\beta \neq \pm 1$, also its antipodal counterpart $-(s_1, s_2)$) is an atom of Γ iff the above ratio is constant on a subset of E with positive measure m.

Now we are ready to make two assertions about Γ.

1. Notice that the set of (τ, w)'s for which $f((t-\tau)/w) - f(-\tau/w) = 0$ and $f((s-\tau)/w) - f(-\tau/w) \neq 0$, and also the set for which $f((t-\tau)/w) - f(-\tau/w) \neq 0$ and $f((s-\tau)/w) - f(-\tau/w) = 0$ have positive measure m. Hence, Γ has atoms at $\pm(1, 0)$ and $\pm(0, 1)$.

2. Similarly, Γ has atoms at $\pm(1/\sqrt{2}, 1/\sqrt{2})$ (corresponding to $f((t-\tau)/w) - f(-\tau/w) = f((t-\tau)/w) - f(-\tau/w) \neq 0$).

REMARK 3.1 The above facts also follow from more intuitive reasoning involving pulse representation. For example, pulses whose amplitude is different at 0 and s, but the same at 0 and t, contribute to the independent from $X(t)$ part of $X(s)$, hence, cause Γ to be concentrated at $\pm(1,0)$ (Fact 1); these which do not change between s and t but change between 0 and s (i.e. start before 0 and die in $(0,s)$) add $\pm(1/\sqrt{2}, 1/\sqrt{2})$ to the support of Γ (Fact 2).

The next theorem draws conclusions from the above facts. In the range of admissible α and H, it compares $\{X(t),\ t \geq 0\}$ to other α-stable H-self-affine processes with stationary increments, namely to stable Lévy motion (with independent increments) with $H = 1/\alpha$, log-fractional stable motion, also with $H = 1/\alpha$ but defined only for $\alpha > 1$, and linear fractional stable motions (LFSM) $\{L_{\alpha,H,a,b}(t),\ t \geq 0\}$ defined for $0 < H < 1,\ H \neq 1/\alpha,\ |a|+|b| > 0$ (cf. Samorodnitsky and Taqqu (1994)).

THEOREM 3.1 *For fixed admissible values of parameters α and H, the dependence structures of an SFSP $\{X(t),\ t \geq 0\}$ is different from that of the following α-stable H-self-affine processes with stationary increments: stable Lévy motion, log-fractional stable motion, and LFSM with any $a, b,\ |a| + |b| > 0$.*

PROOF: Stable Lévy motion, as a unique α-stable process with independent stationary increments (hence, with the spectral measure of two-dimensional distributions concentrated at $\pm(0,1)$ and $\pm(1/\sqrt{2}, 1/\sqrt{2})$) is clearly different from $\{X(t),\ t \geq 0\}$ with $H = 1/\alpha$. Sato ((1992)) described the spectral measure of $(L_{\alpha,H,a,b}(s), L_{\alpha,H,a,b}(t))$ depending on $a,\ b$. It always has an absolutely continuous part and possibly a discrete part. However, there are no atoms at $\pm(0,1)$ which contradicts Fact 1 for $\{X(t),\ t \geq 0\}$. Using an argument similar to one of Sato, one can show that log-fractional stable motion has only absolutely continuous spectrum and must differ from $\{X(t),\ t \geq 0\}$ with $H = 1/\alpha$. ∎

Now we describe sample paths of SFSPs.

THEOREM 3.2 *For fixed admissible values of parameters α and H, and function f satisfying assumptions of Proposition 2.1 or Corollary 2.1, when $0 < H \leq 1/\alpha$, sample paths of an SFSP $\{X(t),\ t \geq 0\}$ are nowhere bounded (i.e. unbounded on every finite interval) with probability one, and when $1/\alpha < H (< 1)$, $\{X(t),\ t \geq 0\}$ has a sample continuous version.*

PROOF: Let us first point out that, by Kolmogorov's moment criterion (cf., for example, Billingsley (1968), p. 95-97), the last statement is true for any α-stable H-self-affine process with stationary increments such that $1/\alpha < H \leq 1$, hence, also for $\{X(t),\ t \geq 0\}$.

To prove the first statement we use the fact that if an α-stable process with an integral representation $\{\int_E f(t,u)M(du),\ t \in T\}$ is sample bounded then, necessarily,

$$\sup_{T^* \subset T} \int_E \sup_{t \in T^*} |f(t,u)|^\alpha m(du) < \infty,$$

where T^* is any countable subset of T. Moreover, a stable process is sample bounded with positive probability iff it is sample bounded with probability 1 (cf. Samorodnitsky, Taqqu (1994), Corollary 9.5.5 and Theorem 10.2.3).

For $\theta \geq 0$, we will show that

$$\int_{R \times R_+} \sup_{t \in T^*} \left| f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right|^\alpha w^{-\theta-1} d\tau dw = \infty \tag{3.1}$$

for $T^* = Q \cap I$, where Q is the set of rational numbers and $I = (a, a+2b)$ is a finite interval in $(0,\infty)$. Indeed,

$$\begin{aligned} &\int_{R \times R_+} \sup_{t \in T^*} \left| f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right|^\alpha w^{-\theta-1} d\tau dw \\ &\geq \int_a^{a+b} d\tau \int_0^b dw \sup_{t \in T^*} \left| f\left(\frac{t-\tau}{w}\right)\right|^\alpha w^{-\theta-1} \\ &= \sup_s |f(s)|^\alpha \int_a^{a+b} d\tau \int_0^b dw w^{-\theta-1} = \infty, \end{aligned}$$

which proves that $\{X(t),\ t \geq 0\}$ is unbounded on I. ∎

Remark 3.2 When $\alpha \leq 1$ ($H < 1/\alpha$), sample paths of SFSP processes are nowhere bounded with probability one. When $\alpha > 1$, sample paths are nowhere bounded with probability one for $0 < H \leq 1/\alpha$ and there exists a sample continuous version of the process for $1/\alpha < H < 1$.

Note that sample paths in general cannot be used to discriminate between the processes listed in Theorem 3.1 and $\{X(t),\ t \geq 0\}$. The only exception is α-stable Lévy motion ($H = 1/\alpha < 1$) whose sample paths possess discontinuities only of the first order. For $1/\alpha < H < 1$ this is implied by the general rule stated in the proof above, but for $H < 1/\alpha$ such a general rule does not apply. There exist, in fact, processes for $H < 1/\alpha$ with more regular sample paths. For example, sub-Gaussian processes (with FBM as the Gaussian process components) always have sample continuous version, and substable processes with stable Lévy motions as stable process components have only first order discontinuity (cf. Kôno and Maejima (1991)). More sophisticated examples based on a construction due to Kesten and Spitzer can be found in Samorodnitsky and Taqqu (1994).

REMARK 3.3 It is clear that the differences in the dependence structure of SFSP processes with the same fixed admissible parameters come from using various template functions f. Nevertheless, it is difficult to give a sufficient condition for differentiation between SFSPs only in terms of fs since it is an "average" of f (its integral) which is used rather than f itself. This is why to discriminate between SFSPs in the examples below we use the technique developed for the proof of Theorem 3.1, i.e. we compare the spectral measures of two-dimensional characteristic functions.

EXAMPLES

1. The first example of ~~stable~~ SFSP processes constitute cylindrical pulse processes introduced in Cioczek-Georges *et al* (1995). The template is the indicator of $[0, 1]$ and SFSPs are defined for $0 < \theta < 1$.

2. The right and isosceles triangles are two shapes satisfying the assumptions of Proposition 2.1 (i) with $\eta = 1$. The right triangular (semi-conical) pulse starts with a jump (a discontinuity) and then decays linearly; the isosceles triangular (conical) pulse increases linearly to a certain point and then decreases with the same rate. According to Proposition 2.1 (i), the fractal sum of semi-conical pulses process $\{X_1(t),\ t \geq 0\}$ is defined for $\max(0, 1-\alpha) < \theta < 1$ and the fractal sum of conical pulses process $\{X_2(t),\ t \geq 0\}$ is defined for $0 < \theta < 1$.

The spectral measure Γ_1 of $(X_1(s), X_1(t))$, $0 < s < t$, is concentrated only in the first and the third quadrants of the circle S_2, as Γ_2 of $(X_2(s), X_2(t))$ is also distributed in the second or the fourth quadrant. This comes from the fact that $f((s-\tau)/w) - f(-\tau/w)$ and $f((t-\tau)/w) - f(-\tau/w)$, for $f(x) = (1-x)I[0 \leq x \leq 1]$, cannot be of the opposite signs, as they are of the same sign and of the opposite signs on the sets with positive measure m for $f(x) = (1/2 - |x - 1/2|)I[0 \leq x \leq 1]$. Hence, for fixed parameters α, θ and c', c'', $\{X_1(t),\ t \geq 0\}$ and $\{X_2(t),\ t \geq 0\}$ are different α-stable $(1-\theta)/\alpha$-self-affine processes with stationary increments. They also differ from the cylindrical pulse processes in Example 1. This follows from the fact that the respective spectral measures for those processes are purely discrete and concentrated at 6 points $\pm(0,1)$, $\pm(1,0)$, $\pm(1/\sqrt{2}, 1/\sqrt{2})$. On the other hand, both Γ_1 and Γ_2 have continuous parts, corresponding, for example, to $f((s-\tau)/w) - f(-\tau/w) = -s/w$ and $f((t-\tau)/w) - f(-\tau/w) = -(w+\tau)/w$.

3. Other shapes which may be used in pulse constructions are discussed in Cioczek-Georges and Mandelbrot (1994b). They include, for example, the singular Cantor distri-

bution function (or rather its part supported in $[0,1]$) which is Hölder continuous with $\eta \le \log 2/\log 3$. The complicated respective spectral measure for an SFSP obtained from this template features a countable number of atoms.

4 SFSP Domain of Attraction

We describe a part of the domain of attraction of a prescribed SFSP of the form defined in Section 2.

We say that a process $\{Y(t),\ t \ge 0\}$ with stationary increments belongs to the domain of attraction (DOA) of a process $\{X(t),\ t \ge 0\}$ if finite dimensional distributions of suitably normalized $\{Y(ut),\ t \ge 0\}$ converge, as $u \to \infty$, to those of $\{X(t),\ t \ge 0\}$.

Usually, the concept of DOA is defined for stationary sequences. However, the equality $Y(ut) = \sum_{j=1}^{ut}(Y(j) - Y(j-1))$, with $Y(0) = 0$, for $ut \in Z$, makes it obvious that the definition of DOA's extends to processes with stationary increments.

When the limit $\{X(t),\ t \ge 0\}$, with non-degenerate $X(1)$, satisfies some mild continuity (in law) conditions, the analytic form of normalizing constants is known. For example, if

$$\frac{Y(ut) - c(u)}{a(u)} \xrightarrow{d} X(t), \qquad u \to \infty,$$

and the process $\{X(t),\ t \ge 0\}$, with $X(0) = 0$, is H-self-affine and has stationary increments, then norming function a is regularly varying with index H. In this case, moreover, $c(u)$ is $o(a(u))$ and hence can be omitted. Indeed, $H > 0$ and the process $\{X(t),\ t \ge 0\}$ is continuous in probability, so that the result in Lamperti (1962) applies giving the form of a (cf. also Vervaat (1986), (1992), or Bingham *et al* (1987), p. 356). The statement about $c(u)$ easily follows from the fact that $X(0) = 0$. Thus, we are looking for processes $\{Y(t),\ t \ge 0\}$ with stationary increments, which are self-affine "in the limit," i.e. satisfy

$$\frac{Y(ut)}{u^H L(u)} \xrightarrow{d} X(t), \qquad u \to \infty, \tag{4.1}$$

for some slowly varying (at ∞) function L.

Given an SFSP process, as defined in Section 2, we restrict our exploration of its DOA to processes $\{Y(t),\ t \ge 0\}$ with $Y(t)$ of the form (2.2), where integration may be conditional. Each pulse is a translated, horizontally and vertically rescaled template function and the template f is the same as in the limiting SFSP $\{X(t),\ t \ge 0\}$. This is not a constraint since a proper change of variables can always make the integrand of $Y(ut)$ independent from the variable u (cf. the equality leading to (4.3)). The Poisson measure N_Y and intensity

$n_Y(d\lambda, d\tau, dw) = EN_Y(d\lambda, d\tau, dw)$ remain to be specified. The question is: what shape can they take to ensure convergence (4.1)? In other words, how much can we modify the intensity n given by (2.1) used to define the SFSP $\{X(t),\ t \geq 0\}$?

Let $\{X(t),\ t \geq 0\}$ be an SFSP with an admissible template function f and fixed parameters $0 < \alpha < 2$, $min(1-\alpha, 0) < \theta < 1$, $c', c'' \geq 0$, $c' + c'' > 0$, i.e. $\{X(t),\ t \geq 0\}$ is well-defined for these values (condition (2.6) of Theorem 2.1 holds). We consider the following general form of n_Y:

$$n_Y(d\lambda, d\tau, dw) = \begin{cases} L_1'(\lambda)\lambda^{-\alpha-1}L_2(w)w^{-\theta-1}d\lambda d\tau dw & \text{if } \lambda > 0, \\ L_1''(|\lambda|)|\lambda|^{-\alpha-1}L_2(w)w^{-\theta-1}d\lambda d\tau dw & \text{if } \lambda < 0, \end{cases} \tag{4.2}$$

for $\tau \in R$, $w \in R_+$, where fixed α and θ are the same as for the limiting $\{X(t),\ t \geq 0\}$, and L_1', L_1'', and L_2 are positive slowly varying measurable functions. We also assume that the process $\{Y(t),\ t \geq 0\}$ with the same f, α, and θ is well-defined, i.e. that the functions L_1, $L_1(\lambda) := L_1'(\lambda)I[\lambda > 0] + L_1''(|\lambda|)I[\lambda < 0]$, and L_2 are such that the Poisson integral of $\lambda(f((t-\tau)/w) - f(-\tau/w))$ with respect to N_Y converges (possibly conditionally, and with zero compensating constants which is implied by the form of n_y in τ under strong enough integrability conditions).

Let us comment on the form (4.2) of the intensity n_Y. The fact that the measure has a product form in the variables λ, τ, and w mirrors the independence between these variables assumed in the sum of pulses construction. Lebesgue measure in τ ensures stationarity of increments for $\{Y(t),\ t \geq 0\}$ in the same way as for $\{X(t),\ t \geq 0\}$. We modified the intensity measure n of $\{X(t),\ t \geq 0\}$ in variables λ and w by allowing regular variation. If one thinks of sum of pulses processes in terms of its "moving average" structure, then L_1 changes random innovations to non-stable and L_2 modifies the dependence structure, while the coefficients (f) remain unchanged.

At $\xi \in R$, the logarithm of the characteristic function of $Y(ut)/(u^H L(u))$ equals

$$\begin{aligned} &\ln E\exp(i\frac{\xi}{u^H L(u)}\int_A \lambda\left(f\left(\frac{ut-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right)N_Y(d\lambda, d\tau, dw)) \\ =\ & \int_A (e^{i\xi\lambda(f((ut-\tau)/w) - f(-\tau/w))/(u^H L(u))} - 1)n_Y(d\lambda, d\tau, dw) \\ =\ & \int_A (e^{i\xi\lambda(f((ut-\tau)/w) - f(-\tau/w))/(u^H L(u))} - 1)L_1(\lambda)\lambda^{-\alpha-1}L_2(w)w^{-\theta-1}d\lambda d\tau dw \\ =\ & n^{1-\theta-\alpha H}L^{-\alpha}(u)\int_A (e^{i\xi\lambda(f((t-\tau)/w) - f(-\tau/w))} - 1) \\ & L_1(\lambda u^H L(u))\lambda^{-\alpha-1}L_2(uw)w^{-\theta-1}d\lambda d\tau dw, \end{aligned} \tag{4.3}$$

where the last equality follows from a simple change of variables. Note that $n^{1-\theta-\alpha H} = 1$ since H is the self-affinity exponent of the process $\{X(t),\ t \geq 0\}$. If it is permissible to change the order of integration and the limit in u, then the sufficient condition for the expression in (4.3) to approach the logarithm of the characteristic function of $X(t)$ is

$$\frac{L_1(\lambda u^H L(u))L_2(uw)}{L^\alpha(u)} \to c'I[\lambda > 0] + c''I'[\lambda < 0], \qquad u \to \infty, \tag{4.4}$$

for any $\lambda \in R_0, w > 0$.

When (4.4) holds and one of the functions L_1 or L_2 is constant then the other one must be slowly varying since L is a slowly varying function. Hence, the assumption that the density of n_y is regularly varying in variables λ and w with the same exponents as their powers in the density of n is not as restrictive as it would seem. On the other hand, given slowly varying functions L_1 and L_2, does there exists an L which satisfies (4.4)? It turns out that the answer is yes.

PROPOSITION 4.1 *For any slowly varying functions L_1 and L_2, there exists a slowly varying function L such that*

$$\frac{L_1(u^H L(u))L_2(u)}{L^\alpha(u)} \to 1, \qquad u \to \infty. \tag{4.5}$$

PROOF: Consider a regularly varying function $h(y) := y^\alpha(L_1(y))^{-1}$ for $y > 0$. Then, by Theorem 1.5.12 of Bingham et al (1987), p. 28, h has an asymptotic inverse g, i.e. $h(g(z)) \sim g(h(z)) \sim z$ as $z \to \infty$, and g is regularly varying with index $1/\alpha$, i.e. there exists a slowly varying L^* such that $g(z) = z^{1/\alpha}L^*(z)$. Hence, function L^* satisfies $(L^*(z))^\alpha(L_1(z^{1/\alpha}L^*(z)))^{-1} \sim 1$. It is determined uniquely to within asymptotic equivalence. Define $L(u) := (L_2(u))^{1/\alpha}L^*(u^{H\alpha}L_2(u))$. We have

$$\lim_{u\to\infty} \frac{L_1(u^H L(u))L_2(u)}{L^\alpha(u)} = \lim_{u\to\infty} \frac{L_1(u^H(L_2(u))^{1/\alpha}L^*(u^{H\alpha}L_2(u)))}{(L^*(u^{H\alpha}L_2(u)))^\alpha} = \lim_{v\to\infty} \frac{L_1(v^{1/\alpha}L^*(v))}{(L^*(v))^\alpha} = 1,$$

and the proposition holds. ∎

Obviously, scaling function L defined in the proof of Proposition 4.1 by $c'^{1/\alpha}$ (or $c''^{1/\alpha}$) makes the limit in (4.5) equal c' (or c''). This and slow variation of L_1 and L_2 imply that there exists L such that (4.4) holds, and hence also under certain "regularity conditions" (4.1) holds.

THEOREM 4.1 *Fix $0 < \alpha < 2$ and $0 < \theta < \max(1, 1/\alpha)$. Let a function f satisfy assumptions of Proposition 2.1 or Corollary 2.1 and slowly varying (measurable) positive* *functions* L_1 *and* L_2 *be bounded on* $(0, M]$ *for some* $M > 0$. *Then a sum of pulses process* $\{Y(t) = \int_A \lambda(f((t-\tau)/w) - f(-\tau/w))N_Y(d\lambda, d\tau, dw),\ t \geq 0\}$, *with* $n_y(d\lambda, d\tau, dw) = EN_Y(d\lambda, d\tau, dw)$ *given by* (4.2), *is in the domain of attraction of an SFSP* $\{X(t) = \int_A \lambda(f((t-\tau)/w) - f(-\tau/w))N(d\lambda, d\tau, dw),\ t \geq 0\}$, *with* $n(d\lambda, d\tau, dw) = EN(d\lambda, d\tau, dw)$ *given by* (2.1).

PROOF: Under the above assumptions, the Poisson integral $\{Y(t),\ t \geq 0\}$ exists as is easy to check by following the proof of existence of stable Poisson integrals in Samorodnitsky and Taqqu (1994), pp. 159-167. Also these assumptions are strong enough to allow to use the Lebesgue Dominated Convergence Theorem and to change order of integration and limit in u in (4.3). The rest of the statement is an obvious consequence of considerations above. ∎

REMARK 4.1 We are not able to find a general regularity condition sufficient for the existence of $\{Y(t),\ t \geq 0\}$ analogous to (2.6), i.e. involving the part of intensity n_y depending only on τ and w. It seems that any such condition would have to use the function L_1 as well.

REMARK 4.2 Clearly, when $L_1 = const(= 1)$, then define $L(u) = (L_2(u))^{1/\alpha}$. When $L_2 = const$, then $L(u) = L^*(u^{H\alpha})$, or—equivalently—one can obtain L as a slowly varying component of an asymptotic inverse of $h(y) = y^{1/H}(L_1(y))^{-1/(\alpha H)}$.

REMARK 4.3 When solving asymptotic equations for L one is reminded of finding norming constants in the central limit problem for distributions in a stable domain of attraction. If the sum of a distribution tails vary regularly as $x^\alpha L_1(x)$, then the norming constants are of the form $a_n = n^{1/\alpha}L(n)$, where L satisfies the relation $L_1(n^{1/\alpha}L(n))/(L(n))^\alpha \to 1$, as $n \to \infty$.

REMARK 4.4. For some simple slowly varying functions L_1 and L_2 it is easy to calculate L directly. For example, when $L_1(u) = \max(1, \ln u)$, then $L(u)$ can be taken as $(H \max(\ln u, 1)L_2(u))^{1/\alpha}$.

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