Parallel cartoons of fractal models of finance

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Summary. Having been crafted to welcome a new scientific journal, this paper looks forward but requires no special prerequisite. The argument builds on a technical wrinkle (used earlier but explained here fully for the first time), namely, the author's grid-bound variant of Brownian motion $B(t)$. While $B(t)$ itself is additive, this variant is a multiplicative recursive process the author calls a "cartoon." Reliance on this and related cartoons allows a new perspicuous exposition of the various fractal/multifractal models for the variation of financial prices. These illustrations do not claim to represent reality in its full detail, but suffice to imitate and bring out its principal features, namely, long tailedness, long dependence, and clustering. The goal is to convince the reader that the fractals/multifractals are not an exotic technical nightmare that could be avoided. In fact, the author's models arose successively as proper, "natural," and even "unavoidable" generalization of the Brownian motion model of price variation. Considered within the context of those generalizations, the original Brownian comes out as very special and narrowly constricted, while the fractal/multifractal models come out as nearly as simple and parsimonious as the Brownian. The cartoons are stylized recursive variants of the author's fractal/multifractal models, which are even more versatile and realistic.

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EVERY SCIENTIST IS – OR SHOULD BE – MOTIVATED by the belief that "the real world" follows rules that are not yet known but can conceivably be identified, and that those rules are not excessively complex but – to the contrary – "parsimonious." This term expresses that a high return – in the form of refined results, many of them not known in advance, is obtained from a low investment – in the form of few and simple a priori assumptions. Thus, parsimony subjects economic modeling to an economic criterion. Ideally, it also expresses that the input manages with few key parameters that are directly accessible because each affects the output in its own specific way. Ideally, once again, the parameters are not simply inherited from older parts of statistics but intrinsic to the main feature of the data. In the context of financial prices, that feature is the roughness of the charts.

Two models of price variation, the Brownian and the fractal/multifractal, have contended since they were advanced near-simultaneously between 1960 and 1972. True and important, the Brownian originated much earlier, in a most remarkable pioneering work, Bachelier 1900. But it attracted little attention until it was either rediscovered or reinvented independently by several authors.

The key goal of this paper is to show by perspicuous illustrations how these models continuously relate to one another. The gist of the paper, therefore, consists in the figures and captions. The remainder consists in comments of diverse kinds.

The famous English polymath J.B.S. Haldane observed that a scientific idea ought to be interesting even if it is not true. The Brownian motion is certainly both. Its Gaussian and independent price increments and its property of varying continuously helped make it central to business school education (though no longer to the exclusive extent that prevailed a while ago). However, being present when the Brownian was revived, I promptly observed that it thoroughly fails to approximate reality on several distinct grounds: all the actual data show clear-cut discontinuities (jumps), non-Gaussianity, unquestionable dependence, and clustering of large changes.

My objections were absolutely fundamental; they were disputed in many publications but have prevailed, as shown by the innumerable methods proposed to answer them. In this sense, it is recognized that the pure Brownian without "fixes" has run its course.

In addition to pointing out the Brownian failings, I have – between 1962 and 1972 – put forward three successively improved models. First, two fractal models, in 1963 [1; see also my book 2] and 1965 [3; see also Chapter H 30 of my book 4] handled long tails and long dependence separately and introduced new issues. A bit later [5; see also 6, 7, 8, 9, 10] the multifractal model managed to handle old and new issues inextricably mixed together. The multifractal model was announced in 1972. Ironically, it responded in advance to certain difficulties present in my 1963 model that motivated other authors to turn back to the Brownian, as witnessed by the Black-Scholes formula. These models use increasingly broad forms of the same parsimonious principle of scaling, and have added to my "fractal view of risk, ruin, and reward," a topic that a recent book [ ] presents in simple terms. It became one of the streams that merged to form fractal geometry. Since the mid-nineteen hundred nineties, my three models have been widely adopted, one after the other, and expanded in diverse directions by torrents of publications.

1. The incomparable virtues of parsimony and perspicuity

Let us return to the ambitious goal of science. It aims to provide a compromise between two very different goals: satisfactory statistical fit to observations and the highest achievable parsimony.

Even the first of these goals often subdivides into parts that clash with one another – the reason being that every interesting phenomenon involves several distinct relations. This difficulty arises even when one considers a single financial price series. A virtue of the Brownian model is that it handles at the same time the distributions of changes over different time spans. This feature is rare. In particular, except under special and unreasonably demanding conditions, the models that yield the best statistical fits for different time spans are either self-defeatingly complex or mutually incompatible. Therefore, nearly all authors only consider a single span, for example one day. To the contrary, the fractal/multifractal models – like the Brownian – apply to all time spans and allow my faith in the virtues of parsimony to hold without reservation.

This paper presents the contrast between the Brownian and the fractal/multifractal models in a way that is parsimonious and also perspicuous. The latter is the case because, by now, pictures of recursively constructed fractals have been seen by nearly everyone, in some context or another. This is a very gratifying fact.

Not gratifying at all, however, is the widespread belief that fractal geometry reduces to pretty pictures that are pretty useless. This opinion reflects a very serious lack of communication. Everyone agrees that graphics is never the last word in science, and never a substitute for analysis. True. But it is a marvelous additional tool one cannot afford to spurn – both for teaching and for further thinking. It is really too bad that the circumstances prevailing in 1996 made me fail to follow my own advice and that my book [2] was insufficiently illustrated. The resulting graphical bareness is surely one reason why the whole is sometimes described as difficult. In particular, the "cartoons" in Chapter E6 of [2] should have been very much more numerous and explicit. This paper is a welcome opportunity for taking a step in that direction.

Multifractals beyond the cartoons. It must be said immediately that the cartoons discussed in this paper are extremely special examples designed for a pedagogical goal. Even the earliest presentation multifractals in [6] distinguished several stages of generality. They cannot be explained here, only listed: binomial, multinomial, microcanonical (or conservative), and cononical. Increasingly broad generalizations are being developed. The cartoons to be discussed are trinomial, therefore close to the lowest level of generality. But for the present purposes, even they suffice, since Figures 5 and 6 already exhibit the actual data's long failedness, long dependence, and clustering. The more refined canonical models allow an even greater control of longtailedness. For example, one can obtain a power law distribution whose exponent is not constrained in any way; this allows variance to be either infinite or finite.

2. "Fixes," "active ingredients," and "homeopathic" irrelevance

To be frank, parsimony ends near-invariably by encountering limitations. Sooner or later, even the best theories fail to account for some newly recorded or newly accepted data. At that point, even the most demanding scientists must agree to "fixes." Each is specifically meant to patch up some inconvenient discrepancy and it is expected that, in due time, the progress of science will make them unnecessary.

In particular, current practical financial recipes invariably incorporate a multitude of mostly proprietary ingredients. They inevitably raise a question that is serious, deserves to be addressed immediately, and is best stated in the vocabulary of pharmacology. After repeated mixing, does the Brownian input learned in business school remain the "active ingredient?" Or – as seems far more likely – has it been diluted to "homeopathic" irrelevance?

One must keep in mind that every "fix" destroys the Brownian's parsimony. To the contrary, the multifractal view of risk, ruin, and return already suffices to account for all the overall features of price variation that everybody knows are present in the data. In the case of the cartoons presented in this paper, the fit is qualitative. More advanced cartoons and fractal models beyond the cartoons move towards quantitative fit.

Here is a useful and colorful metaphor. In the context of a high mountain to be climbed, the counterpart of "fixes" consists in the fact that the last stage invariably proceeds by foot – effectiveness coming before elegance. But every big climb needs a ground base. For finance, a Brownian ground base is very comfortable but located at an excessively low altitude, having much of the hard work of modeling to be done "higher up." My early fractal models of 1963 and 1965 moved the ground base to higher altitude; the multifractal model moved it even higher up, all that without any recourse to "fixes."

3. Contrasts between two kinds of highly parsimonious models in financial mathematics: Brownian and fractal/multifractal

A.) Before 1960, when it came back into economics, the Brownian had been extensively studied in physics and mathematics. Therefore its revival in finance immediately benefitted from extensive knowledge accumulated in the literature. This gave the Brownian an enormous advantage, namely, a running start.

By contrast, the fractal/multifractal models started with little or no intellectual "capital."

B.) The Brownian model comes in a single flavor; this is an enormous advantage from both the pedagogical and the technical viewpoints. But from a scientific viewpoint, it is an enormous drawback, in fact, a lethal one in my opinion. Its implies that – except for a single tunable parameter called volatility – all financial products follow identical rules. This crude simplification is not supported by any evidence and on its face is most likely to be wrong.

To the contrary, the fractal/multifractal approach consists in several successive generalizations of the Brownian. The progression between those stages created a moving target for criticism. Most critics continue to react to features that characterized early stage models but have long since been corrected.

For example, my 1963 model implies infinite variance and independent price increments. Both features continue to be lambasted, and in many contexts (but not in this paper!), I am obliged to respond. P.H. Cootner in 1964 has observed, and many other authors since then have also stressed, that in the study of the fractal/multifractal model the bulk of the traditional tools of statistics is of little, or no use. No question: such is indeed the case: the notion that statistics is naturally suited to economics is indefensible, witness the fact that the average income or the average firm size (think of software firms) are products of the statistical imagination.

My position on this account is continually criticized, the brunt of questioning being addressed to infinite variance.

The task of replacing traditional statistical tools by a very different set seemed prohibitive but eventually attracted attention and fast progress. Chapter E5 of [2] advances the notion that random and other forms of variability can be so varied that it is best to view them as coming in different states that range from mild to wild.

4. Summary

While the multifractal model is increasingly widely accepted, it remains not fully understood. Fortunately, wide lecturing has led me to develop a new presentation of the basic facts. It tackles all the examples in parallel, is brief, and avoids extraneous complication. It is presented in Section 8, my earlier partial models being presented in Sections 6 and 7.

The "hook" on which this presentation hangs is a cartoon of Brownian motion that is recursive, interpolative, and multiplicative. I have introduced and used it widely without drawing to it the focussed attention I now think it deserves. It is discussed in Section 5.

5. Figure 1. An interpolative recursive cartoon of Brownian motion as a multiplicative process

The key content of this section consists in Figure 1. It calls for several comments. Conceived in Bachelier 1900, the standard model of the variation of financial prices has been Brownian motion. There is a wide belief that the two properties that define it – Gaussianity and independent increments – are not only convenient but necessary, namely, accounted for the fundamental result called central limit theorem. That theorem concerns reduced *sums* of increasingly many random processes. It states that the limit behavior is "universal," that is, independent of the nature of the addends.

Of course, every theorem begins with assumptions, but those of CLT, the central limit theorem, are rarely emphasized because they are taken for granted. One first assumption: stationarity. Second assumption: the increments' marginal variance defines a finite volatility. Third assumption: finite memory. This third assumption demands to be amplified. Consider two instants of time separated by the span T. As T , the dependence between price behavior at those two time instants become negligible.

The fractal/multifractal models contradict one or several of those assumptions. This is not a deficiency but the key of their effectiveness. This creates a paradoxical situation. My critique of Brownian motion has generated many alternative models that are meant to preserve the central limit theorem – but for this very reason automatically fail to be effective and parsimonious.

Be that as it may, during the 1960s I presented my pre-multifractal models by stressing their properties of additivity. This was made possible by invoking the so-called "generalized central limit theorems." They preserve the idea of additive components but only at an extraordinary high cost in arbitrariness. As a result, Brownian motion appeared natural while its generalizations appeared contrived. Actually, the arbitrariness was only apparent but appearances do create false impressions. Furthermore, multifractals intrinsic components are not additive but multiplicative, hence they could not be presented via a generalized CLT.

Before moving on to multiplicative cartoons, it is useful to dwell on earlier cartoons that kept close to Brownian motion itself since they were intrinsically generated by adding contributing components. The simplest nonrandom additive cartoon is the Landsberg function which my book [4] studies in Chapter H2, Section 5 and illustrates by Figure H20-2. This old and repeatedly rediscovered expression is the sum of nonrandom periodic sawtooth curves. Each tooth is made of a segment going UP followed by a segment going DOWN. Shuffling consists in choosing at random between these UP, DOWN teeth and DOWN, UP teeth.

To generalize this kind of cartoon turned out to be impossible. To break the resulting conceptual and pedagogical logjam, I found a way to use multiplication for the Brownian. This was non-traditional but made it more convenient and far easier to generalize, and I now think that multiplication is intrinsically the better approach.

The construction illustrated on Figure 1 does not lead to Brownian motion proper. But it generates a very useful "cartoon" that deepens our understanding of Brownian motion and also lends itself easily to be transformed into cartoons of either of my three models. Comparing all those cartoons help underline both the clear differences and the kinship created by scaling/fractality. The relations between the different models are illustrated by the phase diagram shown in this paper's last Figure.

6. Figure 2. The phenomenon of price discontinuity and related long tailed distribution of price changes ("Noah" Effect); the "PLM" (Pareto-Lévy-Mandelbrot) or "mesofractal" model.

The key content of this section consists in Figure 2. It calls for several comments. Brownian notion is continuous but prices are not and as a result the price increments over fixed time increments *t* have very long-tailed distributions. The first model to acknowledge and face the discontinuity of prices and the corresponding long-tailedness was sketched in [ ] (reproduced photographically in [ ]), pioneered in [ ], and developed in half of [2].

A cartoon of this model is provided by the construction illustrated on Figure 2. It preserves the fractal principle underlying Figure 1 but uses a different generator.

An important issue of old or new terminology. Wide recent acceptance of my first model also had an unfortunate by-product: a multiplicity of distinct terms. They became so varied that many observers believe that the alternative terms denote different models – which they do not The confusion began with my own hesitation and was compounded by the numerous authors who have by now adopted "Lévy law" or "Lévy distribution," even in titles of books. Around 1960, I boasted of having combined inspirations from the real world and pure mathematics, namely, from the economist V. Pareto and the pure probablist P. Lévy. This led me to coin the term "Pareto-Lévy" (PL) law. Paul Samuelson replaced it by "Pareto-Lévy-Mandelbrot" (PLM), which skillfully combined a kind of "thesis," a kind of "antithesis," and a kind of "synthesis." Instead, my book [2] was ill-inspired in introducing the term "M 1963 model" and in several papers I used "stable Paretian," "Lévy stable, " and "mesofractal." All failed to take root while Samuelson's PLM has been used by others. To avoid ambiguity and confusion and preempt inappropriately incomplete terminologies like "power-law distributions," the best seems to settle on PLM.

Contrast between the PLM model and a Brownian motion "improved" by the addition of jumps. After the PLM model had drawn attention to price discontinuities, it has been suggested that the Brownian motion $B(t)$ can be made into a better model by adding discontinuities after the fact. Examining Figure 2, it may be observed that at first glance, it too looks like a Brownian cartoon with superposed discontinuities. A first difference is that in PLM the discontinuities are of all sizes – with the small ones merging into the Brownian-like background. A second more important difference is that in PLM those discontinuities are not "fixes" added by hand to correct a defect of $B(t)$ but consequences of a basic "scaling principle." I introduced it into economics around 1960 and it is discussed in Chapter E2 of [2].

7. Figure 3. The phenomenon of long or global dependence in price variation ("Joseph Effect") and the "HHM" ("Hurst-Holder-Mandelbrot) or "unifractal" model

The key content of this section consists in Figure 3. It calls for several comments. It has always been suspected that the price increments exhibit is some non-independence but its strength and long range were first faced in model pioneered in M 1965h{H9} (reproduced photographically in M 1997FE).

Terminology. It is not quite as multiple and confusing as in the "PLM" case – but nearly so. Early on, just as for PLM, I credited the combined influences of a practical man, the hydrologist Hurst and of a mathematician, Hölder. My book [2] used the notation "M

1965 model" and elsewhere I used "unifractal." Both failed to take root. Ultimately, to parallel "PLM," the best may be to settle on "HHM."

One of the main topics of my book [4] is *R* or *R*/*S* analysis. *R* analysis has recently received attention (but without reference to its origin) under the name "detrended range analysis." The new label does not affect the content but destroys communication.

The basic unifractal model is based on Fractional Brownian motion (FBM). It is developed in reprinted papers and specially written chapters collected in [4]. The recursive fractal construction illustrated on Figure 3 steps back to cartoon using a symmetric three segment generator, like in Figure 1 but generalizes it so that, given H satisfying $0 < H < 1$, the restrictive relation (log of height/log of width) $= H$ holds for each of the three intervals.

Global dependence in financial prices is treated in Chapter H30 of [4], which combines excerpts from papers I published between 1969 and 1973 with diverse recent comments. The topic is subtle because, as underlined in Chapter H30 of [4], unifractality goes far beyond FBM. On the other hand, Fractional Brownian motion is especially simple, being defined by a single long-dependence exponent H. As a result, FBM was long shunned as representing an inefficient market. By now, to the contrary, it has began to be widely studied. Most emphatically, I continue to view FBM as the basic illustration of long dependence but only a gross approximation for prices. Therefore, Chapter H30 of [4] adds very substantially to M 1997E and is very strongly recommended.

8. Figures 4 to 6. The combination of the Noah and Joseph Effects and my current "multifractal model" of price variation.

The key content of this section in Figures 4, 5, and 6. They call for several comments. Throughout the 1960s my publications kept emphasizing that most observed instances of the Noah and Joseph effects do not appear separately but in combination. As a result, both the PLM and HHM models could only be viewed as simplifying approximations. In different ways, each greatly improved upon Brownian motion but did not warrant being pursued separately in excessive detail, and a model embodying both effects was needed. This model – eventually labeled multifractal – was first mentioned in the last paragraph of a 1972 paper of mine, reproduced as Chapter N14 of [ ].

Figure 4. To present multifractal cartoons, it is best to take one tack, then another. The point of departure is a Brownian or fractional Brownian cartoon and the key operation is to change the generator by moving the junction points between the UP and DOWN and the DOWN and UP intervals either farther or closer to each other along the horizontal. The outcome, as illustrated by Figure 4, gave intense satisfaction but clearly defined room for improvement. The reason for satisfaction is that a construction of extreme and unexpected parsimony yields a result that is remarkably reminiscent of the actual data, as exemplified for example by Figure of my book [ ]. But improvement becomes increasingly obvious as the junction points move far enough they create long flat periods when almost nothing happens.

Let me digress for the reader familiar with the multifractal formalism developed in [ ]. The flat periods correspond to $_{max} > 1$, which is the case when the junction points are so distant that one of the generator's interval is of slope < 1.

Figure 5. Figure 5 brings the generator's junction points closer to each other than in the Brownian case. The gain is that the flat periods are not present. The loss is that the increment diagrams are "clunky" and less realistic as other grounds. The result: the cartoon's are no more than cartoons and closer fit must be searched in more general multifractals.

Cartoon-drawing is an addictive game; all that the "player" needs is to gain access to a shuffled recursive program in which all that is left to do is to input a generator.

An immediate idea of the resulting variety is provided by the increment records plotted in the preceding figures in this chapter. The Brownian cartoon stands out instantly as a "fake," a far from realistic representation. The PLM cartoon stands out as having long tails and no dependence. The HHM cartoon stands out as having short tails and clear-cut clustering. To the contrary, realistic approximations of real price variation are instantly spotted among the multifractal cartoons, despite the fact that they were chosen to be as simple as can be.

Figure 6. A phase diagram. This paper's figures involve very special "cartoons." Each has a symmetric three interval generator, has a well-defined "address" in the left half of a unit square. This address is the position of the break between the UP and the DOWN interval. Hence it is possible to draw a "phase diagram," Figure 6, that consists in a square within which different regions correspond to one or another of diverse models of price variation. The Brownian model corresponds to one point. The PLM and HHM cases correspond each to a curve. The multifractal corresponds to the remainder of the square and is thereby shown to be far more "generic."

FIGURE C1-1. An interpolative/recursive cartoon of Brownian motion that allows a limited extent of randomization represented by the term "shuffled." The construction begins with an initiator: a straight trend from $P(0) = 0$ to $P(1) = 1$. Three generators are available, as drawn on the top panel: the only differences between them is that they are distinct permutation of two congruent UP intervals and one DOWN interval. The construction's first stage replaces the initiator with one of the three available generators, selected at random. Each further stage starts with a broken line and replaces each of its intervals by choosing a generator at random among the three available ones, and then squeezing it to fit. The second panel shows the first three stages of construction, the third panel a mature (and more detailed) approximation, and the bottom panel, the increments of that mature approximation. Made into sounds, those increments are perceived as a "hum" very much like radio static. They are a cartoon of Gaussian white noise.

FIGURE C1-2. A shuffled recursive cartoon of a Lévy stable process with $= \log 5 / \log 3 = 1....$ In order to obtain this and accommodate up and down discontinuities, the simplest generator combines more than three non-vertical intervals with two discontinuities that illustrate a phenomenon I have called the Noah Effect. The tails and the middle of the distribution were hastily patched up by hand and the match between them leaves room for improvement in the future.

FIGURE C1-3. Shuffled recursive cartoons of fractional Brownian motions. The bottom panel – for $H = 0.75$ – is best appreciated by examining the increments. They exhibit symptoms of a phenomenon I have called the Joseph Effect (seven fat and seven lean cows), namely, long runs of mostly positive or mostly negative values. To the contrary, the upper panel – for $H = 0.3$ is best appreciated by examining the rapid up and down flipping of the function itself.

FIGURE C1-4. This figure and the next collect six shuffled recursive cartoons of multifractal functions. Both figures step back to the symmetric three-segment generator that was used in Figure 1 and is repeated here in the bottom panel. But it is subjected to a deformation different from that used in Figure 3. In the top two panels the junction points between the UP and DOWN and DOWN and UP intervals are positioned farther away than in the bottom panel but on the same horizontal levels. This suffices to create two features that are characteristic of real price data.

FIGURE C1-5. Compared to Figure 4, the difference is that here the positions of the generator's junction points are *not* positioned *farther* from each other than in the Brownian case but *closer*. Result: a different way of achieving large deviations and clustering, less realistic in the case of prices, but bound to be of use elsewhere.

FIGURE C1-6. A phase diagram that provides a broad overall index to my *Selecta* volumes [ ] This caption will have to be beefed-up.

The inescapable need for fractal tools of finance

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Summary. This short paper advances and defends a strong statement concerning financial modeling. It argues that, even when the present fractal models become obsolete, fractal tools are bound to remain central. The reasons are that the main feature of price records is roughness and that the proper language of the theory of roughness in nature and culture is fractal geometry.

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General questions that are often asked need not have immediate answers. Everyone has wondered why – without ever trying explicitly – fractal geometry impinges deeply on so many issues in otherwise very different fields. How to explain that, as early as in the 1960s, my works in natural and social sciences (namely, turbulence, hydrology, and financial prices) could rely on closely analogous tools? Those questions can, at long last, be given the following short answer:

FRACTAL GEOMETRY MEASURES ROUGHNESS INTRINSICALLY.
HENCE IT MARKS THE BEGINNING OF A QUANTITATIVE THEORY
SPECIFIC TO ROUGHNESS IN ALL IT MANIFESTATIONS.

Roughness is ubiquitous in nature and culture (I use the latter term to denote all the works of Man, including financial markets). This is why fractality is also ubiquitous and why fractal geometry will never lack problems to deal with. An *Overview of fractals and multifractals* written in this spirit is featured in [3] in the form of a long Chapter H1.

Despite its ubiquity and antiquity, it is noteworthy that unquestionable that roughness had lagged very far behind other comparable ancient concepts: steepness (of a road or a trend), heaviness, pitch, color, hotness, and the like. The key drawback was that the first intrinsic quantitative measure of roughness had to wait for fractal geometry. Let me elaborate.

The steepness of a smooth incline came, of course, to be defined by the derivative of the height $h(x)$ along the incline. In theory, this definition implies that the increments' ratio dh/dx tends to a limit as dx 0. Custom has made this ratio be viewed as "normal". In practice, it suffices that dh/dx be nearly constant. But – almost by definition – rough profiles and surfaces are such that dh/dx varies all over without limit. A basic feature of many models of price variation – the Bachelier model and my fractal/multifractal models – is that the derivative is not the proper tool. Instead, a limit exists for the highly "anomalous" ratio $\log(dh)/\log(dx)$. According to the Bachelier model, this ratio has the limit $= 1/2$. Being the same at all instants in all financial data is a very important property. It is both a big asset – because of its simplicity, and a big flaw – because a limit equal to 1/2 is not available as parameter to be fitted to the data. To the contrary, the fractal/multifractal models allow $\neq 1/2$.

In inverse historical sequences and decreasing generality, I have originated and investigated three cases. The value of   may vary in some specific way from instant to instant; this characterizes multifractality. The value of   may be the same at all time instants but different from 1/2; this characterizes unifractality or the HHM (Hurst-Hölder-Mandelbrot) model last. There is also a very important intermediate case I call "mesofractality" or the PLM (Pareto-Lévy-Mandelbrot) model. The initials HHM and PLM are motivated in the preceding article.

For the derivative, the intuitive concept of slope long predated mathematics. That is, a quantitative measure of the intuitive notion of steepness came early and the mathematics came late. The concept of   took the opposite path. I devised it for the sake of science by modifying a concept that Hölder introduced in 1870 as being purely mathematical and totally separate from intuition. Over a century later, my work gradually identified it the Hölder exponent with an exponent due to Hurst, and as a key aspect of roughness. Examine, indeed, the various cartoons that illustrate the preceding paper. From one to another, the intuitive, "eyeball," levels of roughness are immediately seen to be different.

To summarize, a key feature of fractal geometry is that it begins by measuring roughness by   or related concepts. Moreover, the value and/or the distribution of   is directly observable. It is not an elusive concept that has to be unscrambled indirectly from many other observations. The predominant role played by this exponent is an aspect of parsimony.

For reasons that can perhaps be guessed, are developed in [2, Chapter E] but cannot be repeated here, the arguments I deployed in the 1960s against the Brownian amounted to the following assertions. Firstly, the value $= 1/2$ characterizes a special "mild" form of roughness. Secondly, the roughness found in financial data takes a "wild" form that excludes $= 1/2$.

While my substitutes for the Brownian have long been resisted, my objections were widely heard and innumerable alternative models reacted by introducing "fixes," each specifically designed to avoid one of the "anomalies" I pioneered, such as discontinuity, divergent moments, and divergent dependence. Examined in the light if the fractal/multifractal approach, the fixes have a common feature: they automatically reset the local roughness to 1/2.

One basic "fix" is variable – stochastic – volatility, namely, the idea that short enough records follow Brownian motion but its variance changes, either continuing by changes observed or every so often. My first criticism is that all the burden of modeling is thereby not eliminated but solely pushed on to the process that rules the variation of volatility. If those variations are fast, the Brownian input is diluted to homeopathic irrelevance.

If the variance changes slowly, a second criticism kicks in: if volatility only measures the scale of Brownian motion, it does not affect , hence this model implicitly assumes $= 1/2$. To my knowledge, this conclusion has not been tested and the evidence I know suggests otherwise.

In any event, my multifractal model is definitely *not* an example of variable volatility model.

On the short run, fixes may be defended as the quickest response to urgent needs. On a longer run, they are unacceptable and they cannot be allowed to multiply forever. When the number of fixes in a recipes exceed a certain number, that recipe collapses under its own weight and the need arises for a new start. Thomas S. Kuhn famously described this process as "a change of paradigm." The paradigm that I introduced and favor as an alternative to the Brownian is the fractal/multifractal model.