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Fractal dimension, dispersion, and singularities of fluid motion

✦ **Abstract.** It is conjectured that turbulent dispersion in a closed vessel involves surfaces whose fractal dimension exceeds 2. The different singularities and quasi-singularities of the motion are carried by a hierarchy of sets whose dimensions are fractions. The quasi-singularities are viewed as being singularities of the Euler equations, after they have been smoothed by viscosity. ✦

1. Introduction

M 1975o{N18} showed that "fractal" sets, whose main feature is a fractional Hausdorff dimension, play a role in numerous branches of science, in particular the study of turbulence. For example, taking the Gaussian approximation to homogeneous turbulence and the Kolmogorov and Burgers velocity spectra, M 1975o shows that the iso-surfaces of passive scalars have dimensions $3 - 1/3$ and $3 - 1/2$.

On the basis of intuitive considerations and experimental measurements, the present Note states two kinds of conjectures, both related to more fundamental but less developed aspects of turbulence. The first conjecture concerns the geometry of turbulent dispersion. Next, generalizing from the models of intermittency described in M 1974f{N15}, I conjecture that each of the multiple aspects of the notion of "turbulence" corresponds to either a singularity or a "quasi- singularity" of the equations of motion, and that this singularity is carried by a space-time set whose dimension is in general a fraction.

Scheffer 1976 shows how he has succeeded in evaluating two of these dimensions, starting from the Navier-Stokes equations.

2. Dispersion.

It is widely accepted that, when turbulence acts on a material line, the length of this line increases exponentially in time. On the other hand, the radius of the smallest sphere containing one of these lines grows only slowly, or even remains bounded when the vessel is closed. Hence, the line must increasingly curl up on itself. The same is true of material surfaces. Let us first consider the effect of a Richardsonian eddy as it cascades in self-similar fashion towards higher and higher frequencies, before it dissipates. Within a critical zone of eddy intensity, one observes that a regular blob of passive contaminant transforms into a kind of octopus, then each arm subdivides into branches, then (repeatedly) into sub-branches, down to the threshold of dissipation. Experimental diagrams (see Corrsin 1959b) remind us of the first steps of Peano's construction of a plane-filling curve (a section of a space-filling surface). Viscosity and molecular diffusion have a regulating character, but in their absence the process in question would imply the following: that if the contaminant and the solvent asymptotically mix in a uniform fashion, the separating surface tends towards a fractal surface with dimension 3.

I conjecture that this picture is indeed applicable if the initial eddy is very strong. To the contrary, if the eddy is weak, the mixture will not be perfect; I conjecture that if the threshold results from an admixture of eddies with Kolmogorov velocity spectrum (like the iso-surfaces in M 1975f), the fractal dimension would be $3 - 1/3$. This last result also holds in the case of an infinite vessel. No conjecture could yet be formulated for the transition from $D = 3 - 1/3$ to $D = 3$ as the force of the initial eddy is increased.

3. Singularities and quasi-singularities

Inspired by Oseen's view of turbulence, Leray 1934 has investigated the singularities of the Navier-Stokes equations, defined as points in $R^3 \times R^+$ where the local dissipation rate is infinite. Another view of turbulence, distinct historically and (no doubt) also logically, leads to the homogeneous turbulence of G. I. Taylor, which implies that the distribution of dissipation is statistically uniform. However, real dissipation is "intermittent," as noted by Batchelor and Townsend and investigated by Kolmogorov 1962 and Obukhov 1962. It may be defined on linear sections of the flow...along the X-axis. The values of the velocity $u(x)$ oscillate moderately around their mean value. But the measurements of $(u/ x)^2$, which is viewed as an approximation of the turbulent dissipation, can take (at irregular intervals) values that are far from the norm, while finite (if only because a theoretically infinite peak would be smoothed out by the process of measurement).

Having analyzed diverse stochastic *ad-hoc* models of this phenomenon (see M 1974f{N15}), I have made a conjecture which was immediately proven in J. Peyrière 1974: that dissipation concentrates on a random set, a variant of either the Cantor set, or the sets I have named after Besicovitch. (An example is the set of points whose decimal representation contains the integers from 0 to 9 with positive and un-equal frequencies.) More precisely, the set in question is assumed to be a finite approximation of a fractal set, smoothed by viscosity. While dimension is an asymptotic concept, yet it remains useful to measure the degree of irregularity in the self-similar zone that is present in those models. An effect of viscosity is that there are no real singularities involved, only what may be called quasi-singularities. Other empirical data lead to suspect that higher derivatives of u have different, thinner, sets of intermittency.

Extrapolating from the models in M 1974f{N15}, I am led to believe that intermittency and the accompanying role of fractal sets constitute the most distinct characteristic of turbulence, and should be placed in the center of its study.

More precisely, I make two groups of conjectures.

- Turbulence in a fluid is the result of a range of different phenomena, each of which is concentrated, *either* on a set with Hausdorff dimension less than 3 (in space) and less than 4 (in space-time), *or* on such a set after it has been subjected to an inner cutoff related to viscosity.

- The most apparent of those phenomena is related to the presence of quasi-singularities, which are smoothed out singularities of the Euler equations.

&&&&&&&&&&&& ANNOTATIONS &&&&&&&&&&&&

The French original was presented to the Académie des Sciences on June 23, 1975 by Jean Leray.