PART I: ADVANCES IN OLD BUT OPEN TOPICS

This book was planned as a collection of reprints preceded – in a specially written Part I – by a few introductory chapters. However, those introductions were gradually displaced into Parts II and III. The reason is that several topics ordinarily viewed as quite mature proved to have unexpected new facets worth mentioning. Making room for those results produced a Selecta *volume that is more than the sum of reprints of papers published between 1965 and 1985. Similar newly written chapters were scattered around* Fractals and Scaling in Finance, M 1997E. *Unfortunately, novelty not being expected in a book of* Selecta, *it was overshadowed by the reprinted material. Choosing the opposite strategy, this Part I brings together what is newest in this book.*

Chapter H1 began as a bland "directory" to three Selecta *volumes, then introduced a powerfully unifying "phase diagram," and a discussion of "globality." It is an "after-thought" addition to M 1997E and M 1999N.*

Chapter H2 began as a bland tutorial; it ended by giving an anything-but-bland view of the still mysterious notion of self-affinity.

Chapter H3 began as a tutorial on Brownian motion and its kin. Of the latter, the Brownian hull has lately attracted much attention; its dimension is 4/3. Least well known are the "trig-fractional Brownian clusters," an example of which is seen on this book's cover and jacket. However, this chapter ended being dominated by the "web of the random walk cluster," a new concept to be explained. Computer experiments lead to the novel conjecture that this web's gap/mass fractal dimension is 5/3, in a sense that cries out to be elaborated.

Chapter H4, began as a bland reprint; it ended by introducing two novelties, Weierstrass functions that are nonchiral and Weierstrass clusters.

Chapter H5 presents R/S and distinguishes between local and global statistical dependence; it ended by introducing a distinction – of not yet proven usefulness – between three distinct "states of diffusion."

Chapter H6, on logarithmic time, tackles a less important topic.

It may be added that when hand diagrams were redrawn by computer, M & Van Ness 1968 demanded elaboration; see Appendix to Chapter H11.

H1

Self-affinity: close-up on a versatile family

• ***Chapter foreword.*** The notion of self-affinity is surprisingly subtle and different reprints in this book involve several of its multiple facets. An introductory presentation of self-affinity, was, therefore, split into this chapter and the next. An abandoned over-ambitious plan envisioned a "Panorama" expanding on Chapters E6 of M 1997E and N1 of M 1999N, which were both too broad and too concise. Instead of seeking breadth, this chapter selects a deliberately narrow family of self-affine functions that can only be viewed as "cartoons" but allow a detailed, "close-up," investigation. The functions in this family can be mapped into a multilayered "phase diagram" that fits in a square. One of those layers has the virtue of providing this and two earlier *Selecta* volumes with a "hieroglyphic" table of contents.

To peek forward, Chapter H2 will describe diverse other forms of self-affinity. This will reveal that this chapter deals with a very special example of the broader notion of "tile self-affinity." But this special example is valuable. Had it been discussed in M 1997E and M 1999N, those works' goals would have been clearer. After the fact, this chapter can act as a preview of those earlier *Selecta* books. This is one reason why it is written in terms of the variation of prices. •

✦ **Abstract.** Of the many forms of self-affinity to be listed in Chapter H2, the simplest is *tile self-affinity*. When a tile self-affine set is the graph of a function $f(t)$, its construction is recursive: starting with an "initiator" that is an interval and using a "generator" that is a broken line made of b "sticks," b being called the *base*. The smallest base that allows $f(t)$ to be either monotone or oscillating is $b=3$.

FIGURE C1-1. Constructing a “Fickian cartoon” of cumulative coin-tossing. A linear trend (“the initiator”) is repeatedly broken up by following a prescribed “generator.” The generator is inverted for each descending piece.

Specializing further, this chapter studies 3-interval generators that are symmetric with respect to the initiator's midpoint. Their iteration is illustrated in Figure 1.

They were singled out after many false starts, because they can be parametrized by a single point, namely the first break in the generator. This break defines an "address-point" P in the "phase diagram" that is drawn within the half–square $\{0 < x \quad 1/2; 0 < y < 1\}$. Thus, this family of generators is simple and compact, but will prove to be very versatile.

This chapter was organized to fit several purposes. Section 1 is a dry detailed summary that will suffice for some readers. Other readers may use it as a concluding summary. ✦

THE CONCEPT OF SELF-AFFINITY IS DELICATE AND NOT UNIQUE, especially as applied to random sets and related functions. This concept is encountered in this book again and again, along different paths that correspond either to different present goals or to different phases of past work. Chapter H2 uses a wide focus to emphasize the diversity of implementations of self-affinity, that concept's open-endedness, and the conditions under which it is most suitable for randomization.

This chapter, on the contrary, focuses narrowly on self-affinities obtained by recursions whose generator is symmetric and made of three intervals. This is a very special family of constructions for which Chapter H2 will propose the (newly coined) term, "tile self-affinities."

For many purposes, however, this very restricted family proves to be sufficiently versatile. In particular, it includes – hence relates to one another – special examples that provide "cartoons" of almost every basic fractal or multifractal behavior examined in M 1997E, M 1999N, this book, and elsewhere.

The reason for choosing this special restricted family is visual: it is intrinsically mapped by points in a "phase diagram" drawn within a unit square. I could not imagine an even simpler family that achieves the same goals at the same low cost. To provide cartoons of the remaining broad behaviors, one needs only two nonsymmetric 3-interval generators; these are not discussed in this book.

1. DETAILED SUMMARY

1.1. Symmetric 3-interval generators and their address area in the plane

A generator could be any broken line made of $b <$ intervals that joins the points (0, 0) and (1, 1) and is the graph of a functon. When the function is discontinuous, its jumps are counted as intervals. The procedure that uses a generator recursively to obtain a limit function *f*(*t*) is illustrated in Figure 1, which is discussed in some detail in Section 2. Each interval is specified by the lengths $\ _m t$ and $\ _m f$ of its projections on the axes. One has $\ _m t$ 0 and $\ _m f$ can be of either sign. The generator is therefore specified by 2*b* parameters constrained by the two conditions $\Sigma\ \ _m t = \Sigma\ \ _m f = 1$. This leaves 2*b* – 2 free parameters. To allow for oscillating functions, one must have b 3. When the generator is restricted to the smallest number of intervals $b = 3$, it is determined by the bottom left and top right corners of the square, plus two other points.

We shall further restrict the generator to be symmetric with respect to the midpoint of the initiator; thus, the number of parameters reduces to 2. We shall take them to be the x and y coordinates of the first break P in the generator. The coordinates of the second break are $1 - x$ and $1 - y$, hence a 3-interval symmetric generator is fully determined by P.

1.2. The fundamental phase diagram for symmetric 3-interval generators

The point P of a symmetric 3-interval generator always falls in the "address area" constituted by the half–square $\{0 < x < 1/2; 0 < y < 1\}$. P will be called the "address" of both the generator and the function *f*(*t*) from which it generates the graph.

To generate an oscillating function, the address must be in the top left quarter, including its boundary to the right. The bottom left quarter yields nondecreasing measures rather than oscillating functions; they will turn up in a later section.

The mathematical properties of the limit *f*(*t*) obtained by recursion are extremely sensitive to the position of P in the address area. Hence the address area splits into a number of "loci." A locus is a point, curve, or domain such that *P*s in different loci lead to *f*(*t*) that differ from one another in some basic mathematical fashion as well as visually. This subdivision into loci is illustrated by Figure 2, which will be called a "fundamental phase diagram" and is elaborated upon in Figure 3.

This chapter explores Figures 2 and 3 in four successive stages. First comes a "Fickian" dot. Second comes the thick curved "unifractal locus" starting at the center of the square. Third comes the straight "mesofractal locus" moving straight up from the center. The final stage of exploration tackles the remaining points in the upper left quarter; they form the "multifractal" locus, which is neither a point nor a curve, but a domain.

A major virtue of this framework is that a nearly one-to-one correspondence exists between, on the one hand, the limit functions $f(t)$ obtained recursively from those generators and, on the other hand, representative examples of "key" self-affine random functions referred to in Figure 3. The recursive construction corresponding to each locus illus-

FIGURE C1-2. The "fundamental phase diagram" for the symmetric 3-interval generator is drawn on the top left quarter of the unit square. A generator restricted to three intervals is determined by the bottom left and top right corners of the square, plus two other points. Symmetry implies that those points are symmetric with respect to the center of the square. If the generated function is to be oscillating, the generator is determined by a point in the top left quarter, including its boundary to the right.

FIGURE C1-3. Two alternative versions of Figure 2. The top panel relabels the loci of Figure 2 by the corresponding basic grid-free functions, when they exist, and indicates when they do not. The bottom panel refers to my *Selecta* for further background material concerning all those grid-free models.

trates many features of the corresponding key random functions. Recursivity implies the presence of a "grid," hence the resulting constructs are called "grid-bound." To the contrary, the key random functions referred to in Figure 3 are grid-free. Therefore, the former constructs will be described as being "cartoons" of the latter.

The key grid-free random functions include those considered in this book, as well as throughout M 1997E and M 1999N. The correspondence shown in the top panel of Figure 3 provides a common "road map" or "directory" for the first three volumes of these *Selecta*.

Digression. For more general generators, the phase diagram is not drawn in the plane. One reason is that when $b=3$ and the generator is not required to be symmetric, the smallest number of parameters is $6-2=4$. A different reason is that every nonsymmetric generator function $G(t)$ is best viewed as defining not one but two distinct generators: $G(t)$ itself is to be called "forward" and $G^*(t)=1-G(1-t)$ is to be called "backward." A suitably general recursion rule must include suitable rules, either random or systematic, that specify at each stage a choice between $G(t)$ and $G^*(t)$. Random choice mitigates the relentless rigidity of systematic iteration. Systematic choice may lead to the dimensional anomalies encountered in M 1985t{H24}.

1.3. List of significant nondecomposable loci in the phase diagram

This list uses many technical terms that are defined later in this chapter.

The diagonal $\mathbf{0 \le x = y \le 1/2}$ ***is the "degenerate address locus."*** Here, the generator is identical to the overall trendline. Recursion does not change this trendline, hence yields a degenerate $f(t)=t$, whose graph is a straight line of slope 1. The key grid-free functions of which this case is a cartoon are all the differentiable functions.

The address point $(4/9, 2/3=\sqrt{4/9})$ ***is the "Fickian locus"; unifractality of exponent*** $\mathbf{H=1/2}$ ***and a cartoon of Wiener Brownian motion (WBM).*** The other break of the generator, obtained by symmetry, is of coordinates 5/9 and 1/3. One observes that

$$\text{for all } m,\quad |\Delta_m f| = \sqrt{\Delta_m t} = (\Delta_m t)^H \quad \text{with} \quad H=1/2.$$

An arbitrary number k of iterations yields a "prefractal" broken line made of 3^k intervals, each satisfying the relation $|\Delta f| = \sqrt{\Delta t}$. This exponent value $H = 1/2$ defines "Fickian diffusion." The key oscillating function that corresponds to the address point of coordinates $(4/9, 2/3 = \ 4/9)$ is the Wiener Brownian Motion (WBM) (or the coin-tossing random walk) discussed in Chapter H3.

The address locus U, a bold curve: oscillating unifractality of exponent H ≠ 1/2 and cartoons of fractional Brownian motion (FBM). In the next step in generality, the generator's m intervals satisfy the relation

$$\text{for all } m, \quad {}_m f = (\ {}_m t)^H \ \text{ with } \ 0 < H < 1/2 \ \text{ or } \ 1/2 < H < 1.$$

The term "unifractal" expresses that the exponent H is independent of m. Because $H \neq 1/2$, the diffusion is said to be "non-Fickian," "anomalous," or "abnormal."

In the case of symmetric 3-interval generators, the necessary and sufficient condition of non-Fickian unifractality constrains the coordinates x and y of the first break point P. They must satisfy $y = x^H$ and $|1 - 2y| = (1 - 2x)^H$. When H is given, $y = x^H$, and x must be the solution of the equation $|1 - 2x^H| = (1 - 2x)^H$.

This equation has a single solution x that must be obtained numerically. The iterated functions $f(t)$ are self-affine, hence fractal. Adding the Fickian case $H = 1/2$, the values of H in the interval $]0, 1[$ parametrize a curve in the address plane, to be denoted by U, that joins two points that are excluded, $\{1/2, 1/2\}$ and $\{2 - 1, 1\}$.

The addresses that belong to U yield oscillating functions called "unifractal." Unifractality and U are often (improperly) redefined as including the Fickian $H = 1/2$. If so, the unifractal functions are cartoons of all Brownian motions: WBM itself and a family of functions $B_H(t)$ central to this book, the "fractional Brownian motions" (FBM.)

The address locus D (the horizontal open interval with* $0 < x < 1/2$ and $y = 1/2$) *and "mesofractal" cartoon measures, namely, devil staircases of fractal dimension $= -\log 2/\log x < 1$; ***cartoons of Lévy dusts of dimension*** **.** The term "mesofractal" was first introduced in M 1999N, to denote objects that previously were not distinguished from those called unifractal. The most familiar value, $x = 1/3$, yields the "Cantor staircase" illustrated in M 1982F{*FGN*} on Plate 83. The corresponding key grid-free constructs are the "Lévy staircases" illustrated in *FGN* by Plates 286-287.

Those functions' are "intermittent" motions that only vary on a fractal dust of fractal dimension $D = -\log 2/\log x$, satisfying $0 < D < 1$. Noises based on those dusts are the topic of half of M 1999N.

The address locus MEM (the vertical open interval with $x = 1/2$ **and** $0 < y < 1/2$**)** ***and "mesofractal" cartoons called "inverse devil staircases"; they are cartoons of Lévy subordinators, which are defined as stable functions of exponent and maximal skewness.*** Those addresses yield monotone increasing functions which M1999N proposed to call "cumulative mesofractal measures." They vary exclusively by positive jumps and provide cartoons of functions known as "maximally skew Lévy stable functions," or "stable subordinators," with the exponent $= -\log_2 y < 1$. The increments of those functions are discontinuous measures I propose to call mesofractal.

The address locus MEF (the vertical open interval with $x = 1/2$ **and** $1/2 < y < 1$**)** ***and "mesofractal" oscillating functions; if 1 <*** $= -$ **log2/log$x$ < 2,** ***they are cartoons of Lévy stable functions of exponent and maximal skewness; if*** **> 2,** ***they are not cartoons of any classical function.*** The "maximally skew Lévy stable motions" whose exponent $= -\log_2 y$ satisfies $1 < \quad < 2$ form the basis of the M 1960 model of the distribution of personal incomes; see Part III of M 1997E.

Prices vary up and down and therefore require L-stable motions that are not maximally skew. These are central to the "M 1963" model in finance, which M 1997E described in its Part IV (made of reprints) and throughout its early chapters (prepared specially for that book).

An interesting complication is that the addresses that belong to the interval $\{x = 1/2, 1/2 < y < 1\}$ yield oscillating functions that cannot be randomized naturally. That is, they are *not* cartoons of any key random function. This failure is linked to the fact that the Lévy stable exponent cannot exceed 2.

The address locus MUM (the open square minus an interval, defined by $\{0 < x < 1/2, 0 < y \neq x < 1/2\}$**)** ***and cartoon "multifractal" times and measures.*** The "multifractal measure locus" is made of addresses that correspond to a "depleted" lower quarter square MUF. They yield monotone increasing functions called "multifractal." Their increments are the grid-bound "multifractal measures." Together with their simple randomized versions, they are the topic of half of M 1999N and also of M 2001T, a planned volume of these *Selecta*.

The proper key multifractal measures are grid-free. The oldest example is described in M 1972j{N14}; other examples are found in Coppens & M1999, Barral & M2001, and forthcoming articles coauthored with J. Barral and M.O. Coppens.

The address locus MUF (the open square defined by $\{\mathbf{0} < x < \mathbf{1/2}, \mathbf{1/2} < y < \mathbf{1}\}$ ***minus the curve U) and cartoon "multifractal" functions; decomposition of those cartoons into cartoon unifractal functions of cartoon multifractal time.*** The "oscillating multifractal locus" is made of the addresses that do not belong to *U* but belong to the open upper quarter square MUM. They yield oscillating functions known as "multifractal." In M 1997E, they are the main topic of one of the chapters consisting of entirely new material, namely Chapter E6.

1.4. Aggregate loci of locality and globality

A rougher classification collects the above loci into two broader ones. Under two distinct meanings of the term "local," the "locality locus" is the aggregate of the degenerate locus – namely, the interval $\{0 \quad x = y \quad 1/2$ – and the Fickian locus – namely, the point (4/9, 2/3). Under several meanings of "global," the "globality locus" consists of the remainder of the phase diagram.

Locality as "improper fractality" and globality as either "proper fractality" or "multifractality." The notions of local and global are clear enough within the above list of special examples. But, in general, they share a feature of the notion of fractal that is mentioned in the Overview, Section 3: they cannot be defined in an indisputable fashion.

There is also a question of usage. Brownian motion is a prototype fractal behavior; nevertheless, most fields, including finance (M 1997E), use the words "fractal/multifractal model" to exclude the Brownian model of Louis Bachelier and denote *alternatives* to it.

In summary, from the viewpoint of usage rather than formal definition, the notions of "proper fractality" or "multifractality" largely coincide with the notion of globality. The passage from local to global is accompanied by a considerable but unavoidable increase in complexity.

1.5. Three forms of concentration: absent, hard (or mesofractal) and soft (multifractal)

1.5.1. Probabilistic description of nonrandom constructions. The main theorems of probability theory concern sums of random variables (R.V.). Best known are the law of large numbers and the central limit theorem. Less well known, but equally important, are the theorems that concern the asymptotic relative negligibility of individual addends as N .

While the cartoons in this chapter are largely nonrandom, an understanding of their structure is assisted by those probabilistic theorems. The classical ones concern the relative size of the largest addend. New ones dealing with multifractals concern the relative size of the sum of the N^D largest addends, with a suitable exponent satisfying $0 < D < 1$.

This section summarizes the detailed treatment that M 2001c addressed to the problem of price variation. Because of this motivation, and also for more intrinsic reasons, the variables under consideration will not be the values of the increments of a cartoon, but of the increments squared.

1.5.2. Aggregate loci of asymptotic nonnegligibility or negligibility of the maximum as determined by the distribution tail. For positive quantities, the usual negligibility theorems concern the ratio

$$\frac{\max_{1 \quad n \quad N} X_n}{1 \quad n \quad N X_n} \text{ to be referred as } \max / \quad .$$

As N , two possibilities arise. When $EX <$, the ratio max /Σ converges to 0. When $EX =$, the ratio fails to converge to 0. In particular, when the distribution of X follows a power-law distribution with < 1, the ratio converges to a limit r.v..

The corresponding results for our cartoons lead to the following dichotomy. The – very small – locus of asymptotic nonnegligibility is the sum of the intervals $\{x = 1/2, 0 < y < 1/2\}$ and $\{x = 1/2, 1/2 < y < 1\}$. The – very large – remainder, namely the sum of the half–square $x < 1/2$ and the point $x = y = 1/2$, is the locus of asymptotic negligibility.

As it should, this locus of negligibility includes the traditional Fickian locus. In terms borrowed from the economics of industrial concentration, the ratio max/Σ is a measure of the degree of monopoly. Therefore, the preceding aggregate loci concern, in the case of large N, the presence or absence of monopoly due to the tail distribution of the addends.

More general is the concept of oligopoly, which asserts that in a sum of many addends, a large proportion is contributed by "only a few," meaning an integer independent of N. The locus of negligibility remains unchanged.

1.5.3. Aggregate loci of "soft" or "multifractal" concentration, as determined by a varying distribution bell; parametrization by an exponent D satisfying $\mathbf{0 < D < 1}$**.** The limit values of this parameter, $D = 1$ and $D = 0$, characterize, respectively, the loci of negligibility and nonnegligibility described in Section 1.5.2. Other values of D concern the ratio

$$\frac{\text{of the } N \quad \text{largest among the } N \text{ addends } X_n}{\text{of all } N \text{ addends } X_n}.$$

Follow this ratio as N . When $> D$, it converges to 1. When $< D$, it converges to 0. Thus, the bulk of the sum of N addends is contributed by the N^D largest among those addends.

2. EXAMPLE OF A CARTOON FUNCTION OF TIME CONSTRUCTED BY RECURSIVE INTERPOLATION

This section amplifies the brief description given early in Section 1.1.

2.1. The process of recursion in an increasingly refined grid

As shown by the top panel of Figure 1, the process of recursion begins with price variation reduced to a "trendline" called the "initiator." Next, a broken line called a "generator" replaces the trend-initiator with a relatively slow up-down-and-up price oscillation. In the following stage, each of the generator's three intervals is interpolated by three shorter ones. One must squeeze the generator's horizontal axis (time scale) and the vertical axis (price scale) in different ratios, whose values will be discussed in Section 2.6. The goal is to fit the horizontal and vertical boundaries of each interval of the generator. To fit the middle interval, the generator must be reflected in either axis. Repeating these steps reproduces the generator's shape at increasingly compressed scales.

Only four construction stages are shown in Figure 1, but the same process continues. In theory, it has no end, but in practice, it makes no sense to interpolate down to time intervals shorter than those between trading transactions, which may be of the order of a minute. Each interval of a finite interpolation eventually ends up with a shape like the whole. This scale invariance is present simply because it was built in.

2.2. The novelty, versatility and surprising creative power of simple recursion

Sections 3 to 6 will show that a recursion's outcome can exhibit a wealth of structure, and that it is extremely sensitive to the exact shape of the generator. Generators that might seem close to one another may generate *qualitatively* distinct "price" behaviors. This will make it necessary to construct a phase diagram in which different parts or "loci" lead to different behaviors. Being sensitive, the construction is also very versatile: it is general enough to range from the coin-tossing model's "mildness" to surrogates of the "wild" and tumultuous real markets – and even beyond.

This finding is compelling and surprising.

It is essential for the number and exact positions of the pieces of the generator to be completely specified and kept fixed. If, to the contrary, the generator fails to be exactly specified or (worse!) one fiddles with it as the construction proceeds, the outcome can be anything one wants. But it becomes pointless.

An analogous construction with a two-interval generator would not simulate a price that moves up and down. When the generator consists of many more than three intervals, it involves many parameters and versatility of the procedure is less surprising.

2.3. Randomly shuffled grid-bound cartoons

The recursion described in the preceding sections is called "grid-bound," because each recursion stage divides a time interval into three. This fixed pattern is clearly not part of reality and was chosen for its unbeatable simplicity. Due to its artificiality and the acknowledged drawbacks described in Section 8, the resulting constructions are called "cartoons." Unfortunately, artefacts remain visible even after many iterations, especially with symmetric generators. To achieve a higher level of realism, the next easiest step is to inject randomness. This is best done in two stages.

Shuffling. The random sequence of the generator's intervals before each use. Altogether, three intervals allow the six permutations

1, 2, 3; 1, 3, 2; 2, 1, 3; 2, 3, 1; 3, 1, 2; and 3, 2, 1, of a die one for each side. Before each interpolation, the die is thrown and the permutation that comes up is selected. A symmetric generator allows only three distinct permutations and shuffling has less effect.

The most desirable proper randomizations. Despite many virtues, the shuffled versions of all the cartoons we shall examine in sequence (Fickian, unifractal, mesofractal, and multifractal) are grid-bound, therefore unrealistic. Fortunately, we shall see that each major category of cartoons was designed to fit a natural random and grid-free counterpart.

2.4. The "Fickian" square-root rule

Moving from qualitative to quantitative examination, the nonshuffled Figure 1 uses a three-piece generator that is very special. Indeed, let the width and height of the initiator-trend define one time unit and one price unit. In Figure 2, each interval height – namely, 2/3, 1/3, or 2/3 – is the square-root of the stick width – namely, 4/9, 1/9, or 4/9.

This being granted, define for each m 3 the quantities

$$\frac{\log(\text{height of the } m\text{th generator interval})}{\log(\text{width of the } m\text{th generator interval})} = H_m.$$

By design, the generator intervals in Figure 2 satisfy the following

FICKIAN CONDITION: $H_k = 1/2$ for all k.

An integer-time form of this "square-root rule" is familiar in elementary statistics. Indeed, the sum of N independent r.v. of zero mean and unit variance has a standard deviation equal to $\sqrt{N}$. Therefore, the sum is said to "disperse" or "diffuse" like $\sqrt{N}$.

In continuous grid-free time the square-root rule characterizes the WBM and "simple diffusion," also called "Fickian."

In our grid-bound interpolation, the square-root rule is nonrandom and only holds for the time intervals that belong to some stage k of the recursive generating grid. The result is a behavior that is only pseudo-Brownian: close to the continuous-time WBM, but not identical to it.

3. NON-FICKIAN 3-INTERVAL CARTOONS; THE PHASE DIAGRAM

Fickian diffusion is classical and extraordinarily important in innumerable fields, but the Brownian model does not fit the financial prices. Fortunately, the square-root does not follow from the recursive character of our construction, only from the special form of the generator.

3.1. Symmetric 3-interval generators beyond the Fickian case; the "phase diagram"

Indeed, let us preserve the idea behind Figure 2 and show that modifying the H_m suffices to open up a wealth of behaviors that differ greatly from the Brownian and from one another. As argued early in this paper, it is essential to keep those generalizations as simple as possible and capable of being followed on a simple two-dimensional diagram. It will suffice to assume that the generator continues to include three intervals symmetric with respect to the center of the original box.

The coordinates of its first break determine those of the second by taking complements to 1, hence a 3-interval symmetric generator is fully determined by the position P of its first break. This point will be called "generator address," and the resulting fundamental "phase diagram" is drawn as Figures 2 and 3.

For curves that oscillate up and down, all the possibilities are covered by points P in the "address space" defined as the top left quarter of the unit square. Instead of oscillating functions, the bottom left quarter yields nondecreasing measures that a later section will use to define multifractal time.

Active actual experimentation is very valuable at this stage and is accessible to the reader with a moderate knowledge of computer programming. Playing "hands-on," that reader will encounter a variety of behaviors that are extremely versatile, hence justifying the attention about to be lavished on 3-interval symmetric generators. Section 3.2 lists rapidly the possibilities that will be discussed in later sections.

3.2. Two fundamental but very special loci, called "unifractal" and "mesofractal," and the "multifractal" remainder of the phase diagram

The terms describing the simplest loci in Figure 3 are recent or new.

The mesofractal cartoons will be seen in Section 5 to correspond to the "M 1963" model of price variation built in M 1963b{E14} using the stable random processes of Cauchy and Lévy. Price increments according to that model are illustrated by the second panel from the top in Figure 4. Panel 2 is less unrealistic than the Fickian Panel 1, because it shows many

FIGURE C1-4. A stack of diagrams, illustrating the successive "daily" differences in at least one actual financial price and some mathematical models. Obviously, the top three panels do not report on data but on models; among the lower five panels, to the contrary, to identify the models is a difficult task.

spikes; however, these are isolated against an unchanging background in which the overall variability of prices remains constant.

The unifractal cartoons will be seen in Section 4 to correspond to the "M 1965" model of price variation in M 1965h{H9}. The corresponding increments are illustrated by panel 3 of Figure 4. Compared to the M 1963 model, the strengths and failings interchange because jumps are absent.

The mesofractal and unifractal models are interesting but inadequate, except under certain special market conditions.

Special regions having been examined, Sections 5 and 7 shall proceed to the remaining multifractal, cartoons which correspond to my current model of financial price variation (see M 1997E), the "M1972/97 model" of FBM in multifractal trading time.

3.3. Definitions of volatility: the traditional "root mean square" and beyond

The coin-tossing economics illustrated on the top panel of Figure 4 is fully specified by a single parameter, therefore, volatility is necessarily an increasing function of standard deviation . It is often the r.m.s. ^{2}, but the intervals between percentiles also come to mind. For example, a strip of total width from –2 to 2 contains 95% of all price changes. If only implicitly, volatility is a relative concept: it concerns the comparison of the observed fluctuations to an ideal economy that has achieved equilibrium and involves no fluctuation at all.

This implicit reference to equilibrium must be elaborated upon. Is economics more complex than the classical core of physics? Almost everyone agrees, but the Brownian model implies the precise contrary. For example, the physical theory closest to coin-tossing finance is that of a perfect gas in thermal equilibrium, for which ^{2} is proportional to temperature. But such a system also depends on either volume or pressure. Could it really be the case that a perfect gas is more complicated than economics?

The unifractal model illustrated on panel 3 of Figure 4 and discussed in Section 4 is specified by and an exponent *H*. This *H* measures how much a constant-width "snake" oscillates along the time axis. *H* must be included in order to specify intuitive volatility quantitatively.

In the mesofractal model illustrated on panel 2 of Figure 4 and discussed in Section 5, the population standard deviation diverges. But the equally classical notion of intervals between percentiles remains meaningful. Hence volatility can be defined as including the two parameters that determine the process. One is the width of the horizontal strip containing 95% of "price" changes. The second specifies the variability of the remaining 5% of large changes, which is ruled by an exponent   or its inverse, $H = 1/$.

4. UNIFRACTAL CARTOONS, NONPERIODIC BUT CYCLIC BEHAVIOR AND GLOBALITY

4.1. The exponent H – satisfying $0 < H < 1$ – and equations that characterize unifractality

Logically, if not quite so historically, cartoons that deserve to be called "unifractal" come immediately after the Fickian ones. Given a single exponent that satisfies $0 < H < 1$, unifractality is defined by the following condition:

CONDITION OF UNIFRACTALITY: $H_m = H$ for every m.

The prefix "uni" refers to the uniqueness of H. Depending on the context, the usage may include or exclude $H = 1/2$. If ambiguity threatens, $H \neq 1/2$ may be called "nontrivially unifractal." (This ambiguity is a perennial issue; real numbers are special complex numbers and one must often specify that a number is "nontrivially complex.")

The example of the Fickian "square-root" rule in Section 2.4 proves that one can implement the unifractality conditions when $H = 1/2$. For other prescribed values of H, the unifractality conditions yield two "unifractality equations:" $y = x^H$ and $2y - 1 = (1 - 2x)^H$.

In particular, x is the unique root of the "generating equation" $2x^H - 1 = (1 - 2x)^H$, which must be solved numerically. In turn, this equation yields a single $y = x^H$. That is, just as in the case $H = 1/2$, each allowable value of H is achieved by choosing for the function address P a single specified point in the address quarter square.

When lumped together, the points P form a "locus of unifractality" that takes the form of the only curve seen in Figure 3. This curve is (a) far more restrictive than the whole allowable quarter square and (b) far less restrictive than the unique Brownian–Fickian address (4/9, 2/3).

Alternative unifractality condition, restated in terms of a quantity   that will become essential in the multifractal case discussed in Sections 6 and 7. The unifractality conditions can be rewritten as $(2y-1)^{1/H} = 1 - 2x$ and $x = y^{1/H}$; eliminating x combines the two conditions into $y^{1/H} + (2y-1)^{1/H} + y^{1/H} = 1$.

This last equation is a property of the sum of the intervals' absolute heights raised to the same power $1/H$. The addends, namely, $_1 = y^{1/H}$, $_2 = (2y - 1)^{1/H}$, and $_3 = y^{1/H}$, satisfy $_1 + {}_2 + {}_3 = 1$. Therefore, they define an auxiliary address point of coordinates x and $y = {}_1$, which will be called the generator's "time address." The time address of a generator fully determines its function address. This unifractal case yields $_1 = x$, therefore the time address is located on the bisector of our diagram, between two limit points to be explained in Section 4.2, namely, (1/2, 1/2) and (2 – 1, 2 – 1).

4.2. Limit points not included in the locus of unifractality

The forbidden limit H 0. This corresponds to $y = 1 -$ exp(–), that is, $2y - 1 = 1 - 2$ exp(– 2) y^2. Hence the generating equation written in terms of $y^{1/H} = x$ becomes $x^2 + 2x - 1 = 0$. It yields the coordinate $x =$ 2 – 1 and $y = 1$ for the generator address, hence, as announced, $x = y =$ 2 – 1 for the time address.

The corresponding intervals of the generator have heights $f = 1$, $f =$ – 1, and $f = 1$. In order to add to 1, the correlations between those three increments are not only *negative,* but as strongly negative as can be. The limit is degenerate. But, after an arbitrary number of recursions, each step in the approximation is equal in absolute value to 1, which is the increment of the function between any two points in the construction grid. This property is extreme but important in a discussion of concentration and asymptotic negligibility M 2001d.

The forbidden limit H 1. It corresponds to a vanishing middle interval, therefore to a straight generator and a straight interpolated curve. In this case, relentless "inertia" makes it "persist" forever in its motion.

4.3. Two forms of persistence, and cyclic but nonperiodic behavior

Three subranges of H must be distinguished.

(A) *The* $0 < H < 1/2$ *part of the unifractal locus*. There is a negative persistence or antipersistence.

(B) *The Fickian* $H = 1/2$. This represents a total absence of persistence.

(C) *(The more important* $1/2 < H < 1$ *part of the unifractal locus.* Persistence is positive and increases as H moves from 1/2 to 1.

Cyclic but nonperiodic behavior. Let us now relate the manifestations of cyclic behavior and globality as they appear in the graphs of a function $f(t)$ itself, rather than of its increments. The phenomenon of persistence manifests itself in patterns of change that are not periodic but are perceived by everyone as "cyclic." As already mentioned, it was observed long ago by Slutzky that the eye decomposes Brownian motion spontaneously into many cycles of having periods that range from very short to quite long. As the total duration of the sample is increased, new cycles appear without end. They correspond to the mere juxtaposition of random changes, nothing real. To appreciate this fact, one should rethink the positive overall trend that is highly visible in Figure 1. Over a time space much shorter than the total space 1, the trend becomes negligible in comparison with local fluctuations. Hence, the up-down-up oscillation represented by the generator will be interpreted as a slow cycle.

As H increases above $1/2$, so does the relative intensity of this longest period cycle. It also ceases to be meaningless (à la Slutzky) and becomes increasingly real. While it does not promise the continuation of a periodic motion, it allows a certain degree of prediction. A nice illustration of what is happening is provided in a closely related context by Plates 264 and 265 of M 1982F{*FGN*}. This is one aspect of the following property common to all values $H \neq 1/2$: the successive movements of $f(t)$ are not simply juxtaposed. In effect, they interact, their interdependence not being short, but long-range, or "global."

In any event, unifractal cartoons fail to generate either a variable volatility or the large spikes of variation that Figure 1 shows to be characteristic of finance. Therefore, the generalization of Fickian square-root must go beyond unifractality, as it will in later sections.

5. MESOFRACTAL CARTOONS AND PRICE DISCONTINUITY

5.1. The locus of discontinuous behavior

In the quarter square that bounds the phase diagrams in Figures 2 and 3, discontinuous functions are associated with the unit length interval characterized by $x = 1/2$ and $0 < y < 1$. Aside from the Fickian point, this locus is the simplest. It also has the oldest roots in finance, insofar as Section 5.3 will link the portion $1/2 < y < 1/\sqrt{2}$ with the M 1963 model of price variation of M 1963b{E14}, M 1967b{E15}.

Recall the quantities H_m defined in Section 2.6. Mesofractality is defined as follows:

CONDITION OF MESOFRACTALITY: $H_2 = 0, \quad H_1 = H_3 \neq 0.$

The middle interval satisfies $H_2 = 0$ if, and only if, $x = 1/2$; if so, the side intervals – by definition of H – satisfy $H_1 = H_2 = \log y / \log(1/2) = H$. There are *two* separate fractal exponents, not one. But early on $H_2 = 0$ used to be disregarded, seemingly qualifying this construction as unifractal. More generally, M 1997E did not single out discontinuities, but M 1999N found it necessary to single them out and coined *mesofractal*. In the present very special generator, both H and H are functions of y, hence of each other; but this feature disappears for more general cartoons.

5.2. The distribution of the jump sizes

Continue the recursion. The next stage adds two smaller discontinuities of size $-y(2y-1)$. Further iterations keep adding increasingly high numbers (4, 8, 16, and higher powers of 2^k for k going to infinity) of increasingly smaller discontinuities of size $= -y^{k-1}(2y-1)$. For small ,

the number of discontinuities of absolute size > $^{-1/H=\ -}$.

Section 5.3 will justify the notation $1/H =$.

5.3. The exponent splits the discontinuity locus into three portions and subportions, to be handled separately; relations with the M 1963 model, and reason for the notation $= 1/H$.

The portion $0 < y < 1/2$. This yields $0 < \ < 1$ and corresponds to positive discontinuities hence to increasing functions. They generate a fractal trading time, a notion that is better discussed in Section 7, as a special case of the multifractal trading time.

The portion $1/2 < y < 1$. This yields > 1, and corresponds to negative discontinuities, hence to oscillating functions. It splits in two.

The subportion $1/2 < y < 1/\sqrt{2}$. This yields $1 < \ < 2$ and justifies the notation for H. The reason is that, in that range of , the distribution of discontinuities is the same in the mesofractal cartoons and the L-stable processes used in the M 1963 model.

More precisely, all the jumps are negative here, while in the M 1963 model of price variation they can take either sign. A distribution with two long tails can be achieved by using generators that include a positive and a negative discontinuity; this requires more than three intervals.

The subportion $1/\sqrt{2} < y < 1$. *Why is it that the L-stable exponent cannot exceed 2*? For all , nonrandom mesofractal cartoons are perfectly acceptable. But the cases < 1 and > 1 differ on a point that seems to involve mathematical nitpicking but turns out to be essential. The *k*th approximation of *f*(*t*) alternates jumps and gradual moves. For < 1, the sum of moves vanishes asymptotically for *k* , and the sum of jumps tends to 1. Hence, the function *f*(*t*) varies only by positive jumps. For > 1, the sum of positive moves exceeds the sum of the negative jumps by the constant 1. However, taken separately, the sums of moves and jumps tend, respectively, to or – as *k* . Therefore, the sum of absolute values of the jumps and moves diverges to infinity, and the function *f*(*t*) is said to be "of unbounded variation."

Unbounded variation causes no harm as long as the construction is nonrandom. But randomization raises a very subtle issue. Replacing fixed numbers of discontinuities by Poisson distributed numbers causes a divergence that recalls the ultraviolet and infrared "catastrophes" in physics. Physicists know how to "renormalize" away many of those infinities. In this case, P. Lévy found, in the 1930s, that infinities can be eliminated when < 2, but not when > 2.

Comment. The case > 2 contributes yet another mismatch between the cartoons and the grid-free processes they imitate. See Section 8.

6. MULTIFRACTAL CARTOONS

6.1. Definition

In the phase diagram in Figure 3, the loci of unifractal and mesofractal behavior are points or curves. If the address is chosen at random with uniform probability, its probability of hitting those loci is zero. The overwhelming majority of address points remains be examined. They satisfy the following condition:

CONDITION OF MULTIFRACTALITY: $H_1 = H_3 \neq H_2 \neq 0.$

One variant of the reason for the prefix "multi," is that the H_m take a multiplicity of values. That perennial question resurfaces again: "Should the Fickian case be called multifractal?" One could either call multifractal all the points in the top left quarter of the address square, or exclude the unifractal and mesofractal loci.

6.2. Variable volatility, revisited

Return to Figure 4 and focus on the five bottom panels. It was said that they intermix actual data with the best-fitting multifractal model. Asked to analyze any of those lines without being informed of "which is which," a coin-tossing economist would begin by identifying short pieces here and there that vary sufficiently mildly to almost belong to white Gaussian noise. These pieces might have been extracted from the first line, then widened or narrowed by being multiplied by a suitable r.m.s. volatility .

Many models view such complex records as the increments of a non-stationary random process, namely, of a Brownian motion whose volatility is defined by , but varies in time. Furthermore, it is tempting to associate those changes in volatility to changes in market activity.

A similar situation occurring in physics should serve as a warning. It concerns the notion of variable temperature. The best approaches are ad hoc and not notable for being either attractive or effective. I took a totally distinct approach to which we now proceed; it consists in "leap-frogging" over nonuniform gases, all the way to turbulent fluids.

Less mathematically oriented observers describe the panels at the bottom of Figure 4 (both the real data and forgeries) as corresponding to markets that proceed at different "speeds" at different times. This description may be very attractive but remains purely qualitative until "speed" and the process that controls the variation of speed are quantified. This will be done in Section 7.

6.3. The versatility of multifractal variation; in a non-Gaussian process, the absence of correlation allows a great amount of structure; this feature reveals a blind-spot of correlation and spectral analysis

Figure 5 illustrates a stack of multifractal cartoons that are shuffled at random before each use. In all cases, the address point P satisfies 2/3, therefore $H=1/2$. The column to the left is a stack of generators; the middle column, the stack of processes obtained as in Figure 2 but with shuffled generators; and the column to the right, the stack of the corresponding increments over identical time-increments t.

The line marked by a star ( ) is the shuffled form of Figure 2. The middle column is a cartoon of Brownian motion and its increments (right

FIGURE C1-5. Stack of shuffled multifractal cartoons with $y = 2/3$ therefore $H = 1/2$ and – from the top down – the following values of x: 0.2222, 0.3333, 0.3889, 0.4444 (Fickian, starred), 0.4556, 0.4667, 0.4778, and 0.4889. Unconventional but true, all the increments plotted in the right column are spectrally white. But only one line in that column is near-Brownian; it is the starred Fickian line for $x = 4/9$.

column) are a cartoon of white Gaussian noise. They do look like a "sample of white noise," as intended and expected.

But the fact is that $H = 1/2$ throughout Figure 5, and this has a surprising (even shocking) implication. The increments plotted on every line in this stack are uncorrelated to one another. That is, they are "spectrally white." As one moves away from the star, up or down the stack, one encounters charts that diverge increasingly from the pseudo-Brownian model. Increasingly, they exhibit the combination of sharp, spiky price jumps and persistently large movements that characterize financial prices.

Mathematicians know that whiteness does not express statistical independence, only absence of correlation. But the temptation existed to view that distinction as mathematical nitpicking. The existence of such sharply non-Gaussian white noises proves that the hasty assimilation of spectral whiteness to independence was understandable but untenable. White spectral whiteness is highly significant for Gaussian processes, but otherwise is a weak characterization of reality.

In the white noises of Figure 5, a high level of dependence is not a mathematical oddity but the inevitable result of self-affinity of exponent $H = 1/2$. By and large, points P close to the Fickian locus of Figure 3 will "tend" to produce wiggles that resemble those of financial markets. As one moves farther from the center, the resemblance decreases and eventually the chart becomes more extreme than any observed reality.

This illustration brings to this old-timer's mind an old episode that deserves to be revived because it carries a serious warning. After the fast Fourier transform became known, the newly practical spectral analysis was promptly applied to price change records. An approximately white spectrum and negligible correlation emerged, and received varied interpretations. Numerous scholars rushed to view them as experimental arguments in favor of the Brownian motion or coin-tossing model. Other scholars, to the contrary, realized that the data are qualitatively incompatible with independence. Finding spectral whiteness to be incomprehensible, they abandoned the spectral tool altogether.

7. MULTIFRACTAL CARTOONS REINTERPRETED AS UNIFRACTAL CARTOONS FOLLOWED IN TERMS OF A TRADING TIME

7.1. Fundamental compound functions representation; the "baby theorem"

Irresistibly, the question arises, can the overwhelming variety of white or nonwhite multifractal cartoons f be organized usefully? Most fortunately, it can, thanks to a remarkable representation that I discovered and called the "baby theorem." It begins by classifying the generators according to the values of H or, equivalently, of y.

In Figure 6, the small "window" near the top left shows the generators of two functions $f_{\text{uni}}(t)$ and $f_{\text{multi}}(t)$. One is unifractal with address coordinates $x = x_u = 0.457$ and $y = 0.603$, hence $H = 0.646$. The other's address coordinates are the same y and H, but $x = x_m = 0.131$. This x_m is so small that the function $f_{\text{multi}}(t)$ is very unrealistic in the study of finance; but an unrealistic x_m was needed to achieve a legible figure.

To change the generator from unifractal to multifractal, the vertical axis is left untouched but the right and left intervals of the symmetric unifractal cartoons are shortened horizontally and provide room for a horizontal lengthening of the middle piece.

Before examining theoretically the transformation from f_{uni} to f_{multi}, it is useful to appreciate it intuitively. The body of Figure 6 illustrates the graphs of $f_{\text{uni}}(t)$ and $f_{\text{multi}}(t)$ obtained by interpolation using the above two generators. Disregarding the bold portions, the dotted lines and the arrows, one observes this: $f_{\text{uni}}(t)$ proceeds, as already known, in measured up and down steps while $f_{\text{multi}}(t)$ alternates periods of very fast and very slow change.

However, the common y and H suffice to establish a perfect one-to-one "match" between "corresponding" pieces of two curves. This feature is emphasized by drawing three matched portions of each curve more boldly. First, toward the right, between a local minimum and a local maximum, a gradual rise of the unifractal corresponds to a much faster rise of the multifractal. Second, in the middle, between a local maximum and the center of the diagram, a gradual fall of the unifractal corresponds to a very slow fall of the multifractal – largely occurring between successive "plateaux" of very slow variation. Third, between two local minima toward the left, a symmetric up and down unifractal configuration corresponds to a fast rise of the multifracal followed by a slow fall which, once again, proceeds by successive plateaux.

More generally, the fact that the two generators share a common y insures that our two curves move up or down through the same values in the same sequence, but not at the same times.

7.2. Compound functions in multifractal trading time and the "power-law" multifractal behavior $f_{multi} = (\ t)^{H(t)}$

One would like to be more specific and say that the functions f_{uni} and f_{multi} proceed at different "speeds," but the fractal context presents the complication. For Brownian motion $B(t)$, the Fickian relation $f \quad t$ implies that, "as a rule," $f / \ t$ tends to as $t \quad 0$. However, a nontraditional expression, $\log \ f / \log \ t$, is well behaved for the WBM $B(t)$. As $t \quad 0$, it

FIGURE C1-6. The small window near the top left shows a unifractal and a multifractal generator corresponding to two address points situated on the same horizontal line in the phase space. The body of the figure illustrates the resulting functions $f_{uni}(t)$ and $f_{multi}(t)$ and the one-to-one correspondence between them governed by the change from clock to trading time.

converges (for all practical purposes) to a quantity called a Hölder exponent. For WBM, it coincides with $H = 1/2$.

More generally, a unifractal cartoon's increments in time t prove to be of the form $f_{\text{uni}}(t)$ $(t)^H$, where the Hölder exponent H is identical to the constant denoted by the same letter that characterizes the unifractal.

Multifractal increments are altogether different. It remains possible to write $f_{\text{multi}}(t)$ $(t)^{H(t)}$, but $H(t)$ is no longer a constant. It oscillates continually and can take any of a multitude of values. This is one of several alternative reasons for the prefix "multi-" in the term "multifractals."

Fortunately, this variety translates easily into the intuitive terms that were reported when discussing variable volatility. The key idea of trading versus clock time has already been announced. One can reasonably describe $f_{\text{uni}}(t)$ as proceeding in a "clock time" that obeys the relentless regularity of physics. To the contrary, $f_{\text{multi}}(t)$ moves uniformly in its own subjective "trading" time, which – compared to clock time – flows slowly during some periods and fast during others. Thus, in the example in Figure 6, one can show that the times taken to draw the generator's first interval are as follows: our unifractal $f_{\text{uni}}(t)$ takes the time 0.457 and our multifractal $f_{\text{multi}}(t)$ takes the extraordinarily compressed time 0.131. In the generator's middle interval, to the contrary, the multifractal is extraordinarily slowed down.

The actual implementation of trading time generalizes the generating equation $y^{1/H} + (2y - 1)^{1/H} + y^{1/H} = 1$. In the unifractal context of Section 4.2, this equation was of no special significance, but here it is essential. Once its root H has been determined, one defines (as before) the three quantities $y^{1/H} = {}_1$, $(2y - 1)^{1/H} = {}_2$, and $y^{1/H} = {}_3$. These quantities satisfy the relation ${}_1 + {}_2 + {}_3 = 1$, already encountered in the unifractal case. Moreover, $f_{\text{multi}} = (\ )^H$ as long as is an increment of that belongs to the hierarchy intrinsic to the generator.

Compared to the unifractal case, the striking novelty brought by multifractality is that, followed as function of t, no longer reduces identically to t itself because the time address $(x, y^{1/H})$ no longer lies on an interval of the main diagonal of the phase diagram. Instead, it lies within a horizontal rectangle that is defined by $0 < x < 1/2$ and $0 < y < \sqrt{2} - 1$. For given H, the rectangle reduces to a horizontal line.

7.3. "Subordination," an extremely special case of compounding, in which (*t*) is a random function with independent increments

Chapter E 21 of M 1997E reproduces a 1967 paper in which Taylor and I pioneered trading time and took for (*t*) a process of independent positive L-stable increments of exponent /2. S. Bochner had called it "subordinator." When followed in this trading time, Brownian motion reduces to the L-stable process postulated by the M 1963 model.

More general independent increments in (*t*) lead to a compound process that has independent increments and is called subordinated. A 1973 paper by P.K. Clark added lengthy irrelevant mathematics and recommended a different subordinator (*t*), but preserved independent increments. Therefore, it also led to a price process with independent increments. Chapter E21 of M 1997E reproduces my sharp criticism of that work.

Many authors elaborated on Clark without questioning independence. From their viewpoint, compounding that allows dependence would be called "generalized subordination." This usage would blur a major distinction. Being associated with independent price increments clearly brands subordination as being unable to account for the obvious dependence in price records. The virtue of multifractal time is that it accounts for dependence while preserving the reliance upon invariances I pioneered in 1963 and proceeds along the path Taylor and I opened in 1967.

Cartoon multifractal measures link and *t* by the simple formula $= (\ t)^{U(t)}$. The resulting "compound function" is an oscillating unifractal cartoon function of exponent H, with the novelty that it proceeds in a trading time that is a nonoscillating cartoon multifractal function of clock time. It follows that $f_{\text{multi}} = (\ )^{H} = (\ t)^{HU(t)} = (\ t)^{H(t)}$. Specifically, when $H = 1/2$, one has a cartoon of a WBM of cartoon multifractal time. When $H \neq 1/2$, one has a cartoon of a FBM of cartoon multifractal time.

7.4. A finer nuance: for fixed *H* and *D*, major differences are associated with the position of min *U*(*t*) with respect to the value that corresponds to unifractality

The next simplest characteristics of a multifractal cartoon are $\min U(t)$ and $\max U(t)$. Both are very important and conspicuous: on graphs like those of Figure 1, $\min U(t)$ measures the degree of "peakedness" of the peaks of , while $\max U(t)$ measures the duration and degree of flatness of the low–lying parts of .

To describe the mathematical situation keep to the Fickian exponent $H = 1/2$ and move x away from the unifractal value $x = 4/9$, either left toward $x = 0$, or right toward $x = 1/2$ – i.e., toward the mesofractal locus of discontinuous variation. One has 0 $\min H(t) < 1$ and the value of min $U(t)$ begins as 1 and tends to 0 in both cases.

To the contrary, max $U(t)$ is not the same left and right.

Scenario to the right of the unifractal locus. Below the starred line in Figure 5, $\max H(t)$ has the finite upper bound $\log 3/\log 2$ 1.5849. Because of this bound, one expects the record to include periods where volatility is near constant and not very small.

Scenario to the left of the unifractal locus. Above the starred line, max $H(t)$ is unbounded and may become arbitrarily large. That is, one expects the record to include periods where $f(t)$ exhibits almost no volatility.

Concretely, this asymmetry creates a sharp difference that is visibly vindicated by Figure 5. Moving from simulatons to real data, the visual appearance of financial records favors the scenario to the left over the right. One needs more exacting tests than those in M, Calvet & Fisher 1997 but the variety of possible behaviors is a major reason for the versatility of the multifractals.

This versatility is welcome, because the data are complex. For example, the study of turbulent dissipation may well favor the second scenario to the left.

To stress the novelty of those predictions, they were formulated after 1999. The comparable Figure N1.4 of M 1999N consisted, in effect, in moving always to the left of the unifractality locus, and never to the right.

The above asymmetry between left and right can be expressed in terms of a theory that warrants a mention here, but only a very brief one: the variation of is "less lacunar" to the right of $x = 4/9$ than to the left.

8. ACKNOWLEDGED LIMITATIONS OF THE CARTOONS, ESPECIALLY THE MESOFRACTAL ONES, AS COMPARED TO THE CORRESPONDING CONTINUOUS–TIME PROCESSES

In every case, I started with grid-free continuous-time models. But when serious difficulties materialized (pedagogical and/or technical), standbys/surrogates became useful or even necessary. They also turned out to be of intrinsic interest and developed in interesting ways. But the cartoons (especially those with a 3-interval symmetric generator) were never meant to reproduce *every* feature of the continuous time models. Of course, neither were the continuous-time processes meant to be the last word on the variation of financial prices.

The cartoons' practical virtue is that they allow a wide range of distinct behaviors compatible with a very simple method of construction.

The cartoons' esthetic virtue is that only a small part of the phase diagram corresponds to nothing of interest. That part reduces to $\{x = 1/2 \text{ and } 1/2 < y < 1\}$ and $\{0 < x < 1/2 \text{ and } 0 < y < 2 - 1\}$, an interval and a rectangle.

The cartoon's major limitations will now be sketched.

8.1. The path to the cartoons, as restated in deeper and broader detail

The unifractal and mesofractal cartoons, respectively, are surrogates for two grid-free models, the M 1965 model based on FBM and the M 1963 model based on Lévy stable processes (LSP). Unifractal cartoons did not appear until papers I wrote in 1985 and 1986, which are reproduced in Part VI.

For multifractal measures, including multifractal time, the original 1972 and 1974 papers are reprinted in M 1999N. A grid-free model came first in 1972. But it proved difficult and lacking in versatility and was replaced in 1974 by cartoons that correspond to the lower left quarter of Figure 3.

The Brownian motion (Wiener or fractional) in multifractal time (BMMT) was conceived in 1972, as described on page 42 of M 1997E. The multifractal cartoons came years after BMMT, as standbys/surrogates. The first investigation of both BMMT and its cartoons was published in Chapter E6 and other early chapters of M 1997E. BMMT (a.k.a. multifractal model of asset returns, MMAR) is described in free-standing fashion in M, Calvet and Fisher 1997 and (as an introduction to tests on actual data) in Calvet and Fisher 2001.

8.2. The multifractal cartoons are too constrained to predict power-law tails; the reason is that they are the counterpart of very constrained measures called multinomial

Power-law distributed tails and divergent moments are very important features of the multifractal model investigated in M 1997E, 2001b. But, except in the mesofractal case – which Section 8.3 will show to be a somewhat peculiar limit – the cartoons fail to predict them.

The reason for this failure is well understood and would deserve to be described in detail. But for lack of space they can only be sketched for the benefit of the reader already familiar with a technical aspect of the multifractal measures whose present status is treated in M 2001b. A well-known heuristic approach to multifractals has nothing to say about the tails, but tails are essential in the three stages I went through in order to introduce the multifractal measures:

(a) The limit lognormal measures introduced in a paper from 1972 reproduced in M 1999N.

(b) The following sequence of less and less constrained cascades: multinomial, microcanonical (or conservative), and canonical, as introduced in papers from 1974 reproduced in M 1999N.

(c) The multifractal products of pulses (MPP) described in Barral and M 2001.

Power-law tails only appear in least-constrained implementations, namely, the limit lognormal case, the canonical cascades, and the pulses. The cartoons, to the contrary, closely correspond to the most constrained special cascades, those called multinomial.

8.3. In multifractal cartoons, *H* and the multifractal time must be chosen together, while the corresponding continuous-time grid-free models allow *H* and the multifractal time to be independent r.v.

In particular, the unifractal cartoon oscillation and the multifractal cartoon time cannot be chosen independently. Indeed, the address (x, y) of the unifractal function determines H and restricts the time address of the multifractal time to have the ordinate $y^{1/H}$ and an abscissa satisfying $x > 0, x \neq y^{1/H}$, and $x < 1/2$. However, those constraints are a peculiar feature of 3-interval symmetric generators. As the number of intervals in the generator increases, those constraints change; I expect them to become less demanding.

8.4. Artifactual singular perturbations present in the mesofractal cartoons

In the mesofractal case, the equation $\Sigma(\text{interval height})^{1/H} = 1$ can take two forms. When the vertical interval is excluded, the equation becomes $2y^{1/H} = 1$ and the solution is $H = 1/$. When the vertical interval is not excluded, the solution is different from H.

To understand the difference, consider a sequence of address points P_k that approximates from the left a mesofractal address point P with $x = 1/2$. This approximation is "singular" in the following sense: the properties of the $f(t)$ corresponding to the limit point P are *not* the limits of the properties of the $f_k(t)$ corresponding to the point P_k.

The singular nature of this approximation is undesirable and reflects a broader unfortunate limitation of the cartoon obtained through symmetric 3-interval generators.

8.5. Failure of the mesofractal and unifractal loci to intersect at the Fickian locus

In continuous-time processes, Brownian motion enters in two ways: as the $= 2$ limit case of LSP and the $H = 1/2$ midpoint of the FBM. In an ideal phase diagram, the $= 2$ limit of the mesofractal cartoons of LSP would coincide with the $H = 1/2$ midpoint of the unifractal cartoons, thus providing two distinct interpolations of the Fickian locus. However, this paper studied a particular generator, yielding a particular phase diagram for which this ideal is not achieved. Hence, the same overall behavior is represented twice: directly by the point (4/9, 2/3) and indirectly by the point (1/2, 1/ 2) .

8.6. The potential threat (or promise?) of cartoons whose values are "localized"

The papers in Part IV investigate some unifractal cartoons in detail, and show that for them the concept of dimension is sharply more complex than for self-similar fractals.

Particularly relevant are the considerations in the abundantly illustrated Foreword of M 1986t{H24}. As explained there, important insights concerning the fine structure of a function are contained in the distribution of its values during a time interval.