18 ⌑ Self-Inverse Fractals, Apollonian Nets, and Soap

The bulk of this Essay is devoted to fractals that are either fully invariant under similitudes or, at least, "nearly" self-similar. As a result, the reader may have formed the impression that the notion of fractal is wedded to self-similarity. Such is emphatically *not* the case, but fractal geometry must begin by dealing with the fractal counterparts of straight lines... call them "linear fractals."

Chapters 18 and 19 take the next step. They sketch the properties of fractals that are, respectively, the smallest sets to be invariant under geometric inversion, and the boundaries of the largest bounded sets to be invariant under a form of squaring.

Both families differ fundamentally from the self-similar fractals. Appropriate linear transformations leave scaling fractals invariant, but in order to generate them, one must specify a generator and diverse other rules. On the other hand, the fact that a fractal is "generated" by a nonlinear transformation, often suffices to determine, hence generate, its shape. Furthermore, many nonlinear fractals are bounded, i.e., have a built-in finite outer cutoff $\Omega<\infty$. Those who find $\Omega=\infty$ objectionable ought to be enchanted by its demise.

The first self-inverse fractals were introduced in the 1880's by Henri Poincaré and Felix Klein, not long after the discovery by Weierstrass of a continuous but not differentiable function, roughly at the same time as the Cantor sets, and well before the Peano and Koch curves and their scaling kin. The irony is that scaling fractals found a durable niche as material for well-known counterexamples and mathematical games, while self-inverse fractals became a special topic of the theory of automorphic functions. This theory was neglected for a while, then revived in a very abstract form. One reason why the self-inverse fractals were half-forgotten is that their actual shape has remained unexplored until the present chapter, wherein an effective new construction is exhibited.

The chapter's last section tackles a problem of physics, whose star happens to be the simplest self-inverse fractal.

BIOLOGICAL FORM AND "SIMPLICITY"

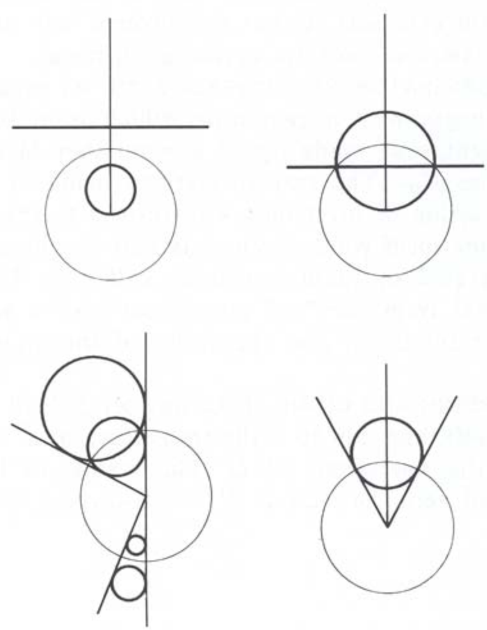
As will be seen, many nonlinear fractals "look organic," hence the present aside concerned with biology. Biological form being often very complicated, it may seem that the programs that encode this form must be very lengthy. When the complication seems to serve no purpose (as is often the case in fairly simple creatures), the fact that the generating programs were not rubbed off to leave room for useful instructions is paradoxical.

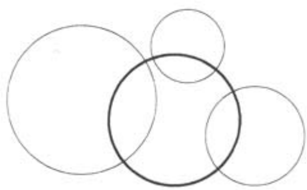
However, the complications in question are often highly repetitive in their structure. We may recall from the end of Chapter 6 that a Koch curve must *not* be viewed as either irregular or complicated, because its generating rule is systematic and simple. The key is that the rule is applied again and again, in successive loops. Chapter 17 extends this thought to the pre-coding of the lung's structure.

In Chapters 18 and 19 we go much further and find that some fractals generated using nonlinear rules recall either insects or cephalopods, while others recall plants. The paradox vanishes, leaving an incredibly hard task of actual implementation.

STANDARD GEOMETRIC INVERSION

After the line, the next simplest shape in Euclid is the circle. And the property of being a circle is not only preserved under similitude, but also under inversion. Many scholars have never heard of inversion since their early teens, hence the basic facts bear being restated. Given a circle C of origin O and radius R, inversion with respect to C transforms the point P into P' such that P and P' lie on the same half line from O, and the lengths $|OP|$ and $|OP'|$ satisfy $|OP||OP'| = R^2$. Circles containing O invert into straight lines not containing O, and conversely (see below). Circles not containing O invert into circles (third figure below). Circles orthogonal to C, and straight lines passing through O, are invariant under inversion in C (fourth figure).





Now consider jointly the three circles C_1, C_2, and C_3. Ordinarily, for example when the open bounded discs surrounded by the C_m are nonoverlapping, there exists a circle Γ orthogonal to every C_m, see above. When Γ exists, it is jointly self-inverse with respect to the C_m.

The preceding bland results nearly exhaust what standard geometry has to say about self-inverse sets. Other self-inverse sets are fractals, and most are anything but bland.

GENERATOR. SELF-INVERSE SETS. As usual, we begin with a *generator*, which is in the present case made up of any number M of circles C_m. The transformations made of a succession of inversions with respect to these circles form what algebraists call the group generated by these inversions; call it $\mathcal{G}$. The formal term for "self-inverse set" is "a set invariant under the operations of the group $\mathcal{G}$."

SEEDS AND CLANS. Take any set $\mathcal{S}$ (call it *a seed*), and add to it the transforms of $\mathcal{S}$ by all the operations of $\mathcal{G}$. The result, to be called here the *clan* of $\mathcal{S}$, is self-inverse. But it need not deserve attention. For example, if $\mathcal{S}$ is the extended plane $\mathbb{R}^*$ (the plane $\mathbb{R}$ plus the point at infinity), the clan of $\mathcal{S}$ is identical to $\mathbb{R}^*=\mathcal{S}$.

CHAOTIC INVERSION GROUPS. Furthermore, given a group $\mathcal{G}$ based upon inversions, it may happen that the clan of every domain $\mathcal{S}$ covers the whole plane. If so, the self-inverse set must be the whole plane. For reasons that transpire in Chapter 20, I propose that such groups be called *chaotic*. The nonchaotic groups are due to Poincaré, but are called Kleinian: Poincaré had credited some other work of Klein's to L. Fuchs, Klein protested, Poincaré promised to label his next great discovery after Klein—and he did!

Keeping to nonchaotic groups, we discuss three self-inverse sets singled out by Poincaré, then a fourth set of uncertain history, and a fifth set whose importance I discovered.

HYPERBOLIC TESSELLATION OR TILING

Few of Maurits Escher's admirers know that this celebrated draftsman's inspiration often came straight from "unknown" mathematicians and physicists (Coxeter 1979). In many cases, Escher added decorations to self-inverse tessellations known to Poincaré and illustrated extensively in Fricke & Klein 1897.

These sets, to be denoted by $\mathcal{T}$, are obtained by merging the clans of the circles C_m themselves.

◅ $\mathcal{G}$ being assumed nonchaotic, the complement of the merged clans of the C_m is a collection of circular polygons called "open tiles." Any open tile (or its closure) can be transformed into any other open (closed) tile by a sequence of inversions belonging to $\mathcal{G}$. In other words, the clan of any closed tile is $\mathbb{R}^*$. More important, the clan of any open tile is the complement of $\mathcal{T}$. And $\mathcal{T}$ is, so to speak, the "grout line" of these tiles. $\mathbb{R}^*$ is self-inverse. $\mathcal{T}$ and the complement of $\mathcal{T}$ are self-inverse and involve a "hyperbolic tiling" or "tessellation" of $\mathbb{R}^*$. (The root is the Latin *tessera* = a square, from the Greek τεσσαρες = four, but tiles can have any number of corners greater than 2.) In Escher's drawings, each tile bears a fanciful picture. ►

AN INVERSION GROUP'S LIMIT SET

The most interesting self-inverse set is the smallest one. It is called the limit set, and denoted by $\mathcal{L}$, because it is also the set of limit points of the transforms of any initial point under operations of the group $\mathcal{G}$. It belongs to the clan of any seed $\mathcal{S}$. To make a technical point clearer: it is the set of those limit points that cannot also be attained by a finite number of inversions. Intuitively, it is the region where infinitesimal children concentrate.

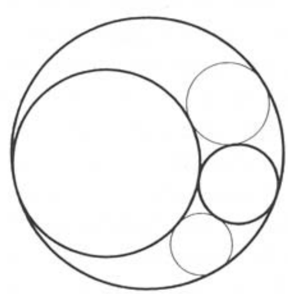
$\mathcal{L}$ may reduce to a point or a circle, but in general it is a fragmented and/or irregular fractal set.

◄ $\mathcal{L}$ stands out in a tessellation, as the "set of infinitesimally small tiles." It plays, with respect to the finite parts of the tessellation, the role the branch tips (Chapter 16) play with respect to the branches. But the situation is simpler here: like $\mathcal{L}$, the tesselation $\mathcal{T}$ is self-inverse *without* residue. ►

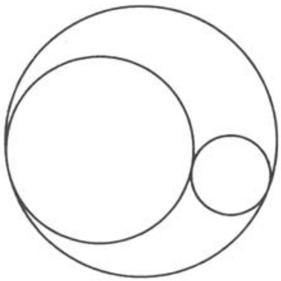
APOLLONIAN NETS AND GASKETS

A set $\mathcal{L}$ is to be called *Apollonian* if it is made of an infinity of circles plus their limit points. In this case, its being fractal is solely the result of fragmentation. This case was understood (though in diffuse fashion) at an early point of the history of the subject.

First we construct a basic example, then show it is self-inverse. Apollonius of Perga was a Greek mathematician of the Alexandrine school circa 200 B.C. and close follower of Euclid, who discovered an algorithm to draw the five circles tangent to three given circles. When the given circles are mutually tangent, the number of Apollonian circles is two. As will be seen momentarily, there is no loss of generality in assuming that two of the given circles are exterior to each other but contained within the third, as follows:



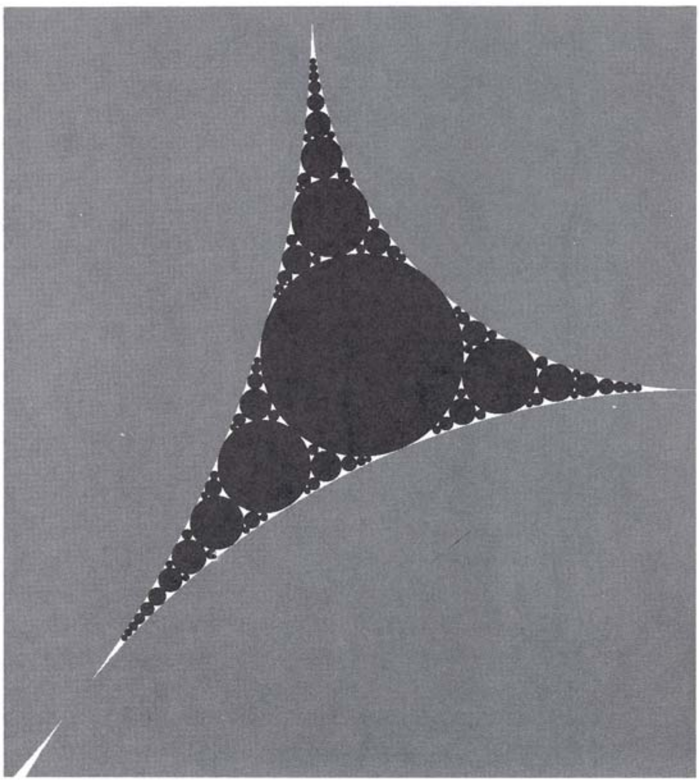
These three circles define two circular triangles with angles of 0°. And the two Apollonian circles are the largest circles inscribed in these triangles, as follows:



The Apollonian construction concludes with five circles, three given and two Apollonian, which together define six circular triangles. Repeating the same procedure, we draw the largest inscribed circle in each triangle. Infinite further repetition is called *Apollonian packing.* To the resulting infinite collection of circles one adds its limit points, and one obtains a set I call *Apollonian net*. A portion of net within a circular triangle, as exemplified to the right, is to be called *Apollonian gasket*.

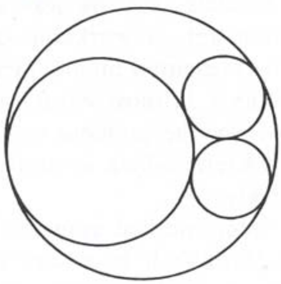
If one of the first generation Apollonian circles is exchanged for either of the inner given circles, the limit set is unchanged. ◅ If said Apollonian circle is made to replace the outer given circle, the construction starts with three given circles exterior to each other, and one of the first stage Apollonian circles is the smallest circle *circumscribed* to the three given circles. After this atypical stage, the construction proceeds as above, proving that our figures involve no loss of generalities ►.

LEIBNIZ PACKING. Apollonian packing recalls a construction I call *Leibniz packing* of a circle, because Leibniz described it in a letter to de Brosses: "Imagine a circle; inscribe within it three other circles congruent to each other and of maximum radius; proceed similarly within each of these circles and within each interval between them, and imagine that the process continues to infinity...."

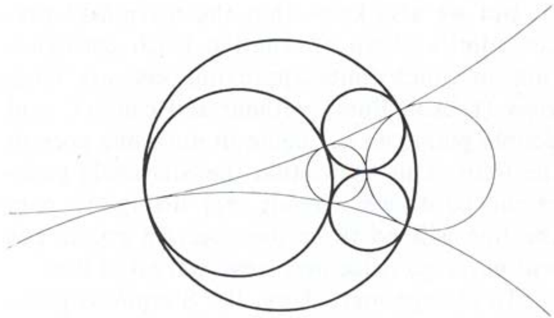


APOLLONIAN NETS ARE SELF-INVERSE

Let us now return to the starting point of the construction of Apollonian net: three circles tangent to each other. Add *either one* of the corresponding Apollonian circles, and call the resulting 4 circles $\mathbf{\Gamma}$ circles. Here they are shown by bold curves.



There are 4 combinations of the $\mathbf{\Gamma}$ circles 3 by 3, to be called triplets, and to each corresponds a circle orthogonal to each circle in the triplet. We take these new circles as our generator, and we label them as C_1, C_2, C_3, and C_4, (the diagram below shows them as thin curves). And the $\mathbf{\Gamma}$ circle orthogonal to C_i, C_j, and C_k will be labeled as $\mathbf{\Gamma}_{ijk}$.



Having set these tedious labels, here is the payoff: Simple inspection shows that the smallest (closed) self-inverse set with respect to the 4 generating circles C_m is the Apollonian net constructed on the 4 circles $\mathbf{\Gamma}$. Curiously, this observation is nowhere explicit in the literature, but it must be widely known.

A more careful inspection shows that each circle in the net transforms into one of the $\mathbf{\Gamma}$ circles through a *unique* sequence of inversions with respect to the C circles. In this way, the circles in the Apollonian net can be sorted out into 4 clans; the clan descending from $\mathbf{\Gamma}_{ijk}$ will be denoted as $\mathcal{G}\,\mathbf{\Gamma}_{ijk}$.

NET KNITTING WITH A SINGLE THREAD

The Apollonian gasket and the Sierpiński gasket of Plate 141 share an imporant feature: the complement of the Sierpiński gasket is a union of triangles, a σ-triangle, and the complement of an Apollonian net or gasket is a union of discs, a σ-disc.

But we also know that the Sierpiński gasket admits of an alternative Koch construction, in which finite approximations are teragons (broken lines) without self-contact, and double points do not come in until one goes to the limit. This shows that the Sierpiński gasket can be drawn without ever lifting the pen; the line will go twice over certain points but will never go twice over any interval of line.

To change metaphors, the Sierpiński gasket can be knitted with a single loop of thread!

The same is true of the Apollonian net.

NON-SELF-SIMILAR CASCADES, AND THE EVALUATION OF THE DIMENSION

The circular triangles of Apollonian packing are *not* similar to each other, hence the Apollonian cascade is not self-similar, and the Apollonian net is not a scaling set. One must resort to the Hausdorff Besicovitch definition of D (as exponent used to define measure), which applies to every set, but the derivation of D proves surprisingly difficult. Thus far (Boyd 1973a,b), the best one can say is that

$$1.300197 < D < 1.314534,$$

but Boyd's latest (unpublished) numerical experiments yield $D \sim 1.3058$.

In any event, since D is a fraction while $D_T = 1$, the Apollonian gasket and net are fractal curves. In the present context, D is a measure of fragmentation. When, for example, the discs of radius smaller than ϵ are "cut off," the remaining interstices have a perimeter proportional to ϵ^{1-D} and a surface proportional to ϵ^{2-D}.

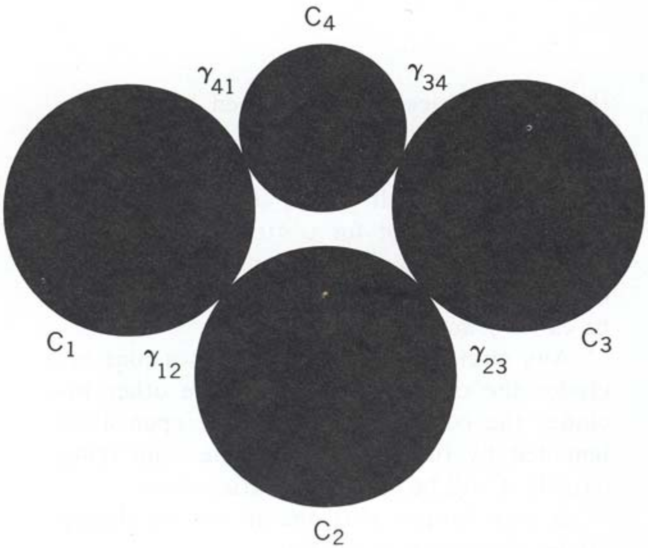
$\mathcal{L}$ IN NON-FUCHSIAN POINCARÉ CHAINS

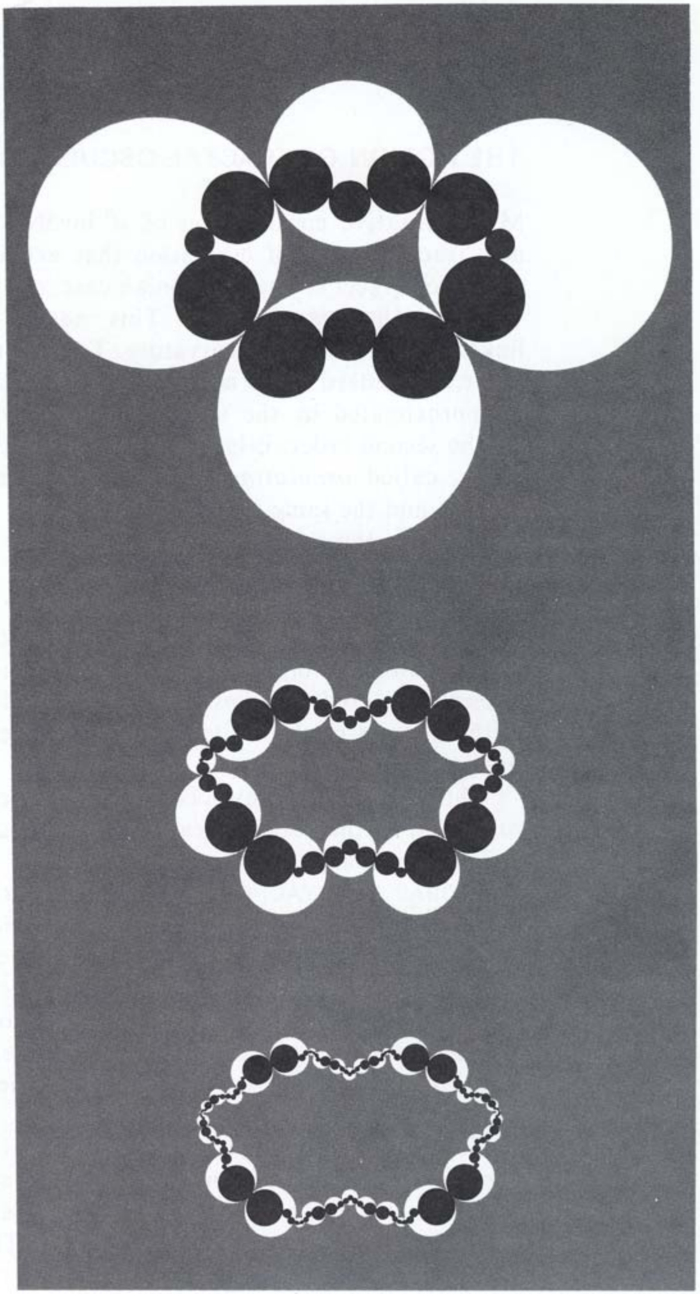
Inversions with respect to less special configuration of the generating circles C_m, lead to self-inverse fractals that are less simple than any Apollonian net. A workable construction of mine, to be presented momentarily, characterizes $\mathcal{L}$ suitably in most cases. It is a great improvement over the previous method, due to Poincaré and Klein, which is cumbersome and converges slowly.

But the older method remains important, so let us go through it in a special case. Let the C_m form a configuration one may call *Poincaré chain*, namely a collection of M circles C_m numbered cyclically, so that C_m is tangent to C_{m-1} and to C_{m+1} (modulo M), and intersects no other circle in the chain. In that case, $\mathcal{L}$ is a *curve* that separates the plane into an inside and an outside. (As homage to Camille Jordan, who first saw that it is not obvious that the plane can thus be subdivided by a single loop, such loops are called Jordan curves.)

When all the C_m are orthogonal to the same circle Γ, $\mathcal{L}$ is identical to Γ. This case, called Fuchsian, is excluded in this chapter.

POINCARÉ'S CONSTRUCTION OF $\mathcal{L}$. The customary construction of $\mathcal{L}$ and my alternative will be fully described in the case of the following special chain with $M=4$:





To obtain $\mathcal{L}$, Poincaré and Fricke & Klein 1897 replace the original chain, in stages, by chains made of an increasing number of increasingly small links. The first stage replaces every link C_i by the inverses in C_i of the links C_m other than C_i, thus creating $M(M-1) = 12$ smaller links. They are shown in the facing column, superimposed on a (gray) photographic negative of the original links. And each stage takes the chain with which it started and inverts it in each of the original C_m. Here several stages are shown in black, each being superposed on the preceding one, shown in white on gray background. Ultimately, the chain thins out to its thread, which is $\mathcal{L}$.

Unfortunately, some links remain of substantial size after large numbers of stages, and even fairly advanced approximate chains give a poor idea of of $\mathcal{L}$. This difficulty is exemplified in horrid fashion in Plate 179.

THE NOTION OF FRACTAL OSCULATION

My alternative construction of $\mathcal{L}$ involves a new fractal notion of osculation that extends an obvious facet of the Apollonian case.

STANDARD OSCULATION. This notion is linked to the concept of curvature. To the first order, a standard curve near a regular point P is approximated by the tangent straight line. To the second order, it is approximated by the circle, called *osculating*, that has the same tangent and the same curvature.

To index the circles tangent to the curve at P, a convenient parameter, u, is the inverse of the (arbitrarily oriented) distance from P to the circle's center. Write the index of the osculating circle as u_0. If $u<u_0$, a small portion of curve centered at P lies entirely on one side of the tangent circle, while if $u>u_0$ it lies entirely on the other side.

This u_0 is what physicists call a *critical value* and mathematicians call a *cut*. And $|u_0|$ defines the local "curvature."

GLOBAL FRACTAL OSCULATION. For the Apollonian net, the definition of osculation through the curvature is meaningless. However, at every point of the net where two packing circles are tangent to each other, they obviously "embrace" the rest of $\mathcal{L}$ between them. It is tempting to call *both* of them *osculating*.

To extend this notion to a non-Apollonian sets $\mathcal{L}$, we take a point where $\mathcal{L}$ has a tangent, and start with the definition of ordinary osculation based on criticality (= cut). The novelty is that, as u varies from $-\infty$ to $+\infty$, the single critical u_0 is replaced by two distinct values, u' and $u''>u'$, defined as follows: For all $u<u'$, $\mathcal{L}$ lies entirely to one side of our circle, while for all $u<u''$, $\mathcal{L}$ lies entirely to the other side, and for $u'<u<u''$, parts of $\mathcal{L}$ are found on both sides of the circle. I suggest that the circles of parameters u' and u'' *both* be called *fractally osculating*.

Any circle bounds two open discs (one includes the circle's center, and the other includes the point at infinity). The open discs bounded by the osculating circles and lying outside $\mathcal{L}$ will be called osculating discs.

It may happen that one or two osculating circles degenerate to a point.

LOCAL VERSUS GLOBAL NOTIONS. Returning to standard osculation, we observe that it is a local concept, since its definition is independent of the curve's shape away from P. In other words, the curve, its tangent, and its osculating circle may intersect at any number of points in addition to P. By contrast, the preceding definition of fractal osculation is global, but this distinction is not vital. Fractal osculation may be redefined locally, with a corresponding split of "curvature" into 2 numbers. However, in the application at hand, global and local osculations coincide.

OSCULATING TRIANGLES. ◅ Global fractal osculation has a counterpart in a familiar context. To define the interior of our old friend the Koch snowflake curve K as a sigma-triangle (σ-triangle), it suffices that the triangles laid at each new stage of Plate 42 be lengthened as much as is feasible without intersecting the snowflake curve. ►

σ-DISCS THAT OSCULATE $\mathcal{L}$

Osculating discs and σ-discs are the key of my new construction of $\mathcal{L}$, which is free from the drawbacks listed on p. 173. This construction is illustrated here for the first time (though it was previewed in 1980, in *The 1981 Springer Mathematical Calendar*!). The key is to take the inverses, not of the C_m themselves, but of some of circles $\mathbf{\Gamma}_{ijk}$, which (as defined on page 171) are orthogonal to triplets C_i, C_j, and C_k. Again, we assume that the $\mathbf{\Gamma}_{ijk}$ are not all identical to a single $\mathbf{\Gamma}$.

RESTRICTION TO M=4. The assumption M=4 insures that, for every triplet i,j,k, either one or the other of the two open discs bounded by $\mathbf{\Gamma}_{ijk}$—namely, either its inside or its outside—contains none of the points γ_{mn} which we define on page 173. We shall denote this γ-free disc by Δ_{ijk}.

My construction of $\mathcal{L}$ is rooted in the following observations: every γ-free Δ_{ijk} osculates $\mathcal{L}$; so do their inverses and repeated inverses in the circles C_m; and the clans built using the Δ_{ijk} as seeds cover the whole plane except for the curve $\mathcal{L}$.

Plate 177 uses the same Poincaré chain as already used on page 173, but is drawn on larger scale. As is true in most cases, the first stage outlines $\mathcal{L}$ quite accurately. Later stages add detail very "efficiently," and after few stages the mind can interpolate the curve $\mathcal{L}$ without the temptation of error present in the Poincaré approach.

GENERALIZATIONS

CHAINS WITH FIVE OR MORE LINKS. When the number of original links in a Poincaré chain is M>4, my new construction of $\mathcal{L}$ involves an additional step: it begins by sorting the $\mathbf{\Gamma}$ circles into 2 bins. Some $\mathbf{\Gamma}$ circles are such that *each* of the open discs bounded by $\mathbf{\Gamma}$ contains at least one point γ_{mn}; as a result, Δ_{ijk} is *not* defined. Such $\mathbf{\Gamma}$ circles intersect $\mathcal{L}$ instead of osculating it. But they are not needed to construct $\mathcal{L}$.

The remaining circles $\mathbf{\Gamma}_{ijk}$ define osculating discs Δ_{ijk} that fall into two classes. Adding up the clans of the Δ_{ijk} in the first class, one represents the interior of $\mathcal{L}$, and adding up the clans of the Δ_{ijk} in the second class, one represents the exterior of $\mathcal{L}$.

The same is true in many (but not all) cases when the C_m fail to form a Poincaré chain.

OVERLAPPING AND/OR DISASSEMBLED CHAINS. When C_m and C_n have two intersection points γ'_{mn} and γ''_{mn}, these points jointly replace γ. When C_m and C_n are disjoint, γ is replaced by the two mutually inverse points γ'_{mn} and γ''_{mn}. The criterion for identifying Δ_{ijk} becomes cumbersome to state, but the basic idea is unchanged.

RAMIFIED SELF-INVERSE FRACTALS. $\mathcal{L}$ may borrow features from both a crumpled loop (Jordan curve), and an Apollonian net, yielding a fractally ramified curve akin to those examined in Chapter 14, but often much more baroque in appearance, as in Plate C7.

SELF-INVERSE DUSTS. It may also happen that $\mathcal{L}$ is a fractal dust.

THE APOLLONIAN MODEL OF SMECTICS

This section outlines the part that Apollonian packing and fractal dimension play in the description of a category of "liquid crystals." In doing so, we cast a glance toward one of the most active areas of physics, the theory of *critical points*. An example is the "point" on a temperature-pressure diagram that describes the physical conditions under which solid, liquid, and gaseous phases can coexist at equilibrium in a single physical system. The analytic characteristics of a physical system in the neighborhood of a critical point are scaling, therefore governed by power laws, and specified by critical exponents (Chapter 36). Many of them turn out to be fractal dimensions; the first example is encountered here.

Since liquid crystals are little known, we describe them by paraphrasing Bragg 1934. These beautiful and mysterious substances are liquid in their mobility and crystalline in their optical behavior. Their molecules are relatively complicated structures, lengthy and chainlike. Some liquid crystal phases are called *smectic*, from the Greek σμηγμα signifying soap, because they constitute a model of a soaplike organic system. A smectic liquid crystal is made of molecules that are arranged side by side like corn in a field, the thickness of the layer being the molecules' length. The resulting layers or sheets are very flexible and very strong and tend to straighten out when bent and then released. At low temperatures, they pile regularly, like the leaves of a book, and form a solid crystal. When temperatures rise, however, the sheets become able to slide easily on each other. Each layer constitutes a two-dimensional liquid.

Of special interest is the focal conics structure. A block of liquid crystal separates into two sets of pyramids, half of which have their bases on one of two opposite faces and vertices on the other. Within each pyramid, liquid crystal layers fold to form very pointed cones. All the cones have the same peak and are approximately perpendicular to the plane. As a result, their bases are discs bounded by circles. Their minimum radius ϵ is the thickness of the liquid crystal's layers. Within a spatial domain such as a square-based pyramid, the discs that constitute the bases of the cones are distributed over the pyramid's base. To obtain an equilibrium distribution, one begins by placing in the base a disc of maximum radius. Then another disc with as large a radius as possible is placed within each of the four remaining pieces, and so on and so forth. If it were possible to proceed without end, we would achieve exact Apollonian packing.

The physical properties of of this model of soap depend upon the surface and perimeter of the sum of interstices. The link is affected through the fractal dimension D of a kind of photographic "negative," the gasket that the molecules of soap fail to penetrate. Details of the physics are in Bidaux, Boccara, Sarma, Sèze, de Gennes & Parodi 1973. ■

PLATE 177 ¤ A SELF-INVERSE FRACTAL (MANDELBROT CONSTRUCTION)

This Plate illustrates page 175.

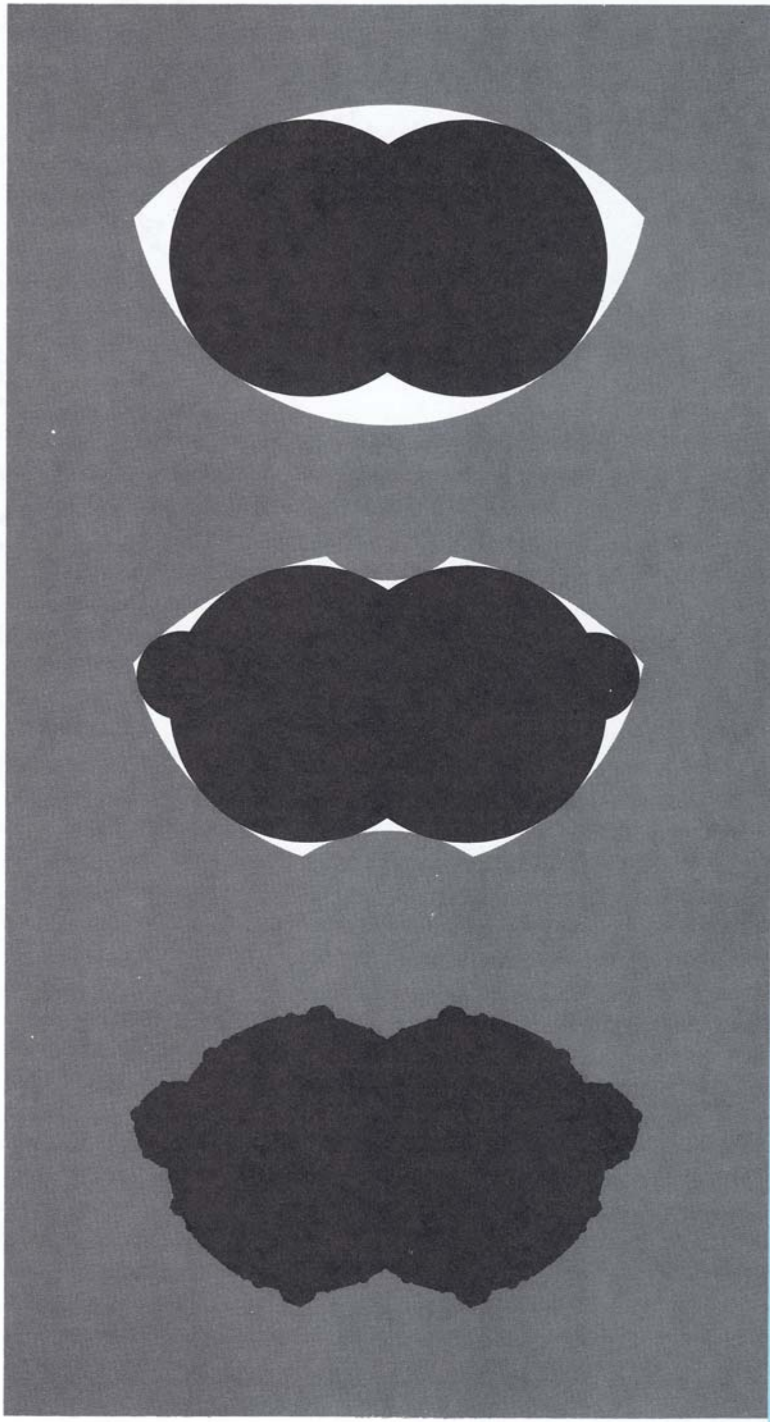
TOP FIGURE. In Poincaré chains with $M=4$, at least one of the discs Δ_{ijk} is always unbounded, call it Δ_{123}, and it intersects the disc Δ_{341}. (Here, Δ_{341} is also unbounded, but in other cases it is not.) The union of Δ_{123} and Δ_{341}, shown in gray, provides a first approximation of the outside of $\mathcal{L}$. It is analogous to the approximation of the outside of Koch's $\mathcal{K}$ by the regular convex hexagon in Plate 43.

The discs Δ_{234} and Δ_{412} intersect, and their union, shown in black, provides a first approximation of the inside of $\mathcal{L}$. It is analogous to the approximation of the inside of $\mathcal{K}$ by the two triangles that form the regular star hexagon in Plate 43.

MIDDLE FIGURE. A second approximation of the outside of $\mathcal{L}$ is achieved by adding to Δ_{123} and Δ_{341} their inverses in C_4 and C_2, respectively. The result, shown in gray, is analogous to the second approximation of the outside of $\mathcal{K}$ in Plate 43.

The corresponding second approximation of the inside of $\mathcal{L}$ is achieved by adding to Δ_{234} and Δ_{412} their inverses in C_1 and C_3, respectively. The result, shown in black, is analogous to the second approximation of the inside of $\mathcal{K}$ in Plate 43.

BOTTOM FIGURE. The outside of $\mathcal{L}$, shown in gray, is the union of the clans of Δ_{123} and Δ_{341}. And the inside of $\mathcal{L}$, shown in black, is the union of the clans of Δ_{234} and Δ_{412}. The fine structure of the inside of $\mathcal{L}$ is seen in the bottom Plate 179, using a different Poincaré chain. Together, the black and gray open regions cover the whole plane, minus $\mathcal{L}$. ■



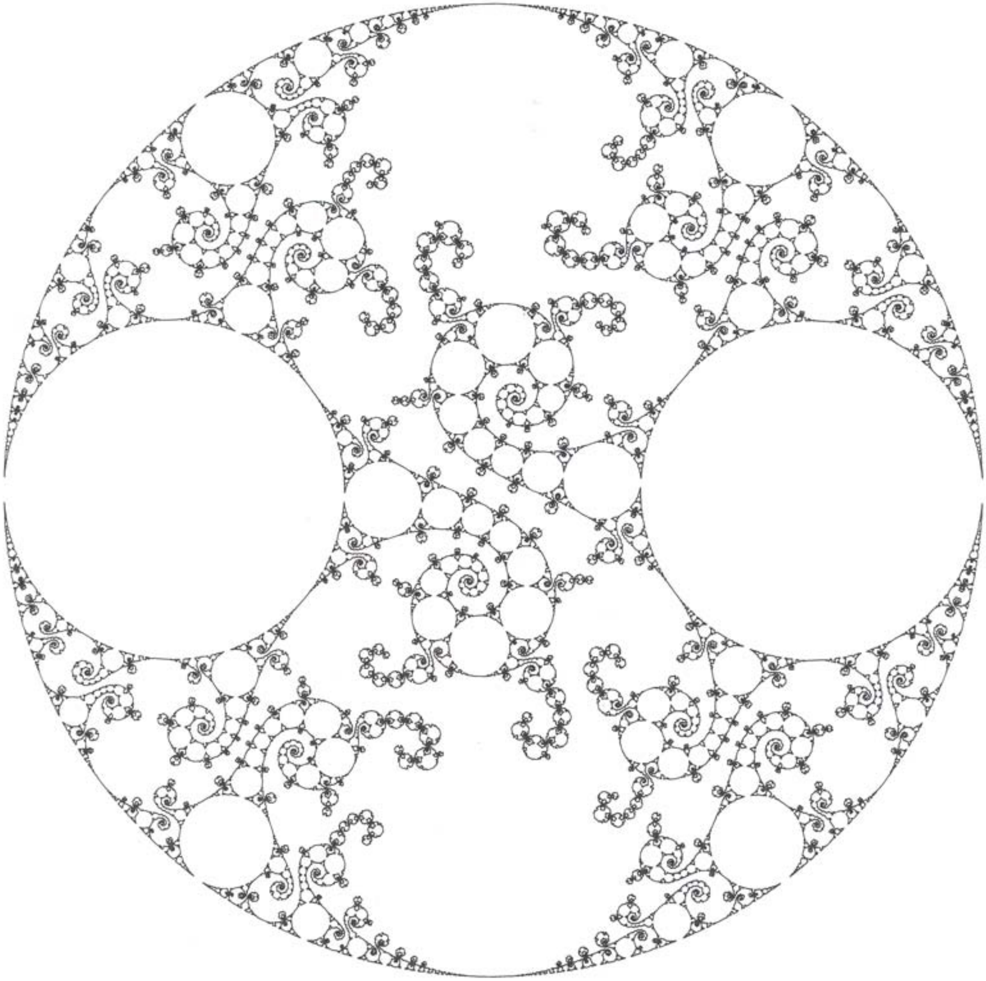
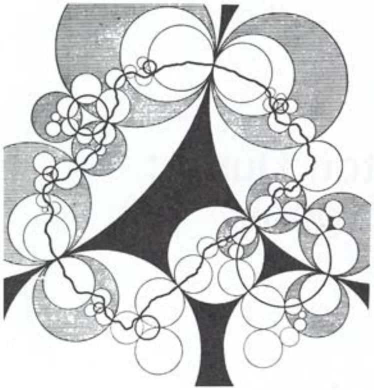


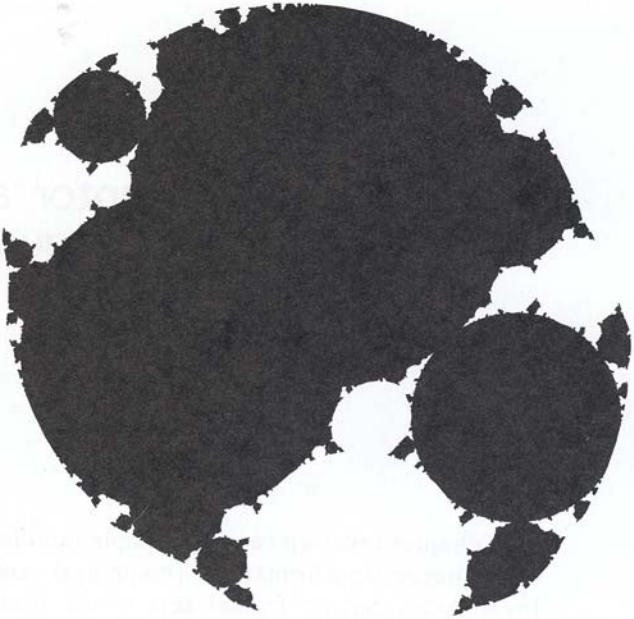
Plate 178 ⌑ SELF-HOMOGRAPHIC FRACTAL, NEAR THE PEANO LIMIT

To the mathematician, the main interest of groups based upon inversions resides in their relation with certain groups of homographies. An homography (also called Möbius, or fractional linear transformation) maps the z-plane by $z \rightarrow (az+b)/(cz+d)$, where $ad-bc=1$. The most general homography can be written as the product of an inversion, a symmetry with respect to a line (which is a degenerate inversion), and a rotation. This is why, in the absence of rotation, the study of homographies learns much from the study of groups based on inversions. But it is obvious that allowing the rotations brings in new riches.

Here is an example of limit set $\mathcal{L}$ for a group of homographies. David Mumford devised it (in the course of investigations inspired by the new results reported in this chapter), and kindly allowed its publication here. This shape is almost plane-filling, and shows uncanny analogies and differences with the almost plane-filling shape in Plate 191.

The fact that the limit set of a group of homographies is a fractal has been proven under wide conditions by T. Akaza, A. F. Beardon, R. Bowen, S. J. Patterson, and D. Sullivan. See Sullivan 1979. ■





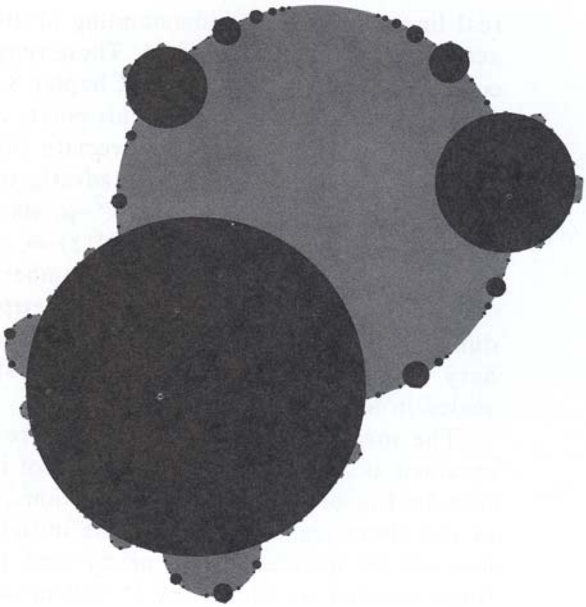


Plate 179 ¤ A CELEBRATED SELF-INVERSE FRACTAL, CORRECTED (MANDELBROT CONSTRUCTION)

The top left reproduces Figure 156 of Fricke & Klein 1897, which claims (in my terminology) to represent the self-inverse fractal whose generator is made of the 5 circles that bound the blackened central region. This Figure has been reproduced very widely.

The outline of the black shape on the top right shows the actual shape of this fractal, as given by my osculating σ-disc construction. The discrepancy is horrid. Fricke knew that $\mathcal{L}$ incorporates circles, and he instructed his draftsman to include them. But otherwise Fricke did not know what sort of very irregular shape he should expect.

The actual $\mathcal{L}$ includes the boundary $\mathcal{L}^*$ of the shape drawn on the bottom right using my algorithm. This $\mathcal{L}^*$ is the self-inverse fractal corresponding to the four among the generating circles that form a Poincaré chain. Transforms of $\mathcal{L}^*$ by other inversions are clearly seen to belong to $\mathcal{L}$. Mandelbrot 1982i elaborates upon this plate. ■