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A crisis of intuition as viewed by Felix Klein and Hans Hahn and its resolution by fractal geometry

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The many ironies in the history of fractal geometry begin with the circumstances under which its first tools originally developed. As was well-known, and may have become more widely known through my efforts, several of these tools had been available for nearly a century, but were viewed in mathematics as monsters that had triggered what Hans Hahn has called a *crisis of intuition*.

Outside of mathematics, they were ignored, or worse, namely, specifically labeled *to be ignored*. Thus, when a very official international committee met around 1977 to discuss the mathematics curriculum for engineers, it recommended that continuous functions without a derivative not be mentioned at all; even a footnote would confuse the student. One of the effects of fractal geometry has been to change the meaning of intuition, by expanding its scope and allowing the eye to link this mathematics with concrete work in economics and physics, and even with everyday experience. This has created a new mood, in which notions that used to be viewed as grossly counter-intuitive have become perfectly natural, and may even appear obvious.

Without knowledge of the above historical background, a great deal of the arguments in my old papers may appear to be incomprehensibly contentious. To help the reader recapture the atmosphere of yesterday, this text begins with excerpts from Klein 1897 and Hahn 1933 (widely available in Newman 1966) and continues with comments.

1. Felix Klein on the mathematical character of space-intuition and the relation of pure mathematics to the applied sciences

Presentation. The name of Felix Klein (1849-1925) is naturally associated with *Kleinian groups*. Oddly enough, that concept is due to Henri Poincar, who chose to use this term to settle a dispute that arose when more special groups he had called Fuchsian turned out to be due to Klein.
Klein's creative career was short, but his influence on mathematics was great and mostly levelheaded.

In the present climate of a fin de siécle turning point in mathematics, it is good to allow our thoughts to travel back to the period when 20th century pure mathematics was being invented. On September 2, 1893, Klein was visiting Northwestern University and devoted the sixth of twelve *Lectures on Mathematics* to the topic of space intuition. It may be that every great person is entitled to a bit of raving silliness, for Klein duly obliged, stating his feeling that, it must be said that the degree of exactness of the intuition of space may be different in different individuals, perhaps even in different races. It would seem as if a strong naive space-intuition were an attribute pre-eminently of the Teutonic race, while the critical, purely logical sense is more fully developed in the Latin and Hebrew races. A full investigation of this subject, somewhat on the lines suggested by Francis Galton in his researches on heredity, might be interesting. With this exception, Klein's hundred-year-old piece is well worth rereading.

Excerpts. The inquiry naturally presents itself as to the real nature and limitations of geometrical intuition....[I distinguish] between what I call the *naive* and the *refined* intuition. It is the latter that we find in Euclid: he carefully develops his system on the basis of well-formulated axioms, is fully conscious of the necessity of exact proofs, and so forth....

The naive intuition, on the other hand, was especially active during the period of the genesis of differential and integral calculus. Thus Newton [did not ask] himself whether there might not be continuous functions having no derivative....

At the present time we are living in a *critical* period similar to that of Euclid. It is my private conviction....that Euclid's period must also have been preceded by a *naive* stage of development....

In my opinion, the *naive intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact*.

The first half of this statement [implies that] we do not picture in our mind an abstract mathematical point, but substitute something concrete for it. In imagining a line, we do not picture a *length without breadth*, but a *strip* of a certain width. [Abstractions] in this case are regarded as holding only approximately, or as far as may be necessary....

I maintain that in ordinary life we actually operate with such inexact definitions. Thus we speak without hesitancy of the directions and curvature of a river or a road, although the *line* in this case certainly has considerable width....

As regards the second half of my proposition, there are actually many cases where the conclusions derived by purely logical reasoning from exact definitions cannot be verified by intuition. To show this, I select examples from the theory of automorphic functions, because in more common geometrical illustrations our judgment is warped by the familiarity of the ideas....

Let any number of non-intersecting circles 1, 2, 3, 4,..., be given, and let every circle be reflected (*i.e.* transformed by inversion, or reciprocal radii vectors) upon every other circle;
then repeat this operation again and again, *ad infinitum*.
The question is, what will be the configuration formed by the totality of all the circles, and in particular, what will be the position of the limiting points? There is no difficulty in answering these questions by purely logical reasoning, but the imagination seems to fail utterly when we try to form a mental image of the result....

When the original points of contact happen to lie on a circle being excluded, it can be shown analytically that the continuous curve which is the locus of all the points of contact *is not an analytical curve.* It is easy enough to imagine a *strip* covering all these points, but when the width of the strip is reduced beyond a certain limit, we find undulations, and it seems impossible to clearly picture the final outcome.
Note that we have here an example of a curve with indeterminate derivatives arising out of purely geometrical considerations, while it might be supposed from the usual treatment of such curves that they can only be defined by artificial analytical series....

Kopcke has [concluded] that our space intuition is exact as far as it goes, but so limited as to make it impossible for us to picture curves without tangents....

Pasch believes - and this is the traditional view - that it is in the end possible to discard intuition entirely, basing all of science on axioms alone. This idea of building up science purely on the basis of axioms has since been carried still farther by Peano, in his logical calculus.... I am of the [firm] opinion that, for the purposes of research it is always necessary to combine intuition with the axioms. I do not believe, for instance, that it would have been possible to derive the results discussed in my [previous] lectures, the splendid researches of Lie, the continuity of the shape of algebraic curves and surfaces, or the most general forms of triangles, without the constant use of geometrical intuition....

What has been said above places geometry among the applied sciences. Let me make a few general remarks on these sciences. I should lay particular stress on the *heuristic value* of the applied sciences as an aid to discovering new truths in [pure] mathematics. Thus I have shown (in my little book on Riemann's theories) that the Abelian integrals can best be understood and illustrated by considering electric currents on closed surfaces. In an analogous way, theorems concerning differential equations can be derived from the consideration of sound-vibrations, and so on....

The ordinary mathematical treatment of any applied science substitutes exact axioms for the approximate results of experience, and deduces from these axioms the rigid mathematical conclusions. [But] it must not be forgotten that mathematical developments transcending the limit of exactness of the science are of no practical value.

It follows that a large portion of abstract mathematics remains without any practical application, the amount of mathematics that can be usefully employed in any science being in proportion to the degree of accuracy attained in that science....

As examples of extensive mathematical theories that do not exist for applied science, consider the distinction between the commensurable and the incommensurable.
It seems to me, therefore, that Kirchhoff makes a mistake when he says in his *Spectral Analyse* that absorption takes place only when there is an *exact* coincidence between the wave-lengths. I side with

Stokes, who says that absorption takes place *in the vicinity* of such coincidences....

All this raises the question of whether it would not be possible to create a, let us say, *abridged* system of mathematics adapted to the needs of the applied sciences, without passing through the whole realm of abstract mathematics....[But no such] system...is...in existence, and we must for the present try to make the best of the material at hand.

What I have said here concerning the use of mathematics in the applied sciences [must] not be interpreted as in any way prejudicial to the cultivation of abstract mathematics as a pure science. Apart from the fact that pure mathematics cannot be supplanted by anything else as a means for developing the purely logical powers of the mind, there must be considered here as elsewhere the necessity of the presence of a few individuals in each country developed in a far higher degree than the rest. Even a slight raising of the general level can be accomplished only when some few minds have progressed far ahead of the average....

Here a practical difficulty presents itself in the teaching, let us say, the elements of the calculus. The teacher is confronted with the problem of harmonizing two opposite and almost contradictory requirements. On the one hand, he has to consider the limited and as yet undeveloped intellectual grasp of his students and the fact that most of them study mathematics mainly with a view to the practical applications; on the other, his conscientiousness as a teacher and man of science would seem to compel him to detract in nowise from perfect mathematical rigor, and therefore to introduce from the beginning all the refinements and niceties of modern abstract mathematics. In recent years, university instruction, at least in Europe, has been tending more and more in the latter direction. [If a work like] *Cours d'analyse* of Camille Jordan is placed in the hands of a beginner a large part of the subject will remain unintelligible, and at a later stage, the student will not have gained the power of making use of the principles in the simple cases occurring in the applied sciences....

It is my opinion that in teaching it is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard to the applications, and to refer to the refinements only gradually as the student becomes able to understand them. This is, of course, nothing but a universal pedagogical principle to be observed in all mathematical instruction....

I am led to these remarks by the consciousness of growing danger in Germany of a separation between abstract mathematical science and its scientific and technical applications. Such separation can only be deplored, for it would necessarily be followed by shallowness on the side of the applied sciences, and by isolation on the part of pure mathematics....

2. Hans Hahn on *the crisis in intuition*

Presentation. This second set of quotes is in sharp contrast with the first. Hans Hahn (1879-1934), best known for the Hahn-Banach theorem never wielded Klein's influence, but the ideas expressed in the following quotes were widely shared and very influential. The text from which we are excerpting (with some minimal reshuffling for the sake of concision) originated in lectures given in Vienna in the 1920's for the purpose of bringing recent scientific advances to a wider public. This text became widely known to the English reading world when it was translated in Newman 1956.

Several shorter passages are quoted in *FGN*, where they are sharply criticized. The goal here is to broaden this criticism from Hahn to Immanuel Kant (1724-1804). This may seem an idle effort, since few present-day scientists know Kant's name, and even fewer are aware of his work. But in fact, some of Kant's ideas have become part of today's unattributed general knowledge. More importantly, Kant's ideas used to be widely known among scientists, early in this century, and their *reaction* to these views has contributed to a widely held view of the meaning of intuition. This view, in my opinion, is both wrong and harmful, and fractal geometry may take pride in having helped break its hold.

Excerpts. Immanuel Kant, in his *Critique of Pure Reason*, [has asserted that]..we conduct ourselves passively when we receive impressions through intuition and actively when we deal with them in our thought. Furthermore, according to Kant, we must distinguish between two ingredients of intuition. One...arises from experience... such as colors, sounds, smells, hardness, softness, roughness, etc. The other is a pure a *priori* part independent of all experience...: [Kant believed that] geometry, as it has been taught since ancient times, deals with the properties of the space that is fully and exactly presented to us by pure intuition....

However plausible these ideas may at first seem, and however well they corresponded to the state of science in Kant's day, their foundations have been shaken by the course that science has taken since then....

[These quotes] narrow the subject to geometry and intuition, and attempt to show how it came about that, even in the branch of mathematics which would seem to be its original domain, intuition gradually fell into disrepute and at last was completely banished....

One of the outstanding events in this development was the discovery [by Weierstrass of] curves that possess no tangent at any point. [That is,]... it is possible to imagine a point moving in such a manner that at no instant does it have a definite velocity. [This] directly affects the foundations of differential calculus as developed by Newton (who started with the concept of velocity) and Leibniz (who started the so-called tangent problem).... The standard curves that have been studied since early times: circles, ellipses, hyperbolas, parabolas, cycloids, etc. [have tangents everywhere. However,] the graph of the function *t cos (1 / t)* demonstrates that a curve does not have to have a tangent at every point. It used to be thought that intuition forced us to acknowledge that such a deficiency could occur only at isolated and exceptional points of a curve [and] that a curve must possess an exact slope, or tangent, at an overwhelming majority of points [The Weierstrass function goes beyond that. By replacing lines with saw-tooth curves, one obtains a simplified variant, the Takagi function...] Its character entirely eludes intuition: indeed, after a few repetitions of the segmenting process, the evolving figure has grown so intricate that intuition can scarcely follow, and it forsakes us completely as regards the curve that is approached as a limit. The fact is that only logical analysis can pursue this strange object to its final form. Thus, had we relied on intuition in this instance, we would have remained in error, for intuition seems to force the conclusion that there cannot be curves lacking a tangent at any point.... [To avoid such advanced] branches of mathematics, I propose to examine an occurrence of failure of intuition at the very threshold of geometry. Everyone believes that...curves are geometric figures generated by the motion of a point. But...Peano... proved that the geometric figures that can be generated by a moving point also include entire plane surfaces. For instance, it is possible to imagine a point moving in such a way that in a finite time it will pass through all the points of a square and yet no one would consider the entire area of a square as simply a curve....This motion cannot possibly be grasped by intuition; it can only be understood by logical analysis.

[For] a second example of the undependability of intuition even as regards very elementary geometrical questions, think of a map showing three countries.

Intuition seems to indicate that corners at which all three countries come together...can occur only at isolated points, and that at the great majority of boundary points on the map only two countries will be in contact. Yet Brouwer showed how a map can be divided into three countries in such a way that at every boundary point all three countries will touch one another....

Intuition cannot comprehend this pattern, although logical analysis requires us to accept it. Once more intuition has led us astray.

Intuition seems to indicate that it is impossible for a curve to be made up of nothing but end points or of branch points. This intuitive conviction as regards branch points was refuted [when] Sierpinski proved that there are curves *all of whose points are branch points*....

Because intuition turned out to be deceptive in so many instances, and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more skeptical of the validity of intuition. They learned that it is unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus, a demand arose for the expulsion of intuition from mathematical reasoning, and for the complete formalization of mathematics. That is to say, every new mathematical concept was to be introduced through a purely logical definition; every mathematical proof was to be carried through strictly by logical means. The task of completely formalizing mathematics, of reducing it entirely to logic, was arduous and difficult; it meant nothing less than a reform in root and branch....

Let us now summarize. Again and again we have found that, even in simple and elementary geometric questions, intuition is a wholly unreliable guide. It is impossible to permit so unreliable an aid to serve as the starting point or basis of a mathematical discipline....

But what are we to say to the often heard objection that only conventional geometry is usable, for it is the only one that satisfies intuition? My first comment on this score...is that *every* geometry...is a logical construct. Traditional physics is responsible for the fact that until recently the logical construction of three-dimensional Euclidean, Archimedean space has been used exclusively for the ordering of our experience. For several centuries, almost up to the present day, it served this purpose admirably; thus we grew used to operating with it. This habituation to the use of ordinary geometry for the ordering of our experience explains why we regard this geometry as intuitive, and every departure from it unintuitive, contrary to intuition, and intuitively impossible. But as we have seen, such intuitional impossibilities, also occur in ordinary geometry. They appear as soon as we no longer restrict ourselves to the geometrical entities with which we have long been familiar, but instead reflect upon objects that we had not thought about before....

The theory that the earth is a sphere was also once an affront to intuition. However, we have got used to the idea, and today it no longer occurs to anyone to pronounce it impossible because it conflicts with intuition.

If the use of [new] geometries for the ordering of our experience continues to prove itself so that we become more and more accustomed to dealing with these logical constructs; if they penetrate into the curriculum of the schools, if we, so to speak, learn them at our mother's knee, as we now learn three-dimensional Euclidean geometry • then nobody will think of saying that these geometries are contrary to intuition. They will be considered as deserving of intuitive status as three-dimensional Euclidean geometry is today. For it is not true, as Kant urged, that intuition is a pure *a priori* means of knowledge, but rather that it is force of habit rooted in psychological inertia.

Comments on the views of Klein and Hahn

My reaction to Klein's 1897 lecture is to marvel at how sensible it was in its time, and how up-to-date it remains today. History tells us that during the period of Klein's greatest political influence, his University of G•ttingen was blessed with a most unusual balance between all the diverse kinds of mathematics. Klein's 1897 lecture helps explain why.

Hahn's shrill manifesto is an altogether different story, a foretaste of horrors to come. To describe Hahn's text, the editor of Newman 1956 states that he discusses the cherished faculty of intuition, its role in mathematics, the nest of paradoxes it got us into, and how successful we have been in crawling out. My reaction is very different: Fractal geometry demonstrates that Hahn was dead wrong. Intuition is not invariable but can and must be trained to perform new tasks. But discontinuous change of character was needed to BUG. Hahn draws a mistaken diagnosis, and suggests a treatment that has indeed been tried, and has proved to be lethal.

It is both funny and pathetic to read that intuition tells man that curves have tangents. My own recollection, together with unsystematic and limited tests, suggests the precise contrary. Before the mathematicians' efforts, intuition must have been based on the shapes of coastlines or of tree bark. This would lead to the conclusion that curves have no tangent. It is the notion of tangent that has to be learned and then made intuitive. Paul Lévy (see *FGN*, p.36) had anticipated me in this view. Didn't Hahn ever have to teach the notion of tangent to a fresh mind? Did he even ask himself what the house of mathematics gains by refusing to open its doors to those who, when introduced to differentials, somehow fail to immediately accept them as *intuitive*?

Hahn's errors were in large part already present in Kant, who distinguished between an empirical and a posteriori intuition arising from experience *such as color, sounds, smells and sensations of touch*, and a pure a priori intuition independent of all experience. Presumably, the pure intuition would have been present at birth. It used to be that the validity of such distinctions was decided by philosophic introspection by the likes of Kant or Hahn. Today, however, popular interest in fractals has led to a spontaneous, very large scale psychological experiment. One can watch children, students, and grown-ups, including many who thought they understood nothing about math, use computers to play with Peano or Sierpinski curves and other fractals. All witnesses recognize that their subjects' initial ignorance does not suggest a fully developed innate ability. These subjects' increasingly secure intuition of these shapes grows visibly in response to concrete manipulation of visible objects. No one can assert that these learners sit on anyone's knee, gradually impregnating themselves with the results of logical analysis.

I must hasten to disclaim any understanding of the workings of intuition and any burning interest in the issue of inborn or innate abilities versus acquired ones. My interest is limited to the *crisis of intuition* and to the justification it had provided for the excesses of *pure mathematics*. It suffices to see this justification evaporate upon examination. Self-appointed devil's advocates have criticized my jaundiced interpretations of the crisis of intuition as being self-serving and blind to history. Their rejoinder is that, even if my own intuition, and the intuition of all of those who now play on the computer, may be the daughter of experience, Weierstrass, Cantor, Peano, Browner and Sierpinski did not respond to intuitive, but to logical needs. I think that this rejoinder is based entirely on supposition and is devoid of empirical evidence. In *FGN*, p 407, I have argued that a different work by the same Immanuel Kant in effect states that the clustering of (conjectural) galaxies is ruled by a set which is almost the set that was (much later) to come to the mind of H.J.S. Smith, Vito Volterra and then Cantor (who gained claim to it by making it the object of an interesting observation). These men lived at a time when a person of culture was fully aware of Kant and had many reasons to refer to him. When our heroes require a response to a logical need, would they have found one in a philosopher's musings, or directly in the works of Mother Nature?