Probab. Theory Relat. Fields 124, 409–430 (2002)
Digital Object Identifier (DOI) 10.1007/s004400200220

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Multifractal products of cylindrical pulses

First received: 14 September 1999 / Resubmited: 27 June 2001 /
Revised version: 30 May 2002 /
Published online: 30 September 2002 –

Abstract. New multiplicative and statistically self-similar measures μ are defined on $\mathbb{R}$ as limits of measure-valued martingales. Those martingales are constructed by multiplying random functions attached to the points of a statistically self-similar Poisson point process defined in a strip of the plane. Several fundamental problems are solved, including the non-degeneracy and the multifractal analysis of μ. On a bounded interval, the positive and negative moments of $\|\mu\|$ diverge under broad conditions.

1. Introduction

This paper deals with a new class of random multifractal measures introduced in [Ma6], to be called "multifractal products of cylindrical pulses" (MPCP). They improve on the familiar "canonical cascade multifractals" (CCM) introduced in [Ma3, Ma4].

As will be recalled, the construction of CCM involves a prescribed artificial b-adic grid of intervals of [0, 1]. The basis b (integer ≥ 2) was introduced to simplify the construction in [Ma1] and allow the conjectures in [Ma3, Ma4] to be proven [KP]. This b-ary tree structure restricts the statistical self-similarity of CCM to b-adic subintervals of [0, 1]. The CCM led to a considerable body of literature (see [K2], [HoWa], [Mol], [B1], [B2] and references therein for extensions).

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be the probability space on which random variables are defined in this paper. To construct CCM, let W be a non-negative random variable having expectation 1, and let W_v, $v \in \cup_{n=1}^{\infty}\{0, \ldots, b-1\}^n$, be a collection of random variables i.i.d. with W. Consider the sequence of random measures μ_n, $n \geq 1$, on [0, 1], defined by

$$\frac{d\mu_n}{d\ell}(t) = \prod_{j=1}^{n} W_{(t_1,\ldots,t_j)},$$

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Mathematics Subject Classification (2000): 28A80, 60G18, 60G44, 60G55, 60G57

Key words or phrases: Random measures – Multifractal analysis – Continuous time martingales – Statistically self-similar Poisson point processes

where $t = (t_1, t_2, \dots)$, $t_j \in \{0, \dots, b-1\}$, is a b-ary expansion of $t \in [0,1]$, and ℓ denotes the Lebesgue measure. The CCM μ is the almost sure (a.s.) vague limit of $(\mu_n)_{n\geq 1}$ (see [KP]).

The mass that μ assigns to the subinterval $[\sum_{j=1}^n t_j b^{-j}, \sum_{j=1}^n t_j b^{-j} + b^{-n}]$ is a product of two statistically independent factors: $b^{-n}\prod_{j=1}^n W_{(t_1,\dots,t_j)}$, and a random variable $Y_\infty(t_1,\dots,t_n)$ that is distributed as the total mass $\mu([0,1])$ (this reflects the self-similarity).

The MPCP provide a continuous parameter extension of CCM. To relate CCM and MPCP, the basic subintervals of the form $[kb^{-j}, (k+1)b^{-j}]$, $k \in \{0, 1, \dots, b^j - 1\}$, should first be reparametrized as $[s-\lambda, s+\lambda]$, where the location and scale parameters s and λ are $s_{k,j} = \dfrac{k+1/2}{b^j}$ and $\lambda_{k,j} = \dfrac{1}{2b^j}$. This notation restates the density of μ_n as a product of random quantities associated, down to a "resolution" $\varepsilon_n = (2b^n)^{-1}$, with the atoms of the "deterministic point process" $S = \{(s_{k,j}, \lambda_{k,j}) : \ k = 0, \dots, b^j - 1, \ j = 1, 2, \dots\}$. Specifically, for $(s,\lambda) \in S$ with $2\lambda = b^{-j}$, one defines the "cylindrical pulse" $P_{(s,\lambda)}$ by

$$t \in \mathbb{R} \mapsto P_{(s,\lambda)}(t) = \begin{cases} W_{(t_1,\dots,t_j)} & \text{if } t \in [s-\lambda, s+\lambda]; \\ 1 & \text{otherwise.} \end{cases}$$

Then μ is the a.s. vague limit (as $\varepsilon \to 0^+$) of the family of measures μ_ε given by

$$\frac{d\mu_\varepsilon}{d\ell}(t) = \prod_{(s,\lambda)\in S,\ \lambda\geq\varepsilon} P_{(s,\lambda)}(t), \ \text{ with } \mu_\varepsilon = \mu_n \text{ if } \varepsilon \in]\varepsilon_{n+1}, \varepsilon_n].$$

Note that for a given $t \in [0,1]$, the number of (non-unit) factors in the previous product is the number of points in S "under" t and is equivalent to $(\log\, b)^{-1} \log\, 1/\varepsilon$. The factor $1/\log\, b$ can be viewed as a formal density for the point process S.

The step from this framework to MPCP consists in replacing the point process S by a Poisson point process $S = \{(s_j, \lambda_j)\}$ on $\mathbb{R} \times (0,1]$, with intensity

$$\Lambda(dt\, d\lambda) = \frac{\delta}{2}\frac{dt\, d\lambda}{\lambda^2} \ \ (\delta > 0).$$

The "cylindrical pulses" associated with S are a denumerable family of functions $P_j(t)$, such that each P_j is identically 1 outside the interval $[s_j - \lambda_j, s_j + \lambda_j]$, and identically equal to a weight W_j within $[s_j - \lambda_j, s_j + \lambda_j]$, so that the W_j's are i.i.d. with W, and independent of S.

The MPCP μ is the a.s. vague limit (as $\varepsilon \to 0^+$) of the family of measures μ_ε defined on $\mathbb{R}$ by

$$\frac{d\mu_\varepsilon}{d\ell}(t) = \prod_{(s_j,\lambda_j)\in S,\ \lambda_j\geq\varepsilon} P_j(t).$$

For every $t \in \mathbb{R}$, the expected number of (non-unit) factors in the previous product is $\delta \log(1/\varepsilon)$. The CCM formal density $1/\log\, b$ is now formally replaced by the MPCP density δ.

The first key virtue of the MPCP's follows from the invariance properties of Λ: these measures are statistically invariant under a continuous change of scale. They involve no b-adic grid. Neither do the limit lognormal multifractals introduced in [Ma1], nor the "fractal sums of pulses" in [Ma5], which inspired the present study.

A second key virtue concerns a deep change in the form of the familiar multifractal function $\tau(q)$. For MPCP, the next sections will show that when $W > 0$

$$\tau(q) = -1 + q - \delta(\mathbb{E}(W^q) - 1).$$

For CCM, it is well known that

$$\tau(q) = -1 + q - \log_b \mathbb{E}(W^q).$$

The condition of divergence of high moments of μ continues to be that $\tau(q) < 0$ for some $q > 1$. The restriction $\tau(q) < 0$ imposes on W is clearly less for MPCP than for CCM.

Section 2 tightens up the construction of the MPCP μ. When $\mathbb{E}(W) \neq 1$, the natural normalization of the products of the pulses is formed, to give the density

$$\frac{d\mu_\varepsilon}{d\ell}(t) = \varepsilon^{\delta(\mathbb{E}(W)-1)} \prod_{(s_j,\lambda_j)\in S,\ \lambda_j \geq \varepsilon} P_j(t).$$

Then the main results are stated and a self-similar property is described. Theorem 1 concerns the conditions under which μ is non-degenerate, i.e., positive with positive probability. Theorems 2 and 3 concern the existence of finite moments for pieces of μ. Theorem 4 concerns the whole multifractal spectrum. Section 3 is devoted to proofs of these theorems.

This paper incorporates, proves and much strengthens the conjectures in [Ma6]. In the absence of a grid, the geometrical properties of MPCP are subtler than those of CCM, and serious mathematical complications arise. The reason why [Ma3, Ma4] singled out CCM for study is that for CCM the mass $\mu([0, 1]) = Y_\infty$ satisfies the now-classical functional equation

$$(\mathcal{E}):\quad Y_\infty = b^{-1} \sum_{j=0}^{b-1} W_{(j)} Y_\infty(j),$$

where the $Y_\infty(j)$ are copies of Y_∞, and these random variables are mutually independent and independent of the $W_{(j)}$. By construction, $b^{-1}W_{(j)}Y_\infty(j) = \mu([jb^{-1}, (j + 1)b^{-1}])$ for each $0 \leq j \leq b - 1$. The properties of μ are controlled by $(\mathcal{E})$ itself or its iterations. For a MPCP, Sect. 2.3 replaces $(\mathcal{E})$ with the far more difficult Eq. (3). The geometry of the Poisson point process S implies that (3) no longer involves random variables having the same distribution as Y_∞. While copies exist, they are implicit in integral terms (by Theorem 5). Moreover, the copies that concern intervals close to one another are correlated. Nevertheless, several non-obvious reductions make it possible to adapt for MPCP some features of the familiar approach developed for CCM.

Products of more general pulses are discussed in [BM].

2. Definitions, results and self-similarity

2.1. Construction of the limit measure and main results

Let W be a positive integrable random variable and denote $\mathbb{E}(W)$ by V.

Let $\{B_k\}_{k\geq 1}$ be a partition of $\mathbb{R}\times]0,1]$ such that for all $k\geq 1$, $0<\Lambda(B_k)<\infty$. For every $k\geq 1$, let $\Lambda_{|B_k}$ denote the restriction of Λ to B_k and choose a sequence $(M_{k,n})_{n\geq 1}$ of B_k-valued random variables with common distribution $\frac{\Lambda_{|B_k}}{\Lambda(B_k)}$; denote by N_k a Poisson variable with parameter $\Lambda(B_k)$, and $(W_{k,n})_{n\geq 1}$ a sequence of copies of W.

Assume that all the random variables $M_{k,n}$, N_k and $W_{k,n}$, $k,n\geq 1$, are mutually independent.

$S=\{M_{k,n};\ 1\leq k,\ 1\leq n\leq N_k\}$ is a Poisson point process with intensity Λ. For $M=(t_M,\lambda_M)=M_{k,n}\in S$, define $W_M=W_{k,n}$, $I_M=[t_M-\lambda_M, t_M+\lambda_M]$, and the *cylindrical pulse* $P_M:\ t\in\mathbb{R}\mapsto W_M\,\mathbf{1}_{I_M}(t)+\mathbf{1}_{I_M^c}(t)$.

For all $\varepsilon\in]0,1]$ and $t\in\mathbb{R}$, define the truncated cone $C_\varepsilon(t)=\{(t',\lambda)\in\mathbb{R}\times]0,1];\ t-\lambda\leq t'\leq t+\lambda,\ \varepsilon\leq\lambda<1\}$ and

$$Q_{C_\varepsilon(t)}=\prod_{M\in S\cap C_\varepsilon(t)} W_M.$$

For every $0<\varepsilon\leq 1$, denote by μ_ε the measure on $\mathbb{R}$ defined by

$$\frac{d\mu_\varepsilon}{d\ell}(t):=Q_\varepsilon(t)=\varepsilon^{\delta(V-1)}\prod_{M\in S\cap\{\lambda\geq\varepsilon\}}P_M(t)=\varepsilon^{\delta(V-1)}Q_{C_\varepsilon(t)}$$

and define $F_\varepsilon=\sigma\,(M, W_M,\ M\in S\cap\{\lambda\geq\varepsilon\})$. In all the text, weak convergence of measures on a locally compact Hausdorff set K means weak*-convergence in the dual of $C(K)$, the space of real continuous functions on K.

The limit measure. By construction, for every $t\in\mathbb{R}$, $(Q_{1/s}(t))_{s\geq 1}$ is a positive right-continuous martingale with respect to $(F_{1/s})_{s\geq 1}$, with expectation 1. Therefore Kahane's theory of T-martingales ([K1]) is applicable. That is, for every $n\in\mathbb{Z}$ and with probability one, the restrictions of the measures μ_ε to the compact $[n,n+1]$ converge weakly, as $\varepsilon\to 0$, to a non-negative measure $\mu^{(n)}$ on $[n,n+1]$. It also follows that the endpoints n and $n+1$ are not atoms of $\mu^{(n)}$.

Consequently (with probability one) there exists a unique non-negative measure $\mu^{\mathbb{R}}$ on $\mathbb{R}$ whose restriction to $[n,n+1]$ is $\mu^{(n)}$ for every $n\in\mathbb{Z}$.

By definition of Λ, the measure $\mu^{\mathbb{R}}$ is statistically invariant by horizontal translations. The sequel will only consider the measure $\mu=\mu^{(1)}$.

Remark 1. The choice of $(B_k, N_k, (M_{k,n})_{n\geq 1})_{k\geq 1}$ and $((W_{k,n})_{n\geq 1})_{k\geq 1}$ affects neither the probability distribution of the stochastic process $(Q_\varepsilon(t))_{\varepsilon\in]0,1],\ t\in\mathbb{R}}$, nor those of the other random variables defined in this paper.

The function $\tau(q)$. Recall that μ denotes the restriction of $\mu^{\mathbb{R}}$ to $[0,1]$. Define $Y=\|\mu\|$. For $q\in\mathbb{R}$ define

$$\tau(q)=-1+q\,(1+\delta(V-1))-\delta(\mathbb{E}(W^q)-1)\in\mathbb{R}\cup\{-\infty\}.$$

Thus τ is concave and finite on $[0, 1]$.

Non-degeneracy of μ and the moments of $\|\mu\|$.

Theorem 1 (Non-degeneracy). *(i) If $\tau'(1^-) > 0$ then $\mathbb{P}(\mu \neq 0) = 1$ and $\mathbb{E}(Y) = 1$. (ii) If $\mathbb{P}(\mu \neq 0) > 0$ then $\mathbb{P}(\mu \neq 0) = 1$, $\mathbb{E}(Y) = 1$, and $\tau'(1^-) \geq 0$. If, moreover, $\mathbb{E}((1 + W)|\log W|^{2+\gamma}) < \infty$ for some $\gamma > 0$, then $\tau'(1^-) > 0$.*

Theorem 2 (Moments of positive orders). *Let $h > 1$.*
(i) If $\tau(h) > 0$ then $0 < \mathbb{E}(Y^h) < \infty$. (ii) If $0 < \mathbb{E}(Y^h) < \infty$ then $\tau(h) \geq 0$.

Remark 2. [Ma6] conjectures that μ is non-degenerate if and only if $\tau'(1^-) > 0$, and that if μ is non-degenerate then for $h > 1$, $\mathbb{E}(Y^h) < \infty$ if and only if $\tau(h) > 0$.

The necessary and sufficient conditions for non-degeneracy and finiteness of moments of positive orders are similar for MPCP and CCM, but, by design, are less restrictive for MPCP. The following proposition characterizes the divergence of high moments.

Proposition 1 (Divergence of high moments for MPCP and CCM). *(i) Assume that μ is non-degenerate. There exists $h > 1$ such that $\mathbb{E}(Y^h) = \infty$ if and only if $\mathbb{P}(W > 1) > 0$ (this is independent of δ) or $\mathbb{P}(W \leq 1) = 1$ and $\mathbb{E}(W) < 1 - 1/\delta$.*
(ii) Assume that $\mathbb{E}(W) = 1$ and the CCM constructed with W is non-degenerate. There exists $h > 1$ such that $\mathbb{E}(Y_\infty^h) = \infty$ if and only if $\mathbb{P}(W > b) > 0$ or $\mathbb{P}(W = b) \geq 1/b$.

Theorem 3 (Moments of negative orders). *Assume Y is non degenerate and fix $a > 0$. Then $\mathbb{E}(Y^{-a}) < \infty$ holds if and only if $\mathbb{E}(W^{-a}) < \infty$.*

Multifractal analysis of μ. New definitions are needed.

For a function $f : \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty\}$, define $f^* : \alpha \in \mathbb{R} \mapsto \inf_{q \in \mathbb{R}}(\alpha q - f(q))$.

For $t \in [0, 1]$ and $r > 0$, denote $[0, 1] \cap [t - \frac{r}{2}, t + \frac{r}{2}]$ by $I_r(t)$, and for $\alpha > 0$ define

$$E_\alpha = \{t \in [0, 1];\ \lim_{r \to 0^+} \frac{\log\ \mu(I_r(t))}{\log\ r} = \alpha\}.$$

The multifractal analysis of μ computes the mapping $\alpha \mapsto \dim_H\ E_\alpha$ on an interval as large as possible, where $\dim_H$ stands for the Hausdorff dimension. Since the geometry of μ does not depend on a particular b-ary tree, the logarithmic density in the definition of the E_α's is not expressed via b-adic intervals as for CCM, but via centered intervals.

Theorem 4 (Multifractal analysis). *Assume that τ is finite on an interval J containing a neighborhood of $[0, 1]$, and that $\tau'(1) > 0$. Define $J' = \{q \in \mathrm{Int}(J);\ \tau'(q)q - \tau(q) > 0\}$, $I' = \{\tau'(q);\ q \in J'\}$, $\alpha_{\inf} = \inf(I')$ and $\alpha_{\sup} = \sup(I')$ ($[0, 1] \subset J'$, $I' \subset]0, \infty[$, $\alpha_{\inf} > 0$). With probability one:*
(i) For all $\alpha \in I'$, $\dim_H\ E_\alpha = \tau^*(\alpha)$.
(ii) If $\tau^*(\alpha_{\inf}) = 0$ then for all $\alpha \in]0, \alpha_{\inf}[$, $E_\alpha = \emptyset$. If $\alpha_{\sup} < \infty$ and $\tau^*(\alpha_{\sup}) = 0$ then for all $\alpha \in]\alpha_{\sup}, \infty[$, $E_\alpha = \emptyset$.

2.2. Additional definitions and a principle of self-similarity

$X \sim X'$ means that the two random variables X and X' are identically distributed.

If B is a Borel subset of $H = \mathbb{R}\times]0,1]$ with $\Lambda(B) < \infty$, define

$$Q_B = \prod_{M \in S \cap B} W_M.$$

If I is a compact subinterval of $[0,1]$, then $|I|$ stands for its length and we define

$$\begin{aligned} T_I &= \{(t,\lambda) \in H;\ 0 < \lambda < |I|,\ \inf(I) - \lambda \le t \le \sup(I) + \lambda\}, \\ T^I &= \{(t,\lambda) \in H;\ |I| \le \lambda < 1,\ t \in [\sup(I) - \lambda, \inf(I) + \lambda]\}, \\ B^I_\gamma &= \{(t,\lambda) \in H;\ |I| \le \lambda < 1,\ t \in [\inf(I) + \gamma\lambda, \sup(I) + \gamma\lambda]\},\ \gamma \in \{-1,1\}, \\ B^I &= B^I_{-1} \cup B^I_1. \end{aligned}$$

Moreover, f_I the affine transformation on $\mathbb{R}$ which maps $\inf(I)$ onto 0 and $\sup(I)$ onto 1.

Then for all $0 < \varepsilon \le 1$ define μ^I_ε as the measure determined on I by

$$\frac{d\mu^I_\varepsilon}{d\ell}(t) = \varepsilon^{\delta(V-1)} \prod_{M \in S \cap \{\varepsilon|I| \le \lambda < |I|\}} P_M(t) = \varepsilon^{\delta(V-1)}\, Q_{C_{\varepsilon|I|}(t) \setminus C_{|I|}(t)}.$$

Theorem 5 examines the strong similarity between the μ^I_ε's and the μ_ε's (see Sect. 2.1).

Theorem 5. *For every non-trivial compact subinterval I of* $[0,1]$ *one has almost surely for all* $0 < \varepsilon \le |I|$

$$\mu_\varepsilon(I) = |I|^{\delta(V-1)} \int_I Q_{C_{|I|}(t)} \mu^I_{\varepsilon/|I|}(dt) = |I|^{\delta(V-1)} Q_{T^I} \int_I Q_{B^I \cap C_{|I|}(t)}\ \mu^I_{\varepsilon/|I|}(dt).$$

Here Q_{T^I} and $t \mapsto Q_{B^I \cap C_{|I|}(t)}$ are independent of one another and of the μ^I_ε's, and, as $\varepsilon \to 0$, the family $(\mu^I_\varepsilon)_{0<\varepsilon\le 1}$ converges a.s. weakly to a measure μ^I.

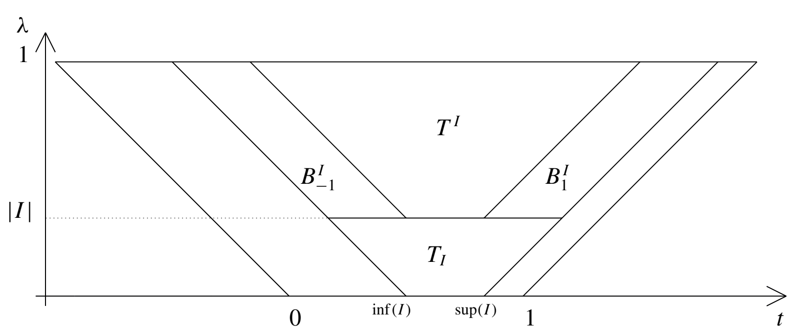

Fig. 1. Illustration of the sets in H defined early

Moreover, the following properties hold for all $f \in C(I)$*:*
(i) $\int_I f(t)\,\mu_\varepsilon^I(dt) \sim |I| \int_{[0,1]} f \circ f_I^{-1}(t)\,\mu_\varepsilon(dt)$ *for all* $\varepsilon \in]0,1]$*. In particular* $\|\mu_\varepsilon^I\| \sim |I|\,\|\mu_\varepsilon\|$*.* (ii) $\int_I f(t)\,\mu^I(dt) \sim |I| \int_{[0,1]} f \circ f_I^{-1}(t)\,\mu(dt)$*. In particular* $\|\mu^I\| \sim |I|\,\|\mu\|$*.*

Proof. The equality $\mu_\varepsilon(I) = |I|^{\delta(V-1)} \int_I Q_{C_{|I|}(t)} \mu^I_{\varepsilon/|I|}(dt)$ follows from the respective definitions of μ_ε and $\mu_{\varepsilon/|I|}$. Because $I \subset I_M$ for all $M \in S \cap T^I$, it follows that $\int_I Q_{C_{|I|}(t)} \mu^I_{\varepsilon/|I|}(dt) = Q_{T^I} \int_I Q_{B^I \cap C_{|I|}(t)}\ \mu^I_{\varepsilon/|I|}(dt)$.

The random variable Q_{T^I} and the stochastic process $t \mapsto Q_{B^I \cap C_{|I|}(t)}$ are independent of one another and of the μ_ε^I's. Indeed they involve mutually disjoint subsets of S, namely, $S \cap T^I$, $S \cap B^I$, and $S \cap T_I$.

The reason for a.s. weak convergence as $\varepsilon \to 0$ is the same for the family $(\mu_\varepsilon^I)_{0<\varepsilon\leq 1}$ as for $(\mu_\varepsilon)_{0<\varepsilon\leq 1}$ restricted to any compact interval.

(i) Fix $\varepsilon > 0$. To show that $\int_I f(t)\,\mu_\varepsilon^I(dt) \sim |I| \int_{[0,1]} f \circ f_I^{-1}(t)\,\mu_\varepsilon(dt)$ for every $f \in C(I)$, it suffices to show that the same holds for the function $f = \mathbf{1}_J$ for every subinterval J of I. Indeed, every $f \in C(I)$ is the limit in $\|\ \|_\infty$ norm of piecewise constant functions. Fixing such a J reduces the problem to showing that $\mu_\varepsilon^I(J) \sim |I|\mu_\varepsilon(f_I(J))$.

f_I is the restriction to the real line $\mathbb{R}$ of the similarity $\tilde{f}_I = h_I \circ \theta_I$ on the plane $\mathbb{R}^2$, where $h_{|I|}$ is the homothety with center $(0,0)$ and ratio $|I|^{-1}$, and θ_I is the horizontal translation by the vector $(-\inf(I), 0)$. Inspired by [Ma2], we use the property that for every subset F of H such that $\tilde{f}_I(F) \subset H$, $\Lambda(F) = \Lambda(\tilde{f}_I(F))$. Together with the equality $\tilde{f}_I(T_I \cap \{(t,\lambda) \in H;\ \lambda \geq \varepsilon|I|\}) = T_{[0,1]} \cap \{(t,\lambda) \in H;\ \lambda \geq \varepsilon\}$, this property implies that the point process $\tilde{f}_I(S \cap T_I \cap \{(t,\lambda) \in H;\ \lambda \geq \varepsilon|I|\})$ has the same distribution as $S \cap T_{[0,1]} \cap \{(t,\lambda) \in H;\ \lambda \geq \varepsilon\}$.

Consider the measure ν_ε constructed on $[0,1]$ like the restriction of μ_ε to $[0,1]$, but with the pairs $(\tilde{f}_I(M), W_M)$, for M in $S \cap T_I \cap \{(t,\lambda) \in H;\ \lambda \geq \varepsilon|I|\}$, instead of the pairs (M, W_M), for M in $S \cap T_{[0,1]} \cap \{(t,\lambda) \in H;\ \lambda \geq \varepsilon\}$. We see that $\nu_\varepsilon(f_I(J)) \sim \mu_\varepsilon(f_I(J))$. Moreover, the change of variable $t' = f_I(t)$ in $\int_I \mathbf{1}_J(t)\,\mu_\varepsilon^I(dt)$ yields $\mu_\varepsilon^I(J) = |I|\,\nu_\varepsilon(f_I(J))$, since, by construction, for every $t \in I$

$$\frac{d\mu_\varepsilon^I}{d\ell}(t) = \frac{d\nu_\varepsilon}{d\ell}(f_I(t)).$$

(ii) The measures μ^I and μ are, respectively, the weak limit of $(\mu_\varepsilon^I)_{0<\varepsilon\leq 1}$ and $(\mu_\varepsilon)_{0<\varepsilon\leq 1}$ as $\varepsilon \to 0$. It follows that (ii) is deduced from (i) by letting ε tend to 0.

Now define $Y_s = \|\mu^{(1)}_{1/s}\|$ for all $s \geq 1$. By construction, $(Y_s, F_{1/s})_{s\geq 1}$ is a right-continuous positive martingale with mean 1 that converges to Y.

If I is a non-trivial compact subinterval of $[0,1]$, define $Y_I = \frac{1}{|I|}\|\mu^I\|$ and, for all $s \geq 1$, define $Y_{s,I} = \frac{1}{|I|}\|\mu^I_{1/s}\|$.

The measure μ will be represented as the image of a measure on the boundary of an homogeneous tree.

2.3. Measure on a tree associated with μ

Given two integers $b \geq 2$ and $m \geq 0$, denote by A_m the set of finite words of length m on the alphabet $\{0, \dots, b-1\}$ ($A_0 = \{\epsilon\}$). Denote $\bigcup_{m=0}^{\infty} A_m$ by A. For $a \in A$, the length of a and the closed b-adic subinterval of $[0, 1]$ naturally encoded by a are, respectively, denoted by $|a|$ and I_a.

For $n \geq 1$ and $a = (a_1, \dots, a_n) \in A_n$, denote $(a_1, \dots, a_{n-1})$ by $a|(n-1)$.

Define $\partial A = \{0, \dots, b-1\}^{\mathbb{N}}$. The set A acts on the disjoint union of A and ∂A by the concatenation operation. For $a \in A$, let C_a denote $a\partial A$, namely, the cylinder generated by a. Denote by $\mathcal{A}$ the σ-field generated by the C_a's in ∂A.

Denote by π the mapping $t = (t_1, \dots, t_i, \dots) \in \partial A \mapsto \sum_{i\geq 1} t_i/b^i \in [0, 1]$.

Denote by $\tilde{\ell}$ the measure on $(\partial A, \mathcal{A})$ such that for all $a \in A$, $\tilde{\ell}(C_a) = b^{-|a|}$.

If ρ is a non-negative measure on $(\partial A, \mathcal{A})$, the measure $D_n.\rho$ will be defined, for $n \geq 1$, by $\dfrac{d(D_n.\rho)}{d\rho}(t) = b^{-n\delta(V-1)} Q_{C_{b^{-n}}(\pi(t))}$. The sequence $(D_n.\rho)_{n\geq 1}$ converges a.s. weakly to a non-negative random measure $D.\rho$. Moreover, by [K1], the operator $L : \rho \mapsto \mathbb{E}(D.\rho)$ on non-negative measures on ∂A is a projection.

Define $\tilde{\mu} = D.\tilde{\ell}$ and $\tilde{\mu}_n = D_n.\tilde{\ell}$ for all $n \geq 1$. By construction, $\mu = \tilde{\mu} \circ \pi^{-1}$ and $\mu_{b^{-n}} = \tilde{\mu}_n \circ \pi^{-1}$ for $n \geq 1$.

The following three relations, (1), (2), and (3), will prove to be fundamental. By Theorem 5, for all $n > m > 1$

$$Y_{b^n} = \sum_{a \in A_m} \mu_{b^{-n}}(I_a) = b^{-m\delta(V-1)} \sum_{a \in A_m} Q_{T^{I_a}} \int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)}\ \mu^{I_a}_{b^{m-n}}(dt), \quad (1)$$

$$\tilde{\mu}(C_a) = b^{-m\delta(V-1)} Q_{T^{I_a}} \int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)}\ \mu^{I_a}(dt) \quad \forall\, a \in A_m. \quad (2)$$

(Proof: $\tilde{\mu}(C_a) = \lim_{n\to\infty} \tilde{\mu}_n(C_a)$ since the space ∂A is totally disconnected; moreover, $\tilde{\mu}_n(C_a) = \mu_{b^{-n}}(I_a)$ for all $n \geq 1$, and $\mu^{I_a}(\{t_M;\ M \in S,\ \lambda_M \geq b^{-|a|}\}) = 0$ a.s.)

$$Y = \sum_{a \in A_m} \tilde{\mu}(C_a) = b^{-m\delta(V-1)} \sum_{a \in A_m} Q_{T^{I_a}} \int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)}\ \mu^{I_a}(dt) \quad \forall\, m \geq 1. \quad (3)$$

3. Proofs of the main results

3.1. Basic lemmas

Lemma 1. *Fix B as a Borel subset of H such that $\Lambda(B) < \infty$, $q \in \mathbb{R}$ and $\beta > 0$.*

- (i) $\mathbb{E}(Q_B^q) = e^{\Lambda(B)(\mathbb{E}(W^q)-1)}$;
- (ii) $\mathbb{E}(Q_B^q \log Q_B) = \Lambda(B)\mathbb{E}(W^q \log W) e^{\Lambda(B)(\mathbb{E}(W^q)-1)}$ *if* $\mathbb{E}(W^q|\log W|) < \infty$;
- (iii) $\mathbb{E}(Q_B^q|\log Q_B|) \leq \Lambda(B)\mathbb{E}(W^q|\log W|) e^{\Lambda(B)(\mathbb{E}(W^q)-1)}$;

(*iv*) *Denote by* $\bar{\beta}$ *the integer such that* $\bar{\beta} \le \beta < \bar{\beta} + 1$. *There exists a constant* $C_\beta > 0$, *independent of* B, *such that*

$$\mathbb{E}(Q_B|\log Q_B|^\beta) \le C_\beta(1+\Lambda(B))^{\bar{\beta}+2}(1+V)^{\bar{\beta}+1}\mathbb{E}(W|\log W|^\beta)e^{\Lambda(B)(V-1)}.$$

Proof. We begin by proving (*iii*) and (*iv*). Conditionally on $\#S \cap B = k \ge 1$, we have $Q_B = \prod_{i=1}^k W_i$, where the W_i's are i.i.d. with W and independent of S. Hence, by using the subadditivity on $\mathbb{R}_+$ of the mapping $x \mapsto x^\beta$ when $0 < \beta \le 1$ and its convexity when $\beta > 1$, for every $\beta > 0$ and $q \in \mathbb{R}$ we get

$$\begin{aligned}\mathbb{E}(Q_B^q|\log Q_B|^\beta|\#S \cap B = k) &\le \mathbb{E}(\prod_{i=1}^k W_i^q[\sum_{i=1}^k |\log W_i|]^\beta)\\ &\le k^{(\max(1,\beta))}\mathbb{E}(W^q|\log W|^\beta)V_q^{k-1},\end{aligned}$$

where $V_q = \mathbb{E}(W^q)$. Since $\mathbb{P}(\#S \cap B = k) = e^{-\Lambda(B)}\frac{(\Lambda(B))^k}{k!}$, taking the unconditional expectation yields

$$\mathbb{E}(Q_B^q|\log Q_B|^\beta) \le \mathbb{E}(W^q|\log W|^\beta)e^{-\Lambda(B)}\sum_{k\ge1}\frac{(\Lambda(B))^k}{k!}k^{(\max(1,\beta))}V_q^{k-1},$$

and in the particular case $\beta = 1$, we get (*iii*). To get (*iv*), put $q = 1$ and define $p = \bar{\beta} + 2$ ($p \ge \max(1, \beta)$) and $x = \Lambda(B)V$. We have

$$\mathbb{E}(Q_B|\log Q_B|^\beta) \le \mathbb{E}(W|\log W|^\beta)e^{-\Lambda(B)}\Lambda(B)\sum_{k\ge0}\frac{(k+1)^p}{(k+1)!}x^k.$$

Define $C_\beta = \max(\sup_{0\le k\le p-1}\frac{(k+1)^p}{(k+1)!}k!,\ \sup_{k\ge p}\frac{(k+1)^p}{(k+1)k\dots(k+2-p)})$.

$$\begin{aligned}\sum_{k\ge0}\frac{(k+1)^p}{(k+1)!}x^k &\le C_\beta\sum_{k=0}^{p-1}\frac{x^k}{k!} + C_\beta\sum_{k\ge p}\frac{x^k}{(k+1-p)!}\\ &\le C_\beta e^x + C_\beta x^{p-1}\sum_{k\ge1}\frac{x^k}{k!} \le C_\beta(1+x)^{p-1}e^x\end{aligned}$$

($p - 1 \ge 1$). Since $1 + x \le (1+\Lambda(B))(1+V)$, it follows that

$$\mathbb{E}(Q_B|\log Q_B|^\beta) \le C_\beta(1+\Lambda(B))^{\bar{\beta}+2}(1+V)^{\bar{\beta}+1}\mathbb{E}(W|\log W|^\beta)e^{\Lambda(B)(V-1)}.$$

Assertion (*ii*) follows from the fact that if $k \ge 1$ then

$$\begin{aligned}\mathbb{E}(Q_B^q\log Q_B|\#S \cap B = k) &= \mathbb{E}(\prod_{i=1}^k W_i^q[\sum_{i=1}^k \log W_i])\\ &= k\mathbb{E}(W^q\log W)V_q^{k-1}.\end{aligned}$$

Then (*i*) follows by a similar computation or simply by integrating the equality given in (*ii*).

Lemma 2. *Fix $t \in \mathbb{R}$. For every $s \geq 1$, $\Lambda(C_{1/s}(t)) = \delta \log s$, and $(Q_{1/s}(t))_{s\geq 1}$ is a right continuous martingale with respect to $(F_{1/s})_{s\geq 1}$, with expectation* 1.

The verification, left to the reader, uses Lemma 1(i) with $B = C_{1/s}(t)$ and $q = 1$.

If $B \subset H$ and I is a non-trivial compact subinterval of [0, 1] and $q \in \mathbb{R}$, we define

$$m_{B,I} = \inf_{u\in I} Q_{B\cap C_{|I|}(u)}, \quad M_{B,I} = \sup_{u\in I} Q_{B\cap C_{|I|}(u)},$$
$$\gamma_I(q) = \mathbf{1}_{\{q<0\}}\, m^q_{B^I,I} + \mathbf{1}_{\{q\geq 0\}}\, M^q_{B^I,I}.$$

Lemma 3. *Fix a non-trivial compact subinterval I of* [0, 1].
$(i)(a)$ $\Lambda(T^I) = \delta(\log \frac{1}{|I|} - \frac{1}{2}(1 - |I|))$; (b) $\Lambda(B_I) = \delta(1 - |I|)$. (c) *For every $t \in I$, $\Lambda(B_I \cap C_{|I|}(t)) = \Lambda(B_I)/2$.*
(ii) *Fix $\beta > 0$. If $\mathbb{E}((1+W)|\log W|^\beta) < \infty$ then there exists $C_\beta > 0$ independent of I such that $\sup_{t\in I} \mathbb{E}\left(Q_{|I|}(t)\left|\log |I|^{-1}\int_I Q_{B^I\cap C_{|I|}(u)}du\right|^\beta\right) \leq C_\beta$.*
(iii) (a) $\mathbb{E}(M_{B^I,I}) \leq e^{\delta(\mathbb{E}(\max(1,W))-1)}$;
(b) $\mathbb{E}(\sup_{q\in K} \gamma_I(q)) \leq e^{\delta(\mathbb{E}(\max(1,W^{\inf(K)}+W^{\sup(K)}))-1)}$ *for every compact subinterval K of $\mathbb{R}$.*

Proof. (i) The computations are left to the reader.
(ii) Fix $t \in I$ and define

$$T_1 = \left|\log |I|^{-1} \int_I Q_{B^I\cap C_{|I|}(t)\cap C_{|I|}(u)} Q_{(B^I\setminus C_{|I|}(t))\cap C_{|I|}(u)}\, du\right|^\beta.$$

It follows from the definitions of $m_{B,I}$ and $M_{B,I}$ that

$$\prod_{\varepsilon\in\{-1,1\}} m_{B^I_\varepsilon\cap C_{|I|}(t),I}\, m_{B^I_\varepsilon\setminus C_{|I|}(t),I} \leq |I|^{-1}\int_I Q_{B^I\cap C_{|I|}(t)\cap C_{|I|}(u)} Q_{(B^I\setminus C_{|I|}(t))\cap C_{|I|}(u)} du$$
$$\leq \prod_{\varepsilon\in\{-1,1\}} M_{B^I_\varepsilon\cap C_{|I|}(t),I}\, M_{B^I_\varepsilon\setminus C_{|I|}(t),I}.$$

Hence, $T_1 \leq 4^\beta \sum_{\varepsilon\in\{-1,1\}}(T_{2,\varepsilon} + T_{3,\varepsilon})$ with

$$\begin{cases} T_{2,\varepsilon} = |\log m_{B^I_\varepsilon\cap C_{|I|}(t),I}|^\beta + |\log M_{B^I_\varepsilon\cap C_{|I|}(t),I}|^\beta \\ T_{3,\varepsilon} = |\log m_{B^I_\varepsilon\setminus C_{|I|}(t),I}|^\beta + |\log M_{B^I_\varepsilon\setminus C_{|I|}(t),I}|^\beta. \end{cases}$$

Therefore

$$\begin{aligned} T := {} & Q_{|I|}(t)\left|\log |I|^{-1}\int_I Q_{B^I\cap C_{|I|}(u)}\, du\right|^\beta \\ &= |I|^{\delta(V-1)} Q_{C_{|I|}(t)} T_1 \\ &\leq T_4 + T_5 \end{aligned}$$

with

$$\begin{cases} T_4 = 4^\beta |I|^{\delta(V-1)} Q_{C_{|I|}(t)\setminus B^I}[Q_{B^I\cap C_{|I|}(t)}(T_{2,-1}+T_{2,1})] \\ T_5 = 4^\beta |I|^{\delta(V-1)} Q_{C_{|I|}(t)}(T_{3,-1}+T_{3,1}). \end{cases}$$

Then the identity $|I|^{\delta(V-1)}\mathbb{E}(Q_{C_{|I|}(t)\setminus B^I})\mathbb{E}(Q_{B^I\cap C_{|I|}(t)}) = 1$, together with the fact that the sets $C_{|I|}(t)\setminus B^I$ and $B^I \cap C_{|I|}(t)$ are disjoint, as well as $C_{|I|}(t)$ and $B^I \setminus C_{|I|}(t)$, yield

$$\mathbb{E}(T) \le 4^\beta [\frac{1}{\mathbb{E}(Q_{B^I\cap C_{|I|}(t)})}\mathbb{E}(Q_{B^I\cap C_{|I|}(t)}(T_{2,-1}+T_{2,1})) + \mathbb{E}(T_{3,-1}+T_{3,1})],$$

where by $(i)(c)$ and Lemma $1(i)$ $(\mathbb{E}(Q_{B^I\cap C_{|I|}(t)}))^{-1} = e^{-\delta(1-|I|)(V-1)/2}$ is bounded independently of I.

It remains to show that $\mathbb{E}(Q_{B^I\cap C_{|I|}(t)}T_{2,\varepsilon})$ and $\mathbb{E}(T_{3,\varepsilon})$ are bounded independently of I and t for $\varepsilon \in \{-1, 1\}$.

First, we estimate $\mathbb{E}(Q_{B^I\cap C_{|I|}(t)}T_{2,-1})$. Conditionally on $\#S \cap B^I \cap C_{|I|}(t) = k \ge 1$, we write $S \cap B^I \cap C_{|I|}(t) = \{N_1, \dots, N_k\}$. Conditionally on $\#S \cap B^I_{-1} \cap C_{|I|}(t) = l \in [1, k]$ (if k or $l = 0$ then $T_{2,-1} = 0$), we can assume that $N_1, \dots N_l \in B^I_{-1}$ and $t_{N_1} + \lambda_{N_1} \le \cdots \le t_{N_l} + \lambda_{N_l}$. Then, for every $u \in I$, we have

$$Q_{B^I_{-1}\cap C_{|I|}(t)\cap C_{|I|}(u)} \in \{\prod_{i=j}^{l} W_{N_i};\ 1 \le j \le l\} \cup \{1\}$$

according to whether or not $u \in \cap_{i=j}^{l} I_{N_i}$ for some $1 \le j \le l$. This implies that

$$T_{2,-1} \le 2k^{\max(0,\beta-1)} \sum_{i=1}^{k} |\log\ W_{N_i}|^\beta.$$

Consequently for $\varepsilon \in \{-1, 1\}$ and $k \ge 1$ we obtain

$$\begin{aligned}\mathbb{E}(Q_{B^I\cap C_{|I|}(t)}T_{2,\varepsilon}|\#S \cap B^I \cap C_{|I|}(t) = k) &\le 2k^{\max(0,\beta-1)}\mathbb{E} \\ &\quad\times\Big(\prod_{j=1}^{k} W_{N_j} \sum_{j=1}^{k} |\log\ W_{N_j}|^\beta\Big) \\ &= 2k^{\max(1,\beta)}\mathbb{E}(W|\log\ W|^\beta)V^{k-1}.\end{aligned}$$

Similarly we obtain

$$\mathbb{E}(T_{3,\varepsilon}|\#S \cap B^I \setminus C_{|I|}(t) = k)) \le 2k^{\max(1,\beta)}\mathbb{E}(|\log\ W|^\beta).$$

Since $\Lambda(B^I \cap C_{|I|}(t))$ and $\Lambda(B^I \setminus C_{|I|}(t))$ are bounded independently of I and t (by $(i)(b)$), taking the unconditional expectations in the previous inequalities (as in the proof of Lemma 1) yields the conclusion.

$(iii)(a)$ One obtains $\mathbb{E}(M_{B^I,I}) \le (e^{\Lambda(B^I_{-1})(\mathbb{E}(\max(1,W))-1)})^2$ as follows. Use the inequality $M_{B^I,I} \le M_{B^I_{-1},I}.M_{B^I_1,I}$ and the equality $\mathbb{E}(M_{B^I_{-1},I}.M_{B^I_1,I}) = (\mathbb{E}(M_{B^I_{-1},I}))^2$

(B_{-1} and B_1 are disjoint, and orthogonally symmetric with respect to the line $\{t = (\inf(I) + \sup(I))/2\}$). Then use computations very similar to those done in (ii) to estimate $\mathbb{E}(Q_{B^I \cap C_{|I|}(t)} T_{2,\varepsilon})$: conditionally on $\# S \cap B^I_{-1} = k \geq 1$ and $S \cap B^I_{-1} = \{N_1, \ldots, N_k\}$

$$M_{B^I_{-1},I} \leq \sup_{L \subset \{1,\ldots,k\}} \prod_{i \in L} W_{N_i} \leq \prod_{i=1}^{k} \max(W_{N_i}, 1).$$

This yields

$$\mathbb{E}(M_{B^I_{-1},I} | \# S \cap B^I_{-1} = k) \leq [\mathbb{E}(\max(W, 1))]^k.$$

This estimate also holds if $k = 0$. Taking the unconditional expectation yields $\mathbb{E}(M_{B^I_{-1},I}) \leq e^{\Lambda(B^I_{-1})\mathbb{E}(\max(1,W))-1)}$. As $\Lambda(B^I_{-1}) \leq \delta/2$ by $(i)(b)$, we have the conclusion.
$(iii)(b)$ Notice that $\sup_{q \in K} \gamma_I(q) \leq \tilde{M}_{B^I,I}$, where $\tilde{M}_{B^I,I}$ is defined as $M_{B^I,I}$ but with $\tilde{W} = W^{\inf(K)} + W^{\sup(K)}$ instead of W. Conclude by using $(iii)(a)$.

Lemma 4. *Fix b, an integer ≥ 2, and $q \in \mathbb{R}$ such that $\mathbb{E}(W^q) < \infty$. There exists $C_q = C_q(W) > 0$ such that for $n > m \geq 1$ and $a \in A_m$:*
$(i)(a)$ $\mu^q_{b^{-n}}(I_a) \leq w_q(I_a) Y^q_{b^{n-m},I_a}$ with $w_q(I_a) = b^{-mq[1+\delta(V-1)]} Q^q_{T^{I_a}} \gamma_{I_a}(q)$ and $\mathbb{E}(w_q(I_a)) \leq C_q b^{-m(\tau(q)+1)}$;
(b) $\sum_{a \in A_m} \mathbb{E}(\mu^q_{b^{-n}}(I_a)) \leq C_q b^{-m\tau(q)} \mathbb{E}(Y^q_{b^{n-m}})$;
(c) if $q \geq 1$ then $\mathbb{E}(Y^q_{b^n}) \geq b^{-m\tau(q)} e^{\frac{\delta}{2}(1-b^{-m})(q(V-1)-(\mathbb{E}(W^q)-1))} \mathbb{E}(Y^q_{b^{n-m}})$.
$(ii)(a)$ $\tilde{\mu}^q(C_a) \leq w_q(I_a) Y^q_{I_a}$; (b) $\sum_{a \in A_m} \mathbb{E}(\tilde{\mu}^q(C_a)) \leq C_q b^{-m\tau(q)} \mathbb{E}(Y^q)$.

Proof. Fix $n > m \geq 1$ and $a \in A_m$.
$(i)(a)$ By Theorem 5

$$\mu^q_{b^{-n}}(I_a) = b^{-mq\delta(V-1)} Q^q_{T^{I_a}} \left(\int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)}\ \mu^{I_a}_{b^{m-n}}(dt) \right)^q.$$

Hence, $\mu^q_{b^{-n}}(I_a) \leq w_q(I_a) Y^q_{b^{n-m},I_a}$ follows from the definitions of $\gamma_{I_a}(q)$ and Y_{b^{n-m},I_a}.

We have $\mathbb{E}(w_q(I_a)) = b^{-mq[1+\delta(V-1)]} \mathbb{E}(Q^q_{T^{I_a}}) \mathbb{E}(\gamma_{I_a}(q))$ since $Q^q_{T^{I_a}}$ and $\gamma_{I_a}(q))$ are independent. Moreover, by Lemma 3$(i)(a)$ and Lemma 1(i) applied with $B = T^{I_a}$,

$$\mathbb{E}(Q^q_{T^{I_a}}) = b^{m\delta(\mathbb{E}(W^q)-1)} e^{-\frac{\delta}{2}(1-b^{-m})(\mathbb{E}(W^q)-1)}, \tag{4}$$

and by Lemma 3$(iii)(b)$ applied with $K = \{q\}$, $\mathbb{E}(\gamma_{I_a}(q)) \leq e^{\delta(\mathbb{E}(\max(1,2W^q))-1)}$.

Therefore

$$\begin{aligned}\mathbb{E}(w_q(I_a)) &\leq e^{-\frac{\delta}{2}(1-b^{-m})(\mathbb{E}(W^q)-1)} e^{\delta(\mathbb{E}(\max(1,2W^q))-1)} b^{-m[q(1+\delta(V-1))-\delta(\mathbb{E}(W^q)-1)]} \\ &\leq C_q\, b^{-m(\tau(q)+1)}\end{aligned}$$

where $C_q = e^{\delta(\mathbb{E}(\max(1,2W^q))-1)} \sup_{m \geq 1} e^{-\frac{\delta}{2}(1-b^{-m})(\mathbb{E}(W^q)-1)}$.

$(i)(b)$ Follows from $(i)(a)$ and the independence between $w_q(I_a)$ and Y_{b^{n-m},I_a}.
$(i)(c)$ The super-additivity of $x \geq 0 \mapsto x^q$ applied in (1) yields

$$\mathbb{E}(Y_{b^n}^q) \geq \sum_{a \in A_m} b^{-mq\delta(V-1)} \mathbb{E}(Q_{T^{I_a}}^q) \mathbb{E}\left(\left[\int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)}\ \mu_{b^{m-n}}^{I_a}(dt)\right]^q\right). \quad (5)$$

The Jensen inequality applied in $T = \mathbb{E}([\int_{I_a} Q_{B^{I_a} \cap C_{b^{-m}}(t)} \mu_{b^{m-n}}^{I_a}(dt)]^q \,|\, \cup_{0<\varepsilon\leq b^{-m}} F_\varepsilon)$ yields

$$\begin{aligned} T &\geq \left(\int_{I_a} \mathbb{E}(Q_{B^{I_a} \cap C_{b^{-m}}(t)} \,|\, \cup_{0<\varepsilon\leq b^{-m}}\ F_\varepsilon) \mu_{b^{m-n}}^{I_a}(dt)\right)^q \\ &= \left(\int_{I_a} \mathbb{E}(Q_{B^{I_a} \cap C_{b^{-m}}(t)}) \mu_{b^{m-n}}^{I_a}(dt)\right)^q \\ &= \left(e^{\frac{\delta}{2}(1-b^{-m})(V-1)} b^{-m} Y_{b^{n-m},I_a}\right)^q \end{aligned}$$

by Lemma $3(i)(c)$ and Lemma $1(i)$ applied with $B = B^{I_a} \cap C_{b^{-m}}(t)$ and $q = 1$. Then, by using (4) and the previous computation in (5), we get

$$\begin{aligned} \mathbb{E}(Y_{b^n}^q) \geq \sum_{a \in A_m} & b^{-mq\delta(V-1)} b^{m\delta(\mathbb{E}(W^q)-1)} e^{-\frac{\delta}{2}(1-b^{-m})(\mathbb{E}(W^q)-1)} \\ & \times e^{\frac{\delta}{2}(1-b^{-m})q(V-1)} b^{-mq} \mathbb{E}(Y_{b^{n-m}}^q), \end{aligned}$$

and the conclusion follows.
$(ii)(a)$ and $(ii)(b)$ are deduced from $(i)(a)$ and $(i)(b)$ by letting n tend to ∞.

The random function $f_{a,n,m}$ involved in Lemma 5 is defined in the proof of Theorem $1(i)$ in Sect. 3.2.

Lemma 5. *(i) $\mathbb{E}(f'_{a,n,m}(1^-)) = b^{-m}\left(-m\ \log(b)\, \tau'(1^-) + \mathbb{E}(Y_{b^{n-m}}\ \log\ Y_{b^{n-m}})\right)$.*
(ii) $\sum_{a \neq a' \in A_m} \mathbb{E}\left(\mu_{b^{-n}}^{\frac{1}{2}}(I_a) \mu_{b^{-n}}^{\frac{1}{2}}(I_{a'})\right) \leq 5\,C + C\, b^m\, (\mathbb{E}(Y_{b^{n-m}}^{\frac{1}{2}}))^2$ for some $C > 0$ independent of m and n.

Proof. (i) Differentiate $f_{a,n,m}$ at 1^- yields $\mathbb{E}(f'_{a,n,m}(1^-)) = T_1 + T_2 + T_3$ with

$$\begin{aligned} T_1 &= -m \log(b)[\delta(V-1)] \mathbb{E}(f_{a,n,m}(1)) = -b^{-m} m \log(b) \delta(V-1), \\ T_2 &= b^{-m\delta(V-1)} \mathbb{E}\left(\int_{I_a} \mathbb{E}(Q_{C_{b^{-m}}(t)} \log(Q_{C_{b^{-m}}(t)})) \mu_{b^{m-n},I_a}(dt)\right) \\ &= b^{-m} m \log(b) \delta \mathbb{E}(W \log\ W), \\ T_3 &= b^{-m\delta(V-1)} \mathbb{E}\left((\log(Y_{b^{n-m},I_a}) - m \log(b)) \int_{I_a} \mathbb{E}(Q_{C_{b^{-m}}(t)}) \mu_{b^{m-n},I_a}(dt)\right) \\ &= b^{-m}(\mathbb{E}(Y_{b^{n-m}} \log\ Y_{b^{n-m}}) - m \log(b)), \end{aligned}$$

by using Lemma $1(i)$ and (ii) with $B = C_{b^{-m}}(t)$ and $q = 1$. As $\tau'(1) = 1 + \delta(V - 1) - \delta\mathbb{E}(W \log\ W)$, we have the conclusion.

(ii) By the Cauchy–Schwarz inequality and Lemma 4$(i)(a)$, for every $(a,a')\in A_m^2$ we have

$$\mathbb{E}\left(\left(\mu_{b^{-n}}(I_a)\mu_{b^{-n}}(I_{a'})\right)^{\frac{1}{2}}\right)\le C_1\,b^{-m}\,\mathbb{E}(Y^{\frac{1}{2}}_{b^{n-m},I_a}Y^{\frac{1}{2}}_{b^{n-m},I_{a'}}).$$

Moreover, Y_{b^{n-m},I_a} and $Y_{b^{n-m},I_{a'}}$ are independent when $T_{I_a}\cap T_{I_{a'}}=\emptyset$, otherwise we have $\mathbb{E}(Y^{\frac{1}{2}}_{b^{n-m},I_a}Y^{\frac{1}{2}}_{b^{n-m},I_{a'}})\le 1$ since $\mathbb{E}(Y_{b^{n-m}})=1$. As $\#\{a'\in A_m;\ T_{I_a}\cap T_{I_{a'}}\neq\emptyset\}\le 5$ for every $a\in A_m$, we get

$$\sum_{a\neq a'\in A_m}\mathbb{E}(\mu^{\frac{1}{2}}_{b^{-n}}(I_a)\mu^{\frac{1}{2}}_{b^{-n}}(I_{a'}))\le 5\,b^m\times C_1\,b^{-m}+b^{2m}\times C_1\,b^{-m}(\mathbb{E}(Y^{\frac{1}{2}}_{b^{n-m}}))^2.$$

The probability measures $\mathbb{P}_t$ involved in Lemma 6 are defined in the proof of Theorem 1(ii) in Sect. 3.2.

Lemma 6. *If* $\tau'(1^-)=0$ *and* $\mathbb{E}((1+W)|\log W|^{2+\gamma})<\infty$ *for some* $\gamma>0$ *then for every* $t\in[0,1]$, $\mathbb{P}_t(\limsup_{n\to\infty}Y_{b^n}=\infty)=1$.

Proof. Fix $t\in[0,1]$. For $n\ge 1$, denote by $I_n(t)$ the b-adic subinterval of $[0,1]$ of the n^{th} generation which contains t. One has $Y_{b^n}=\|\mu_{b^{-n}}\|\ge\mu_{b^{-n}}(I_n(t))$ so it suffices to show that $\mathbb{P}_t(\limsup_{n\to\infty}\mu_{b^{-n}}(I_n(t))=\infty)=1$.

Define

$$\begin{cases}R_{1,n}(t)=-\log\ Q_{C_{b^{-n}}(t)\setminus T^{I_n(t)}}\\ R_{2,n}(t)=\log\ b^n\displaystyle\int_{I_n(t)}Q_{B^{I_n(t)}\cap C_{b^{-n}}(u)}du.\end{cases}$$

We have

$$\log\ \mu_{b^{-n}}(I_n(t))=\log\ Q_{b^{-n}}(t)\ -n\log(b)+R_{1,n}(t)+R_{2,n}(t),$$

so the conclusion results from the two following properties:

(1) $\mathbb{P}_t(\limsup_{n\to\infty}\dfrac{\log\ Q_{b^{-n}}(t)\ -n\log(b)}{(n\log\log n)^{1/2}}>0)=1$: for every $k\ge 1$ define the random variable $X_k=\log(Q_{b^{-k}}(t)/Q_{b^{-(k-1)}}(t))-\log(b)$. By construction, the X_k are i.i.d. with respect to $\mathbb{P}_t$ and by Lemma 1(i), (ii) and $(iv)(q=1,\ \beta=2)$ applied with $B=C_{b^{-1}}(t)$

$$\mathbb{E}_{\mathbb{P}_t}(X_k)=\mathbb{E}_{\mathbb{P}_t}(X_1)=\mathbb{E}(Q_{b^{-1}}(t)\log\ Q_{b^{-1}}(t))-\log(b)=-\log(b)\tau'(1^-)=0$$

and $\mathbb{E}_{\mathbb{P}_t}(X_k^2)<\infty$. Moreover $\mathbb{E}_{\mathbb{P}_t}(X_k^2)>0$, otherwise $\mathbb{P}(Q_{C_{b^{-1}}(t)}=b^{1+\delta(V-1)})=1$, implying that $W=1$ a.s. and $\tau'(1^-)=0$. One concludes using the law of the iterated logarithm.

(2) $\mathbb{P}_t(\lim_{n\to\infty}\dfrac{|R_{1,n}(t)|+|R_{2,n}(t)|}{(n\log\log n)^{1/2}}=0)=1$: this holds if $\sup_{i\in\{1,2\},n\ge1}\mathbb{E}_{\mathbb{P}_t}(|R_{i,n}(t)|^{2+\gamma})$ is finite. We have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_t}(|R_{1,n}(t)|^{2+\gamma})&=\mathbb{E}(Q_{b^{-n}}(t)|\log\ Q_{C_{b^{-n}}(t)\setminus T^{I_n(t)}}|^{2+\gamma})\\&=b^{-n\delta(V-1)}\mathbb{E}(Q_{T^{I_n(t)}})\\&\quad\times\mathbb{E}(Q_{C_{b^{-n}}(t)\setminus T^{I_n(t)}}|\log\ Q_{C_{b^{-n}}(t)\setminus T^{I_n(t)}}|^{2+\gamma}),\end{aligned}$$

and $\mathbb{E}_{\mathbb{P}_t}(|R_{2,n}(t)|^{2+\gamma})=\mathbb{E}(Q_{b^{-n}}(t)|\log\ b^n\int_{I_n(t)}Q_{B^{I_n(t)}\cap C_{b^{-n}}(u)}du|^{2+\gamma})$.

These expectations are uniformly bounded over $\mathbb{N}^*$. This results from Lemma 1(i) applied with $B = T^{I_n(t)}$ and $q = 1$ and Lemma 1(iv) applied with $B = C_{b^{-n}}(t) \setminus T^{I_n(t)}$ and $\beta = 2 + \gamma$, together with Lemma 3(ii) applied with $\beta = 2 + \gamma$.

Now we consider the assumptions of Theorem 4. Fix an integer $b \geq 2$. For $q \in J'$, let $\tilde{\mu}_q$ be the measure on $(\partial A, \mathcal{A})$ obtained a.s. as the weak limit of $(\tilde{\mu}_{q,n})_{n\geq 1}$, where $\frac{d\tilde{\mu}_{q,n}}{d\tilde{\ell}}(t) = b^{-n\delta(\mathbb{E}(W^q)-1)}\, Q^q_{C_{b^{-n}}(\pi(t))}$. The total mass of $\tilde{\mu}_q$ is denoted by Y_q and for every $a \in A$, Y_{q,I_a} denotes $b^{|a|}\|\tilde{\mu}_q^{I_a}\|$ and is a copy of Y_q.

Lemma 7. *With probability one*
(i) *For all $a \in A$, the sequence of functions $(q \mapsto \tilde{\mu}_{q,n}(C_a))_{n\geq 1}$ converges uniformly on the compact subsets of J' to $q \mapsto \tilde{\mu}_q(C_a)$, which is positive. Consequently the measures $\tilde{\mu}_q$, $q \in J'$, are defined simultaneously and have ∂A as support.*
(ii) *For every $q \in J'$, for $\tilde{\mu}_q$-almost every $t = (t_1, \dots, t_n, \dots) \in \partial A$*

$$\lim_{n\to\infty} \frac{\log\, \tilde{\mu}_q(C_{(t_1,\dots,t_n)})}{-n \log\, b} \geq \tau^*(\tau'(q)).$$

Proof of (i). The next few lines will assume the following property, $(\mathcal{P})$, whose validity will be proven momentarily. $(\mathcal{P})$: there exists a deterministic complex neighborhood of J', to be denoted by $\mathcal{V}$, such that for every $a \in A$ and $n > |a|$, the mapping $q \in J' \mapsto \tilde{\mu}_{q,n}(C_a) = \sum_{a'\in A_{n-m}} \tilde{\mu}_{q,n}(C_{aa'})$ possesses the analytic extension

$$z \in \mathcal{V} \mapsto \psi_n^{(a)}(z) = \sum_{a'\in A_{n-m}} b^{-n\delta(\mathbb{E}(W^z)-1)} \int_{I_{aa'}} Q^z_{C_{b^{-n}}(t)}\, dt.$$

Moreover, given $a \in A$, for every compact subinterval K of J', there exist three constants $h > 1$, $c < 0$, $C > 0$ and a complex neighborhood U of K, such that for all $n \geq 1$, $\sup_{z\in U} \mathbb{E}(|\psi_{n+1}^{(a)}(z) - \psi_n^{(a)}(z)|^h) \leq Cb^{(n+1)c}$.

For every $a \in A$, the Cauchy formula applied as in [Bi] gives a.s. the uniform convergence of $(\psi_n^{(a)})_{n>m}$ on the compact subsets of a complex neighborhood of J', and so the one of $(q \mapsto \tilde{\mu}_{q,n}(C_a))_{n\geq 1}$, on the compact subsets of J', to $q \mapsto \tilde{\mu}_q(C_a)$. This happens almost surely simultaneously for all the a's in A because A is countable, so the measures $\tilde{\mu}_q$ are defined simultaneously.

To see that $q \mapsto \tilde{\mu}_q(C_a)$ is almost surely positive on J' for every $a \in A$, so that the support of the $\tilde{\mu}_q$'s is ∂A, adapt the proof of Corollary 5 (ii) (β) of [B2] by using Theorem 5 and Eqs. (2) and (3).

Proof of $(\mathcal{P})$. J' is an open subinterval of J. Consequently, there exists a deterministic complex neighborhood $\mathcal{V}$ of J' so that the mapping $z \mapsto \mathbb{E}(W^z)$ is defined and analytic on $\mathcal{V}$. Moreover, for every $n \geq 1$, the piecewise constant function $t \in [0, 1] \mapsto Q^z_{C_{b^{-n}}(t)}$ is almost surely defined for all $z \in \mathbb{C}$ and depends analytically on z. This implies that for every $a \in A$ the $\psi_n^{(a)}$, $n > |a|$, are all defined and analytic on $\mathcal{V}$. The fact that $\tilde{\mu}_q(C_a) = \psi_n^{(a)}(q)$ on J' follows from the definition of $\tilde{\mu}_q$.

Fix $a \in A$. For every z in $\mathcal{V}$, $\psi_{n+1}^{(a)}(z) - \psi_n^{(a)}(z)$

$$= \sum_{a' \in A_{n-m}} \int_{I_{aa'}} b^{-n\delta(\mathbb{E}(W^z)-1)} Q^z_{C_{b^{-n}}(t)} [b^{-\delta(\mathbb{E}(W^z)-1)} Q^z_{C_{b^{-(n+1)}}(t) \setminus C_{b^{-n}}(t)} - 1]\, dt. \tag{6}$$

Let the b-adic intervals of the n^{th} generation involved in the previous sum be numbered from 0 to $b^{n-m} - 1$ as they appear on the real line, and denoted by J_k's, $0 \leq k < b^{n-m}$.

For $t \in \cup_{k=0}^{b^{n-m}-1} J_k$, define

$$\begin{cases} u_n(z,t) = b^{-n\delta(\mathbb{E}(W^z)-1)} Q^z_{C_{b^{-n}}(t)}, \\ v_n(z,t) = b^{-\delta(\mathbb{E}(W^z)-1)} Q^z_{C_{b^{-(n+1)}}(t) \setminus C_{b^{-n}}(t)} - 1. \end{cases}$$

Then for $i \in \{0, 1, 2\}$ and $t \in J_0$ define

$$\Gamma_i(z,t) = \sum_{0 \leq 3k+i < b^{n-m}} u_n(z, t + \frac{3k+i}{b^{n-m}})\, v_n(z, t + \frac{3k+i}{b^{n-m}}).$$

It follows from (6) and a Hölder inequality that for $h > 1$,

$$\mathbb{E}(|\psi_{n+1}^{(a)}(z) - \psi_n^{(a)}(z)|^h) \leq 3^{h-1} |J_0|^{h-1} \int_{J_0} \sum_{i \in \{0,1,2\}} \mathbb{E}(|\Gamma_i(z,t)|^h)\, dt. \tag{7}$$

For each $t \in J_0$, in $\Gamma_i(z,t)$, the $v_n(z, t + \frac{3k+i}{b^{n-m}})$'s are mutually independent since the $T_{J_{3k+i}}$'s are pairwise disjoint. Moreover, they are by construction of mean 0 and independent of the $u_n(z, t + \frac{3k+i}{b^{n-m}})$'s. Then, it follows from Lemma 1 in [Bi] that

$$\begin{aligned} \mathbb{E}\left(|\Gamma_i(z,t)|^h\right) \leq 2^h \sum_{0 \leq 3k+i < b^{n-m}} & \mathbb{E}\left(\left|u_n\left(z, t + \frac{3k+i}{b^{n-m}}\right)\right|^h\right) \\ & \times \mathbb{E}\left(\left|v_n\left(z, t + \frac{3k+i}{b^{n-m}}\right)\right|^h\right) \end{aligned} \tag{8}$$

for every $1 < h \leq 2$. By using Lemma 1(i) with $|W^z|$ instead of W and $B \in \{C_{b^{-(n+1)}}(t) \setminus C_{b^{-n}}(t),\ C_{b^{-n}}(t)\}$ we get

$$\mathbb{E}(|u_n(z,t)|^h)\mathbb{E}(|v_n(z,t)|^h) \leq 2^h b^{(n+1)\theta(z,h)} \tag{9}$$

independently of t, where $\theta(z,h) = -h\delta[\mathbb{E}(\Re(W^z)) - 1] + \delta[\mathbb{E}(|W^z|^h) - 1]$.

It follows from (7), (8), and (9) that (with $C_{h,m} = 12^h b^{-(m+1)(1-h)}$)

$$\mathbb{E}(|\psi_{n+1}^{(a)}(z) - \psi_n^{(a)}(z)|^h) \leq C_{h,m} b^{(n+1)(1-h+\theta(z,h))}.$$

Finally, if K is a compact subinterval of J', a study of function using the definition of J' yields $h \in]1, 2]$ and a complex neighborhood U of K such that $c = \sup_{z \in U} 1 - h + \theta(z,h) < 0$.

Proof of (ii). Define $E_{q,n,\varepsilon} = \{t \in \partial A;\ \dfrac{\log \tilde{\mu}_q(C_{(t_1,\dots,t_n)})}{-n \log b} \le \tau^*(\tau'(q)) - \varepsilon\}$ for $q \in J'$, $\varepsilon > 0$ and $n \ge 1$. It suffices to show that for every compact subinterval K of J' and every $\varepsilon > 0$, a.s. for every $q \in K$, $\sum_{n\ge 1} \tilde{\mu}_q(E_{q,n,\varepsilon}) < \infty$.

Fix such a K and ε. For every $\eta > 0$ and $n \ge 1$, by definition of $E_{q,n,\varepsilon}$ and by Lemma $4(ii)(a)$, we have

$$\tilde{\mu}_q(E_{q,n,\varepsilon}) \le \sum_{a\in A_n} \tilde{\mu}_q^{1+\eta}(C_a) b^{n\eta(\tau^*(\tau'(q))-\varepsilon)} \le f_{n,\eta,\varepsilon}(q)$$

with $f_{n,\eta,\varepsilon}(q) = b^{n\eta(\tau^*(\tau'(q))-\varepsilon)} \sum_{a\in A_n} [\sup_{q'\in K} \gamma_{I_a}((1+\eta)q')]\, b^{-n(1+\eta)(1+\delta(\mathbb{E}(W^q)-1))}$ $(Q^q_{T^{I_a}} Y_{q,I_a})^{1+\eta}$.

Then, using Lemma $1(i)(ii)$ and Lemma $3(iii)(b)$ together with computations patterned after those in the proof of Corollary 1 in [B2] lead to the following conclusion: for η small enough, there exist two positive constants $C_K > 0$ and $C'_K > 0$ such that

$$\forall\, n \ge 1,\ \sup_{q\in K} \mathbb{E}(f_{n,\eta,\varepsilon}(q)) + \sup_{q\in K} \mathbb{E}(|\frac{d}{dq} f_{n,\eta,\varepsilon}(q)|) \le C_K n b^{-nC'_K}.$$

This implies that a.s. the series $\sum_{n\ge 1} f_{n,\eta,\varepsilon}(q) < \infty$ converges uniformly on K.

3.2. *Proofs of the results in Sect. 2.1*

Proof of Theorem 1(i). Fix an integer $b \ge 2$.
Define $c = \mathbb{E}(Y)$ (≤ 1). Λ being invariant by horizontal translations, the definition of $\tilde{\mu}$ implies that, for every $n \ge 1$, $\mathbb{E}(\tilde{\mu}(C_a))$ does not depend upon $a \in A_n$. Consequently, (3) yields $\mathbb{E}(\tilde{\mu}(C_a)) = cb^{-|a|} = c\tilde{\ell}(C_a)$ for every $a \in A$. In the notations of Sect. 2.3, this implies that $L(\tilde{\ell}) = c\tilde{\ell}$. Moreover, $c^2 = c$ since L is a projection.

Moreover, as $W > 0$, by using (3) with $b = 4$ and $m = 1$ we see that $\{Y = 0\} \subset \{\mu^{I_0} = 0,\ \mu^{I_3} = 0\}$. By Theorem 5 this implies that $\{Y = 0\} \subset \{Y_{I_0} = 0,\ Y_{I_3} = 0\}$, where Y_{I_0} and Y_{I_3} are copies of Y, and Y_{I_0} and Y_{I_3} are independent since $T_{I_0} \cap T_{I_3} = \emptyset$. It follows that $\mathbb{P}(Y = 0) \le (\mathbb{P}(Y = 0))^2$. Finally, all that remains to prove is $\mathbb{P}(Y > 0) > 0$.

Fix $n > m > 1$ two integers. By Lemma C of [KP], if $h < 1$ is large enough, expression (1) yields

$$Y^h_{b^n} \ge \sum_{a\in A_m} \mu^h_{b^{-n}}(I_a) - (1-h) \sum_{a\ne a'\in A_m} \mu^{\frac{h}{2}}_{b^{-n}}(I_a)\mu^{\frac{h}{2}}_{b^{-n}}(I_{a'}).$$

Moreover, Theorem 5 and the Jensen inequality yield $\mu^h_{b^{-n}}(I_a) \ge f_{a,n,m}(h)$, with

$$f_{a,n,m}(h) = b^{-mh\delta(V-1)} Y^{h-1}_{b^{n-m},I_a} b^{-m(h-1)} \int_{I_a} Q^h_{C_{b^{-m}}(t)}\ \mu^{I_a}_{b^{m-n}}(dt).$$

Write

$$\frac{\mathbb{E}(Y_{b^n}^h) - \sum_{a\in A_m} \mathbb{E}(f_{a,n,m}(h))}{h-1} \le \sum_{a\neq a'\in A_m} \mathbb{E}(\mu_{b^{-n}}^{\frac{h}{2}}(I_a)\mu_{b^{-n}}^{\frac{h}{2}}(I_{a'})).$$

By letting h tend to 0 and by using the fact that $\mathbb{E}(Y_{b^n}) = \sum_{a\in A_m} \mathbb{E}(f_{a,n,m}(1)) = 1$ and Lemma 5, we get $C > 0$ independent of m and n such that

$$\begin{aligned} &m \log(b)\, \tau'(1^-) + \mathbb{E}(Y_{b^n} \log Y_{b^n}) - \mathbb{E}(Y_{b^{n-m}} \log Y_{b^{n-m}}) \\ &\quad \le 5C + C\, b^m\, (\mathbb{E}(Y_{b^{n-m}}^{\frac{1}{2}}))^2. \end{aligned}$$

By the martingale nature of $(Y_{b^n})_{n\ge1}$, $\mathbb{E}(Y_{b^n} \log Y_{b^n}) - \mathbb{E}(Y_{b^{n-m}} \log Y_{b^{n-m}}) \ge 0$. Hence, $m \log(b)\, \tau'(1^-) \le 5C + C\, b^m\, (\mathbb{E}(Y_{b^{n-m}}^{\frac{1}{2}}))^2$. Moreover, as $\tau'(1^-) > 0$, we can choose m to have $m \log(b)\, \tau'(1^-) - 5C > 0$. Consequently $\inf_{n\ge1} \mathbb{E}(Y_{b^n}^{\frac{1}{2}}) > 0$. We conclude as in the proof of Theorem 1 in [KP] for CCM.

Proof of Theorem 1(ii). (i) shows that $\mathbb{P}(\mu \neq 0) > 0$ implies $\mathbb{P}(\mu \neq 0) = 1 = \mathbb{E}(Y)$. Fix $h \in]0,1[$. For all $m > 1$, we have $Y^h \le \sum_{a\in A_m} \tilde{\mu}^h(C_a)$ by (3), and by Lemma $4(ii)$ there exists $C > 0$ such that $\mathbb{E}(Y^h) \le C.b^{-m\tau(h)}\mathbb{E}(Y^h)$ for all $m > 1$. So if Y is non-degenerate then $\tau(h) \le 0$ near 1^- and $\tau'(1^-) \ge 0$ since $\tau(1) = 0$.

Now assume that $\tau'(1^-) = 0$ and $\mathbb{E}((1+W)|\log W|^{2+\gamma}) < \infty$ for some $\gamma > 0$. For every $t \in [0,1]$ and $n \ge 1$, define the measure $\mathbb{P}_{t,n}$ on $F_{b^{-n}}$ by $\frac{d\mathbb{P}_{t,n}}{d\mathbb{P}}(\omega) = Q_{b^{-n}}(t)(\omega)$. By Lemma 2 $(Q_{b^{-n}}(t), F_{b^{-n}})_{n\ge1}$ is a martingale with expectation one. So $\mathbb{P}_t$, the Kolmogorov extension of $(\mathbb{P}_{t,n})_{n\ge1}$ to $\sigma(F_{b^{-n}},\ n\ge1)$, is defined, and $\mathbb{P}_t(\limsup_{n\to\infty} Y_{b^n} = \infty) = 1$ by Lemma 6. This yields $\mathbb{P}(Y = 0) = 1$ by adapting the proof of Theorem $4.1(i)$ of [WaWi] for CCM.

Proof of Theorem 2(i). It suffices to show that $(Y_{3^n})_{n\ge1}$ is bounded in L^h norm.

Number the intervals I_a, $a \in A_m$ (here $b = 3$) as they follow one another from 0 on the real line, and write $\{I_a;\ a\in A_m\} = \{J_i;\ 0\le i < 3^m\}$. Then, for $i \in \{0,1,2\}$ and $n > m$ define

$$Z_{i,n} = \sum_{0\le 3k+i<3^m} \mu_{3^{-n}}(J_{3k+i}).$$

By construction the $Z_{i,n}$'s have the same distribution, so $\mathbb{E}(Y_{3^n}^h) \le 3^h\mathbb{E}(Z_{0,n}^h)$.

Let $\tilde{h}$ be the integer such that $\tilde{h} < h \le \tilde{h}+1$ and use the sub-additivity of $x \mapsto x^{h/(\tilde{h}+1)}$ on $\mathbb{R}_+$ to write $Z_{0,n}^h \le [\sum_{0\le k<3^{m-1}} \mu_{3^{-n}}^{h/(\tilde{h}+1)}(J_{3k})]^{\tilde{h}+1}$ and obtain

$$\mathbb{E}(Y_{3^n}^h) \le 3^h \sum_{0\le k<3^{m-1}} \mathbb{E}(\mu_{3^{-n}}^h(J_{3k})) + 3^h \sum \alpha_{j_0\ldots j_{3^{m-1}-1}} \mathbb{E}(\prod_{0\le k<3^{m-1}} \mu_{3^{-n}}^{j_k\frac{h}{\tilde{h}+1}}(J_{3k})),$$

where in the last sum the j_i's are $\le \tilde h$, $j_0+\cdots+j_{3^{m-1}-1}=\tilde h+1$, $j_i\ge 0$ and $\sum \alpha_{j_0\dots j_{3^{m-1}-1}} = 3^{(m-1)(\tilde h+1)}-3^{m-1}$.

On the one hand, given such a $j_0\dots j_{3^{m-1}-1}$ we have (with the notations of Lemma 4)

$$\prod_{0\le k<3^{m-1}} \mu_{3^{-n}}^{j_k\frac{h}{\tilde h+1}}(J_{3k}) \le \prod_{0\le k<3^{m-1}} (w_1(J_{3k}))^{j_k\frac{h}{\tilde h+1}} \prod_{0\le k<3^{m-1}} Y_{3^{n-m},J_{3k}}^{j_k\frac{h}{\tilde h+1}},$$

where the $Y_{3^{n-m},J_{3k}}$'s are i.i.d. ($T_{J_{3k}}\cap T_{J_{3k'}}=\emptyset$ if $k\ne k'$) and are also independent of $\prod_{0\le k<3^{m-1}}(w_1(J_{3k}))^{j_k\frac{h}{\tilde h+1}}$. Then, Lemma 4$(i)(a)$ and computations similar to those made in the proof of Theorem 2 of [KP] yield a constant $C_h>0$ (independent of m and n) such that

$$\mathbb{E}\left(\prod_{0\le k<3^{m-1}} \mu_{3^{-n}}^{j_k\frac{h}{\tilde h+1}}(J_{3k})\right) \le C_h 3^{-m(\tau(h)+1)}\mathbb{E}(Y_{3^{n-m}}^{\tilde h})^{h/\tilde h}.$$

On the other hand,

$$3^h\sum_{0\le k<3^{m-1}} \mathbb{E}(\mu_{3^{-n}}^h(J_{3k})) \le 3^{h-1}C_h 3^{-m\tau(h)}\mathbb{E}(Y_{3^n}^h)$$

by Lemma 4$(i)(b)$ and the submartingale property of $(Y_{3^n}^h)_{n\ge1}$. Since for a fixed m large enough we have $3^{h-1}C_h3^{-m\tau(h)}<1$ ($\tau(h)>0$), we conclude that $\sup_{n\ge1}\mathbb{E}(Y_{3^n}^h)<\infty$ by induction on $\tilde h$, as in the proof of Theorem 2 in [KP].

Proof of Theorem 2(ii). Fix an integer $b\ge2$. By letting n tend to ∞ in Lemma 4$(i)(c)$ we get $\mathbb{E}(Y^h)\ge b^{-m\tau(h)}e^{\frac{\delta}{2}(1-b^{-m})(h(V-1)-(\mathbb{E}(W^h)-1))}\mathbb{E}(Y^h)$ for all $m\ge1$. This yields $\tau(h)\ge0$.

Proof of Proposition 1. (i) Due to Theorem 2 and the concavity of the function τ, the divergence of high moments holds if and only if $\lim_{h\to\infty}\tau(h)=-\infty$. If $\mathbb{P}(W>1)>0$ it is immediate that $\lim_{h\to\infty}\tau(h)=-\infty$. If $\mathbb{P}(W\le1)=1$ then $\delta(\mathbb{E}(W^h)-1)$ is bounded over $\mathbb{R}_+$ and $\lim_{h\to\infty}\tau(h)=-\infty$ if and only if $1+\delta(\mathbb{E}(W)-1)<0$.
(ii) See Theorem 3 in [KP].

Proof of Theorem 3. If $\mathbb{E}(W^{-a})<\infty$ then $\mathbb{E}(Y^{-a})<\infty$: write (3) with $b=4$ and $m=1$ and define $B_i=4^{-\delta(V-1)-1}Q_{T^{I_i}}m_{B^{I_i},I_i}$ for $i\in\{0,3\}$ (with the notations preceding Lemma 3). We have

$$Y\ge B_0Y_{I_0}+B_3Y_{I_3}$$

where $Y_{I_0}\sim Y_{I_3}\sim Y$, and Y_{I_0}, Y_{I_3} and (B_0,B_3) are mutually independent. Moreover $\mathbb{E}(B_0^{-a})<\infty$ (use Lemmas 1 and 3) and $B_0\sim B_3$. Consequently the approach [Mol] uses for generalized CCM yields $\mathbb{E}(Y^{-a})<\infty$.
Conversely, by using (3) with $b=2$ and $m=1$ we get

$$Y\le 2^{-\delta(V-1)}Q_{T^{I_0}\cap T^{I_1}}[Q_{T^{I_0}\setminus T^{I_1}}M_{B^{I_0},I_0}+Q_{T^{I_1}\setminus T^{I_0}}M_{B^{I_1},I_1}](Y_{I_0}+Y_{I_1}),$$

the random variables $Q_{T^{I_0}\cap T^{I_1}}$, $[Q_{T^{I_0}\setminus T^{I_1}}M_{B^{I_0},I_0}+Q_{T^{I_1}\setminus T^{I_0}}M_{B^{I_1},I_1}]$ and $Y_{I_0}+Y_{I_1}$ being mutually independent. Hence, $\mathbb{E}(Y^{-a})<\infty$ yields $\mathbb{E}[(Q_{T^{I_0}\cap T^{I_1}})^{-a}]<\infty$ and as $\Lambda(T^{I_0}\cap T^{I_1})>0$, Lemma 1$(i)$ gives the conclusion.

Proof of Theorem 4. Theorem 4 is a consequence of Proposition 2.2(a) of [F] and the following Propositions 2–4.

For $q\in J'$, let μ_q be the measure obtained as μ by replacing the W_M's by the W_M^q's.

Proposition 2. *With probability one: (i) the measures μ_q, $q\in J'$, are defined simultaneously and have $[0,1]$ as support; (ii) for every $q\in J'$, for μ_q-almost every $t\in[0,1]$, $\liminf_{r\to 0}\dfrac{\log\,\mu_q(I_r(t))}{\log r}\geq\tau^*(\tau'(q))$ and $\lim_{r\to 0}\dfrac{\log\,\mu(I_r(t))}{\log r}=\tau'(q)$.*

Proof. (i) Direct consequence of Lemma 7(i) since $\mu_q=\tilde{\mu}_q\circ\pi^{-1}$.
(ii) Result on $\liminf_{r\to0}\frac{\log\,\mu_q(I_r(t))}{\log r}$: fix an integer $b\geq 2$ and for $\varepsilon>0$, $q\in J'$ and $n\geq 1$ define

$$F_{q,n,\varepsilon}=\{t\in[0,1];\ \frac{\log\,\mu_q(I_{b^{-n}}(t))}{\log b^{-n}}\leq\tau^*(\tau'(q))-\varepsilon\}.$$

It suffices to show the property $(\mathcal{P}')$: for every $\varepsilon>0$, a.s. for every $q\in J'$, $\sum_{n\geq1}\mu_q(F_{q,n,\varepsilon})<\infty$.

From the covering $\bigcup_{t\in F_{q,n,\varepsilon}}I_{b^{-n}}(t)$ of $F_{q,n,\varepsilon}$, we extract two finite unions of intervals, namely $\bigcup_i J_i$ and $\bigcup_j J_j'$, so that two distinct J_i's or J_j''s have at most one point in common, and $F_{q,n,\varepsilon}\subset\bigcup_i J_i\cup\bigcup_j J_j'$.

Then, since $1\leq\mu_q^\eta(I)b^{n\eta(\tau^*(\tau'(q))-\varepsilon)}$ when $I\in\{J_i;\,J_j'\}$, for $\eta>0$ we have

$$\mu_q(F_{q,n,\varepsilon})\leq\sum_i\mu_q^{1+\eta}(J_i)b^{n\eta(\tau^*(\tau'(q))-\varepsilon)}+\sum_j\mu_q^{1+\eta}(J_j')b^{n\eta(\tau^*(\tau'(q))-\varepsilon)}.\quad(10)$$

Moreover for every $I\in\{J_i,J_j';\,i,j\}$ we have $I\subset I_a\cup I_{a'}$ for some a and $a'\in A_n$, and consequently $\mu_q^{1+\eta}(I)\leq 2^\eta(\mu_q^{1+\eta}(I_a)+\mu_q^{1+\eta}(I_{a'}))$. So we deduce from (10) that if $\eta\leq1$ then

$$\mu(F_{q,n,\varepsilon})\leq 8\sum_{a\in A_n}\mu_q^{1+\eta}(I_a)b^{n\eta(\tau^*(\tau'(q))-\varepsilon)}.$$

Since $\mu_q(I_a)=\tilde{\mu}_q(C_a)$ for every $a\in A$ ($\mu_q=\tilde{\mu}_q\circ\pi^{-1}$ and $\tilde{\mu}_q$ has no atoms by Lemma 7(ii)), $(\mathcal{P}')$ comes from the proof of Lemma 7(ii).

Result concerning $\lim_{r\to0}\frac{\log\,\mu(I_r(t))}{\log r}$: define

$$\begin{cases}F_{q,n,\varepsilon}^{-1}=\{t\in[0,1];\ \dfrac{\log\,\mu(I_{b^{-n}}(t))}{-n\log b}\geq\tau'(q)+\varepsilon\}\\ F_{q,n,\varepsilon}^{1}=\{t\in[0,1];\ \dfrac{\log\,\mu(I_{b^{-n}}(t))}{-n\log b}\leq\tau'(q)-\varepsilon\}.\end{cases}$$

It suffices to show $(\mathcal{P}'')$: for every $\varepsilon > 0$ a. s. for every $q \in J'$, $\sum_{n\geq 1} \mu_q(F^{-1}_{q,n,\varepsilon}) + \mu_q(F^{1}_{q,n,\varepsilon}) < \infty$.

The sets $F^{-1}_{q,n,\varepsilon}$ and $F^{1}_{q,n,\varepsilon}$ admit the same kind of covering as the one used for $F_{q,n,\varepsilon}$, and for $\eta > 0$ and $\gamma \in \{-1, 1\}$

$$\mu_q(F^{\gamma}_{q,n,\varepsilon}) \leq \sum_i \mu_q(J_i)\mu^{\gamma\eta}(J_i)b^{n\gamma\eta(\tau'(q)-\gamma\varepsilon)} + \sum_j \mu_q(J'_j)\mu^{\gamma\eta}(J'_j)b^{n\gamma\eta(\tau'(q)-\gamma\varepsilon)}.$$

Therefore if $\eta \in]0, 1[$ we get

$$\begin{cases} \mu_q(F^{1}_{q,n,\varepsilon}) \leq 4b^{n\eta(\tau'(q)-\varepsilon)} \displaystyle\sum_{a\in A_n} \mu_q(I_a) \sum_{c\in A_n;\ I_a\cap I_c\neq\emptyset} \mu^{\eta}(I_c) \\ \mu_q(F^{-1}_{q,n,\varepsilon}) \leq 2b^{-n\eta(\tau'(q)+\varepsilon)} \displaystyle\sum_{\bar a\in A_{n+1}} \mu^{-\eta}(I_{\bar a}) \sum_{c\in A_{n+1};\ I_{\bar a|n}\cap I_{c|n}\neq\emptyset} \mu_q(I_c) \end{cases}$$

since, for every I in these coverings, we have $I_{\bar a} \subset I \subset I_a \cup I_{a'}$ for some $a, a' \in A_n$ and $\bar a \in A_{n+1}$. Then $(\mathcal{P}'')$ comes from computations very similar to those needed for the proof of Lemma 7(ii), by using the additional remark: $\sup_{a,c\in A_n;\ I_a\cap I_c\neq\emptyset} \frac{\Lambda(T^{I_a})}{\Lambda(T^{I_a}\cap T^{I_c})}$ tends to 1 as n tends to ∞.

Proposition 3. *Let b be an integer ≥ 2. For $(q, t) \in \mathbb{R}^2$, define*
$C_b(q, t) = \limsup_{n\to\infty} C_{b,n}(q, t) = \sum_{a\in A_n} \mu^q(I_a)|I_a|^t$ *and*
$C(q, t) = \lim_{\delta\to 0} \inf\{\sum_{i\geq 1} \mu^q(I_i)|I_i|^t; [0, 1] \subset \bigcup_{i\geq 1} I_{r_i}(t_i), t_i \in [0, 1], |r_i| \leq \delta\}$.
(i) *For all $q \in \mathbb{R}$, $\varphi_b(q) = \inf\{t \in \mathbb{R};\ C_b(q, t) = 0\}$ and $\varphi(q) = \inf\{t \in \mathbb{R};\ C(q, t) = 0\}$ are defined, the function φ_b is convex and $\varphi \leq \varphi_b$.*
(ii) *Fix $\alpha > 0$. If $(-\varphi)^*(\alpha) \geq 0$ then $\dim_H E_\alpha \leq (-\varphi)^*(\alpha)$ else $E_\alpha = \emptyset$.*

This Proposition is deduced from [BMP] and [O].

Proposition 4. *With probability one, $(-\varphi)^*(\alpha) \leq \tau^*(\alpha)$ for every $\alpha \in I'$.*

Proof. It adapts the beginning of the proof of Theorem VI.A.a in [B1].

Fix $q \in J'$. By using Lemma 4$(ii)(b)$ with $\mu(I_a)$ instead of $\tilde\mu(C_a)$ (μ has no atoms by Proposition 2) we get $C_q > 0$ such that for every $n \geq 1$ and $t \in \mathbb{R}$

$$\mathbb{E}(C_{b,n}(q, t)) \leq C_q b^{-n(\tau(q)+t)}\mathbb{E}(Y^q). \tag{11}$$

Moreover $\mathbb{E}(Y^q) < \infty$ by Theorem 2 (resp. 3) if $q \geq 0$ (resp. $q < 0$). It follows from (11) that for every $t > -\tau(q)$, $C_b(q, t) = 0$ a.s., and by definition of $\varphi_b(q)$ we get $\varphi_b(q) \leq -\tau(q)$ a.s.

Since τ is continuous on J' and φ_b is by definition almost surely continuous, we obtained more: a.s. for every $q \in J'$, $\varphi_b(q) \leq -\tau(q)$, so by Proposition 3$(i)$, a.s. for every $q \in J'$, $-\varphi(q) \geq \tau(q)$. The conclusion follows by taking the Legendre transforms $(-\varphi)^*$ and τ^* on the previous inequality.

Acknowledgements. The authors thank J. Peyrière and R. Riedi for valuable discussions. They are also indebted to the referee; his remarks and advice considerably improved this text.

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