Preface

THIS BOOK'S SUBJECT MATTER IS NONTRADITIONAL and its style and organization are unconventional.

While the subject matter is tightly interconnected, it addresses several old and new concrete problems and combines several old and new lines of abstract thought. The goals Dirac assigned to theoretical science, namely, to remove inconsistencies and unite theories that were previously disjointed, have been observed here. The book is summarized by the sequence of introductions (printed in italics) that precede each Part. Several often-raised general issues relevant to this book, and to fractal and multifractal geometry, are discussed in the Overview chapter that follows.

Lessening the burden placed on the index, this Preface uses ***bold italic*** letters to draw attention to selected occurrences of key words. The reference style is explained on the first page of the Bibliography. For example, my book-length "Essays" on fractals are denoted as M 1975O, M 1977F, and M 1982F; the third one is also denoted as *FGN* and a new edition is expected in 2002. Also, many phrases of the form "model presented in M 1972j{N14}" are often shortened to "M 1972 model."

Informally, fractals are irregular shapes, in either mathematics or the real world, wherein each small part is very much like a reduced-size image of the whole. The best one-word characterization of the scope of fractal geometry, as of today, is described in Section 1 of the Overview. Fractal geometry is the first step towards *rational roughness*, in other words, the first ever scientific approach to Man's sensation of rough versus smooth.

Roughness is ubiquitous in nature and culture – that is, the works of Man. This suffices to account for the surprising diversity of the uses of fractals. The fact that roughness was late in being tackled is clearly due to the high level of complexity along this new frontier of science.

The many geometric constructions considered in this book exhibit a constant interplay between ***abstraction*** and ***realism***. Some – as of now – are mostly of mathematical interest. Others are multipurpose investigations that range from the ***construction of widely usable tools,*** to ***mathematical theorems*** and ***conjectures***, to intuition-building ***illustrations***. Throughout, the major motivation is to provide ***models of observable phenomena.*** Most concern the physical world, but a few concern the man-made structures of ***financial markets*** and ***written discourse.***

The term, ***Gaussian fractal***, denotes any geometric fractal shape generated by a Gaussian random process. The term ***cartoon*** mostly denotes shapes that share many of the properties of Gaussian fractals but are simple limits of recursive constructions that translate directly into computer programs. The "cartoons" involve a basis, or a pattern of subdivision, that is prescribed in advance but has no concrete significance.

This book's cover and jacket follow the lead of M 1982F{*FGN*} and illustrate both the abstract and the realistic aspects. The back-cover illustrations include a "signature" Mandelbrot set and an "angel" that is the graph of a Weierstrass function. The abstract front-cover illustration is a new construct introduced in Chapter H3, namely, a form of ***fractional Brownian cluster***. It announces that much of this volume discusses the fractal sets related to the ***fractional Brownian motions***, FBM, denoted by $B_H(t)$, where the exponent satisfies $0 < H < 1$. In the key case $H = 1/2$, FBM reduces to WBM. This Preface will shortly elaborate on FBM.

The lower portion of the front-cover illustration is meant to alert the reader to the fact that much of this book builds up toward the representation of ***Earth's reliefs*** by ***self-affine fractional Brownian surfaces.*** Those thoroughly artificial models imitate real reliefs and contribute to ***geomorphology.*** The word ***Earth*** in the subtitle also refers to extensive discussions of ***geophysical*** phenomena such as ***river discharges***.

The word ***R/S*** in the subtitle refers to a new statistical technique that shows promise in the study of FBM and many other globally dependent processes and phenomena.

The word ***1/f*** in the subtitle, due to electrical engineers and adopted by physicists, is meant to alert the reader to close connections with the study of certain prevalent and troublesome fluctuations, metaphorically called ***noises***, that range from ***electricity*** to ***human disease.*** A ***1/f noise*** – stated more carefully, a f^{-B} *noise* of exponent B not too far from 1 – is any fluctuation with a Fourier spectral density given in terms of the frequency f by the power law f^{-B}. Those noises, including the multifractals (which are a special case), are studied in M 1999N. Combining the challenge and the good news: (a) $1/f$ noises exemplify random or nonrandom variability that is wild rather than mild; and (b) they belong to the broad and unified mathematical notion of *self-affine fractal variation*. An aspect of the gustiness of the wind having eluded a primitive "uniscaling" $1/f$ noise, I took the step to "multiscaling" self-affinity by introducing the multifractals.

This book's organization will now be described. Like the dress worn by a traditional English bride, it combines something old, something new, something borrowed, and something blue.

The *new,* Parts I and II, consist of specially written chapters whose numbers are starred. While all were originally meant to be expository, Part I triggered ***substantial new research***. For example, the study of random walk (RW) in Chapter H3 reports on a surprising new visual observation leading to a new ***mathematical conjecture*** that involves an exponent 5/3, M 1982F{*FGN*} conjectured that the Brownian motion's boundary is of dimension 4/3 became classic and was proven in 2000.

The *old,* Parts III to VII, consist of ***reprints*** of every paper that I authored or co-authored on the topic between 1965 and 1986. A fresh reading after 20 to 35 years showed that overlaps were minimal and wholesale repeats nonexistent, even when the originals appeared in hard-to-find journals or Proceedings. Some of those texts have never been rediscovered and/or investigated seriously, therefore many ideas that they include are crying out for new research. Furthermore, there are new annotations, and/or appendices, and the different parts, as well as many chapters, are preceded by forewords. Corrections and afterthoughts that seemed necessary are set out in curly brackets { } and begin with the anachronistic mention "P.S. 2000."

The *borrowings* are the contributions of several co-authors.

As to the *blue,* I teach at Yale, worked for IBM for half of my life, and started long ago at the Paris Laboratory of Philips of Eindhoven. Those institutions are known for three different shades of a single color.

This is one of several books described together as ***Selecta,*** a Latin word meaning *selected things,* and an old-fashioned way to refer to an author's

Selected Papers. *By their very nature,* Selecta *are neither monographs, nor treatises, nor textbooks.* In chronological sequence, this book takes place after M 1997E and M 1999N and before the planned M 2001C and M 2001T.

Reviewing M 1997E, Goldenfeld 1998 called it "a characteristically idiosyncratic work. At once a compendium of ... pioneering work and a sampling of new results, the presentation seems modeled on the brilliant avant-garde film *Last Year in Marienbad,* in which the usual flow of time is suspended, and the plot is gradually revealed by numerous but slightly different repetitions of a few underlying events ... [This] presentation allows the reader unusual freedom of choice in the order in which the book is read." The repeats are more than "slightly different," but these comments also apply to this book.

My first scientific paper came out on April 30, 1951, and in due time the accumulation of diverse works created two needs. My book-length Essays elucidated the strong unity of purpose that had always directed my work, though it developed slowly. However, the more technical material was necessarily left out. The *Mathematical Backup,* found in Chapter 39 of M 1982F{*FGN*}, was rushed and sketchy and its abundant references were hard to trace and relate to one another. Placing these belated technical *Selecta* next to the book-length Essays was meant to establish a balance between overall thrust and technical detail.

Each *Selecta* volume evolved in its own special way and they are not numbered, but denoted by letters. This is Volume H, because an essential exponent denoted by H is deeply rooted in two fields of knowledge that were thoroughly removed from each other until fractal geometry spanned the abyss between them: old mathematics due to L. Hölder and more recent experimental findings due to the hydrologist H.E. Hurst.

The relations between those books are complex; therefore, special efforts were made to include cross-references. In Chapter H1 those relations are made visual with the help of a "map," namely, the "phase diagram" provided by Figures 2 and 3. It associates the main concerns of the first three *Selecta* volumes with different "loci" belonging to a square. The overambitious Chapters E6 and N1 were described as "Panorama;" Chapter H1 is a more focused "close-up."

Volume E, *Fractals in Finance; Discontinuity, Concentration, Risk*, includes a detailed discussion of the concept of ***states of randomness***, from ***mild*** to ***wild***. The subtle connections between Volumes E and H are discussed in the long Foreword of Chapter H30.

Volume N, *Multifractals & 1/f Noise*, shares with this volume a common concern with *1/f noises,* but focuses on different forms of these noises.

The prototype Gaussian fractals are generated by the original ***Brownian motion***, for which Wiener provided the mathematical theory. It will be abbreviated to WBM and denoted by $B(t)$. Both t and B can be scalar, complex, or vectorial. The resulting geometric shapes include surfaces, curves, and "dusts" of points. The most thoroughly studied Gaussian fractals are: (a) the ***record*** of the scalar $B(t)$; (b) the ***zeroset*** of $B(t)$; (c) the ***graph*** of a complex function of time whose real and imaginary coordinates are independent WBM $B_R(t)$ and $B_I(t)$; and (d) the ***trail*** defined as the complex plane curve with coordinates $B_R(t)$ and $B_I(t)$, or more generally defined in E-dimensional Euclidean space of the E-dimensional $B(t)$, after the times of transit have been erased.

Less classical are scalar Brownian bridges and the scalar functions $B(t)$ of a multidimensional time t. The new Chapter H3 in this book is wholly devoted to Wiener Brownian Gaussian fractals and includes recent topics like Brownian and random walk clusters (RWC).

The need for constructions beyond WBM arose independently in finance and physics. It is in finance that WBM was first introduced, in Bachelier 1900. This was a splendid innovation well ahead of its time, but Louis Bachelier himself soon realized that as a model of the variation of prices, WBM fails miserably on all counts. In physics, where it was rediscovered around 1905, WBM triumphed in the study of equilibrium, but proved far off the mark in the study of turbulence and $1/f$ noises.

Taking account of those failures, much of my scientific work was devoted to providing alternatives to WBM, first in finance, then largely in physics and recently again in finance. M 1997E and M 1999N abandoned Gaussianity in several distinct ways. This book largely preserves Gaussianity. In mathematical terms, once again, the major topic is the geometric study of FBM.

To fulfill their intended purpose, FBM and other alternatives to WBM invariably present novelties and severe complications that used to be viewed as ***anomalous, unnatural,*** even ***pathological.***

Therefore, I shall argue that *much in nature is ruled by what used to be called pathology.* This theme is amplified in Section 3.1 of the Overview.

In a related technical sense described in M 1997E, and sketched in Sections 3.2 and 3.3 of the Overview, my proposed alternatives to WBM are invariably ***wild*** and (a word included in the subtitle) ***global.*** These features always bring a sharp increase in difficulty, therefore are best first faced under optimal conditions. A partly counteracting piece of good news is provided by a surprisingly ubiquitous empirical finding: in the natural rough phenomena that we shall examine, the wildness is *not* as general as it might have been; the "pathology" of those particular aspects of Nature is *not* unmanageable. This is so because it obeys a form of invariance or symmetry that overlaps Nature and mathematics, and is called ***scale invariance,*** or ***scaling,*** that is central to my life work. The more specific term ***self-affinity,*** first used in M 1977F and M 1982F{*FGN*}, expresses invariance under some linear reductions and dilations, which ordinarily implies *uniformly global statistical dependence*. *Self-affinity* takes at least three distinct forms: *unifractality,* examined in this book, *mesofractality*, examined in M 1997E, and *multifractality*, examined in M 1999N.

The far better known notion of ***self-similarity*** is the special case corresponding to isotropic reductions. A perspicuous illustration of both invariances is provided by the widely known fractal computer landscapes pioneered in M 1975O and M 1982F{*FGN*}, Chapter 28. There, the lake and island coastlines are self-similar, while the relief itself is self-affine. That is, the Gaussian records that represent reliefs are invariant under linear contractions whose ratios are different along the axes of the free variables and the axis of the function. Coastlines are iso-surfaces of reliefs: invariant by isotropic contractions.

The progression from self-similarity to self-affinity is by no means automatic, quite to the contrary. It brings in a large number of novel features explored in this book. In particular, the notion of ***fractal dimension*** undergoes a very extensive generalization and becomes less directly compelling than in self-similar contexts.

The general form of roughness combines wildness with failure of self-affinity. I chose to postpone their study, in a loose analogy with the fact that the theory of heat began with bodies of uniform temperature.

A technical study of self-affinity introduces a formalism made of ***scaling laws, renormalizations, and fixed points.*** I developed this apparatus in 1960–63 for the needs of my early work in finance (reprinted in M 1997E), and Berger & M 1963{N6} immediately applied it to physics. Starting around 1965, the same apparatus – with considerable extensions that only physics allows – was developed quite independently in the theory of the critical phenomena of physics; it succeeded brilliantly, as is recalled in M 1999N, Chapter N3.

Faced with phenomena I view as self-affine, economists, "hard scientists," and engineers from diverse fields, often begin by contrasting alternating periods of quiescence and activity. Examples of such contrasts are: turbulent flow versus laminar inserts, error-prone periods in communication versus error-free periods, and periods of orderly versus agitated ("quiet" and "turbulent") Stock Market activity. Such subdivisions are widely accepted without obvious mutual consultation, hence must be natural to human thinking. Descartes endorsed them by recommending that every difficulty be decomposed into parts to be handled separately.

However, the great success of such subdivisions in the past does not guarantee their continuing success. Successful past investigations concerned a form of variability and randomness that Section 3.2 of the Overview characterizes as very special, namely, as mild and local. In every field where variability/randomness is wild, my view is that such subdivisions hide the important facts, and cannot provide understanding. My alternative to this technique is to move to procedures centered on scaling.

Acknowledgments. Original ***acknowledgments*** printed after each paper were preserved. Warm thanks go to the coauthors of the joint papers for permission to reprint them.

The earliest fractal landscapes, those published in M 1975w{H19} and M 1975O, were drawn under my close supervision by S.W. Handelman.

The landscapes in the next batch – M 1977F and M 1982F{*FGN*} – are due to R.F. Voss and represented a major breakthrough in computer graphics. The greatest form of success – acceptance into the mainstream – is often synonymous with anonymity. Having invented and implemented many basic techniques for this purpose, Voss published only scant documentation. Today, relief forgeries are routine in computer graphics teaching, therefore many Madison Avenue masterpieces include a fractal background. Besides, anyone can draw them on their laptop computer.

Voss's pictures in M 1977F inspired L. Carpenter, the leader of a group working at Lucasfilms on *Star Trek II: The Wrath of Khan*. This inspiration involved two uncanny transfers of knowledge: from esoteric mathematical treatises to the deep subtleties that burdened M 1975w{H19}, and on to the unforgiving simplicity of Hollywood. This episode brought durable wonder and awe (and transient reasons for serious irritation).

The long–drawn actual preparation of this book was performed by several long–term secretaries, L. Vasta, P. Kumar, K. Tetrault, and C. McCarthy. In a late stage of preparation, M.L. Frame undertook the formidable task of commenting on much of this book with a sharp pencil.

Several short-term assistants were of great help. Clumsy English having made some old papers disappointingly difficult to read, the reprints were copy-edited by diverse hands, in particular, by H. Muller-Landau and N. Eisenkraft. Of course, extreme care was taken to never modify the meaning; the originals are available in libraries and may be provided, if need arises, by the author. To make those reprints useful, the notation and terminology were systematically unified and updated; most significantly, *self-affine* was substituted, when appropriate, for the original *self-similar*.

For over 35 years, the Thomas J. Watson Research Center of the International Business Machines Corporation provided a unique haven for mavericks and a variety of investigations that science and society forcefully demanded but academia neither welcomed nor rewarded. I worked at that haven through most of that period, and completed this book there as IBM Fellow Emeritus. This may be one of the last projects typeset on IBM's pioneering *Script* word-processing language under VM.

I am deeply indebted to the Yorktown of its scientific heyday. Among long-term friends and colleagues, it will remain most closely associated with Richard F. Voss, Martin Gutzwiller, Rolf Landauer (1927–1999), and Philip E. Seiden (1934–2001). Voss came to IBM in 1975 when most of the papers reprinted in this book were already out, lured by a shared hope of bringing together his physicist's view of $1/f$ noise and my geometer's view. Our collaboration took a very different tack, perhaps even more productive, and continued after IBM.

As to management, at a time when the old papers in this book were being written and my work was widely perceived as a wild gamble, it received whole-hearted support from Ralph E. Gomory, to whom I reported in his successive capacities as Group Manager, Department Director, and IBM Director of Research and Senior Vice-President. Gomory reminisced on the old times in the *Foreword* he wrote for M 1997E.

Last but not least, this book is dedicated to Aliette, my wife. Neither its parts nor the whole would have been written without her constant and extremely active participation and her unfailingly enthusiastic support.

Benoit B. Mandelbrot

New Haven, CT & Yorktown Heights, NY