Infinite beauty

Fractals and Chaos: The Mandelbrot Set and Beyond
By Benoit B. Mandelbrot
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Kenneth Falconer

It was just over 20 years ago that the Mandelbrot set took the world by storm. Pictures of extraordinary complexity and beauty appeared in scientific and glossy magazines, on the walls of art galleries and classrooms, on posters and even on tablemats. With the increasing availability of personal computers, drawing the Mandelbrot set became a standard exercise for those learning programming, and it was frequently an addiction for computer buffs, who were able to explore its intricacy by forever homing in on parts of the structure. Perhaps it is not surprising that such a simple procedure enabling almost anyone to produce an object of immense sophistication and attractiveness caught the public imagination.

The definition of the Mandelbrot set, denoted by M, is indeed extremely simple. Given a complex number c, start at the origin 0 and follow the trail of points obtained by repeatedly applying the transformation $f(z) = z^2 + c$, that is, the sequence 0, c, $c^2 + c$, $(c^2 + c)^2 + c$... If these points never go far away from the origin then c is in M, but if they wander off to infinity, c is not in M. This straightforward check allows one to scan across a region of the complex plane to determine the extent of M.

Crude pictures of M show a main cardioid surrounded by circular 'buds' of decreasing size. But more detailed investigation, pioneered by Benoit Mandelbrot in 1980, reveals much, much more: the buds are all surrounded by smaller buds, which in turn support even smaller ones, and so on. Homing in on the boundary of M reveals a menagerie of multi-branched spirals, dragons and seahorses. Hairs of imperceptible fineness extend from the buds, holding along their lengths minute replicas of the entire Mandelbrot set.

Is the Mandelbrot set just a pretty curiosity? Far from it. It is a fundamental parameter set that encodes an enormous amount of information about nonlinear processes. First, the position of a complex number c relative to M tells us a great deal about the iteration of the quadratic mapping $f(z) = z^2 + c$. The (filled-in) 'Julia set' at c consists of those complex numbers z whose iterates under repeated application of $f(z) = z^2 + c$ never wander far from the origin. This Julia set comprises a single piece precisely when c lies in the Mandelbrot set. (Interestingly, this topological dichotomy was noted by Pierre Fatou and Gaston Julia in 1918–19, but it was many years before its real significance and delicacy was appreciated.)

Much more than this, the exact position of c in M, such as the bud in which it lies, gives a very full description of how the iterates of $f(z)$ behave: for example, whether there are periodic cycles. Even more surprising is that, although defined in terms of the simplest of nonlinear maps, $f(z) = z^2 + c$, the Mandelbrot set is 'universal' in that it underlies the behaviour of very large classes of more complicated nonlinear mappings, the likes of which crop up throughout modern mathematics and its applications.

The emergence of the Mandelbrot set in 1980 led to a flurry of activity among mathematicians trying to understand its structure and significance, resulting in some of the most impressive advances in pure mathematics in recent years. In 1982, Adrien Douady and John Hubbard proved the (far from trivial) fact that M is connected, though it is still unknown whether M is locally connected — can you travel between nearby points of M staying inside M without making too long a detour? In 1998, Mitsuhiro Shishikura showed that the boundary of M has fractal dimension 2, which means that it is just about as complicated as can be, though it is still not known whether this boundary has positive area.

This is the fourth volume of Mandelbrot's *Selecta*, comprising edited reprints of the author's papers. Largely from the 1980s, these include the series of seminal papers that revealed the magnificence and omnipresence of the Mandelbrot set, together with other papers related to the iteration of functions. Several sections provide an overview of the work along with its scientific and historical background. One chapter has been written specifically to help the non-expert appreciate the rest of the book.

Much of the material does not require particularly technical knowledge, so the book should be accessible to a wide readership. It provides a fascinating insight into the author's journey of seeing and discovering as the early pictures of the Mandelbrot set started to reveal a whole new world. It gives a feeling for his philosophy and approach of experimental mathematics — an approach that has changed the way we think about mathematics and science. ■

Kenneth Falconer is a professor of pure mathematics at the School of Mathematics and Statistics, University of St Andrews, North Haugh, St Andrews, Fife KY16 9SS, UK.