Translated from Comptes Rendus (Paris) 278A, 1974, 289-292 & 355-358 **N16**

Iterated random multiplications and invariance under randomly weighted averaging

• ***Chapter foreword. First mention of the Legendre transform in the context of multifractals.*** This paper's original had few readers: it was in French, was overly concise, and appeared to be a summary of M 1974f{N15}.

Actually, it was written after M 1974f{N15} and went beyond it on several accounts. Most significantly, as mentioned in Section N2.2, this paper includes for the first time an argument that became the basis of my approach to multifractals: Indeed, an argument reproduced in French in Chapter N2, Section 5.5.1, injected the function "$f(\alpha)$" and the Legendre transforms via the Cramèr theory of large deviations of sums of random variables. This method preceded the alternative approach due to Frisch & Parisi 1985 and Halsey et al. 1986. Details are found in Chapter N2 and the *Annotations on Section 21* of this chapter. •

✦ **Abstract.** The iteration of random multiplications yields new random functions that are interesting theoretically and practically. For example, they represent intermittent turbulence, M 1974f{N15}, and the distribution of minerals. This paper also generalizes the stable random variables: Lévy's criterion of invariance under non-random averaging is replaced by the criterion of invariance under randomly weighted averaging. ✦

1. Construction of a multiplicative measure

Consider a sequence of "weights" W, which are independent and identically distributed random variables (i.i.d. r.v.'s). We shall write $F(w) = \Pr\{W < w\}$. The base is a given integer $b > 1$, the first b weights are denoted by $W(i_1)$, $0 \leq i_1 \leq b-1$, the following b^2 weights by $W(i_1, i_2)$. etc. Let t be a real number in the interval $]0, 1]$, expanded in base b in the form

$t = 0, i_1, i_2, \ldots$ Starting from $X'_0(t) \equiv 1$, the sequence of random densities $X'_n(t)$ will be defined iteratively as

$$X'_n(t) = W(i_1)W(i_1, i_2)\ldots W(i_1, i_2, \ldots, i_n).$$

Let $X_n(t) = \int_0^t X'_n(s)\,ds$. We shall primarily study the random function (r.f.) $X_\infty(t) = \lim_{n\to\infty} X_n(t)$. However, in the case where $X_\infty(t)$ itself is degenerate, we shall instead study $Y_\infty(t) = \lim_{n\to\infty} Y_n(t)$, where $Y_n(t) = X_n(t)/A_n$ and A_n is an appropriate normalizing non-random sequence.

This construction is closely related to the construction of the limit lognormal random functions, M 1972j{N14}. These functions are of the form

$$L_\infty(t) = \lim_{n\to\infty} L_n(t),$$

where $\log L'_n(t)$ is normal. In the interesting case, $L_\infty(t)$ is non-degenerate and singular. A general procedure to construct $\log L'_n(t)$ consists in decomposing it into a sum of random functions $\log L'_{n+1}(t) - \log L'_n(t)$, each with a bounded spectrum. Unfortunately, the theory of $L'_\infty(t)$ is far removed from the familiar theories of Kolmogorov and Yaglom, and the theory of $L_\infty(t)$ presents formal difficulties that the present construction is designed to avoid.

Remark. When $b = \Gamma^2$, with an integer $\Gamma > 1$, the construction generalizes to the case where t is a vector of co-ordinates $t' \in\ ]0, 1]$ with $t' = 0, i'_1, i'_2, \ldots$ and $t'' \in\ ]0, 1]$ with $t'' = 0, i''_1, i''_2, \ldots$ In this case, each i_n is a vector whose coordinates i'_n and i''_n are integers that range from 0 to $\Gamma - 1$.

2. Special cases: birth-and-death and symmetric binomial

The most interesting case (considering the number of applications and the precision of theorems) is when $F(0) = 0$ and $0 < EW < \infty$; in this case, we assume that $EW = 1$. Every case where $EW < 0$ can be reduced to a case where $EW > 0$ by replacing A_n by $(-1)^n A_n$.

A second case is interesting because it reduces to a classical theory. When W is binomial, with $\Pr\{W = 1\} = p > 0$ and $\Pr\{W = 0\} = 1 - p > 0$, $bX_1(1)$ is the sum of b i.i.d. r.v.'s of the form $W(i_1)$; $b^2X_2(1)$ is obtained by replacing every term of $bX_1(1)$ by a r.v. which has the same distribution as $bX_1(1)$. Consequently, $b^nX_n(1)$ results from a birth-and-death process for which the number of descendants in each generation is given by the r.v. $bX_1(1)$. Classically, if $pb > 1$, the ratio

$$\frac{b^n X_n(1)}{[bEX_n(1)]^n} = \frac{X_n(1)}{p^n}$$

converges almost surely (a.s.) towards a non-degenerate limit.

Conclusion. If $0 < EW < \infty$, we expect to encounter problems involving a.s. convergence.

A third case that also goes back to a classical theory occurs when W is binomial with $\Pr\{W = 1\} = \Pr\{W = -1\} = 1/2$. This case brings back the central limit theorem: for Bernoulli variables the ratio $b^n X_n(1)/\sqrt{b^n}$ converges in distribution to a reduced Gaussian limit, the r.v. $X_n(1)b^{n/2}$ being independent.

Conclusion. If $EW = 0$, we expect to encounter problems involving convergence in distribution.

3. Fundamental recursion rule between distributions

One has the relation:

$$X_{n+1}(1) = b^{-1} \sum_{g=0}^{b-1} W_g X_{n,g}(1),$$

where the r.v.'s W_g and $X_{n,g}$ are independent, and, for every g,

$$\Pr\{W_g < w\} = F(w) \quad \text{and} \quad \Pr\{X_{n,g}(1) < x\} = \Pr\{X_n(1) < x\}.$$

4. Fundamental invariance (fixed point) property of the limit measure

If $X_n(1)/A_n$ converges in distribution towards $Y_\infty(1)$, one must have

$$\lim_{n \to \infty} \frac{A_{n+1}}{A_n} = A \quad (0 < A < \infty),$$

and one has the following identity *between distributions*:

$$Y_\infty(1) = \sum_{g=0}^{b-1} [Ab^{-1} W_g]\, Y_{\infty,g}(1),$$

where the W_g and $Y_{\infty,g}$ are independent, and, for all g,

$$\Pr\{W_g < w\} = F(w) \quad \text{and} \quad \Pr\{Y_{\infty,g}(1) < y\} = \Pr\{Y_\infty(1) < y\}.$$

It is easy to rewrite the above invariance in terms of the characteristic function of $Y_\infty(1)$.

Remark. The invariance that defines Lévy stability corresponds to the special case where the W_g are identical real numbers. The above invariance implies a functional equation in $\Pr\{Y_\infty(1) < y\}$. Its solutions depend on both $F(w)$ and b. Some are given by the construction of Section 1; the problem of the existence of other solutions remains open.

{P.S. 1998. Additional solutions were discovered in Durrett & Liggett 1983 and Guivarc'h 1987, 1990. See the *Scientific Comment on Sections 5 and 17* at the end of the Chapter.}

5. Statement of the fundamental martingale property

Let $EW = 1$; for all t, $X_n(t)$ is a martingale. For all finite n and integer $q > 1$, $EX_n^q(t) > 0$.

6. Condition for convergence of martingales

Let $EW = 1$. For every integer $h > 1$, the necessary and sufficient condition for $0 < \lim_{n\to\infty} EX_n^q(t) < \infty$ is $EW^q < b^{q-1}$.

7. Conjectured generalization of the result in Section 6

The result of Section 6 is expected to hold for all real $q > 1$.

8. Two sufficient conditions of convergence

Let $EW = 1$. In order for $X_n(t)$ to converge almost surely, two sufficient (not mutually exclusive) conditions are as follows: (a) $F(0) = 0$ (b) $EW^2 < b$ {P.S. 1996, meaning that the martingale is positive and of bounded variance}. When this second condition holds, $X_\infty(t)$ does not reduce to 0, and the equation of Section 4, with $A = 1$, has at least one non-degenerate solution, namely $X_\infty(1)$.

Proof. Theorem on convergence of martingales (see for example Doob 1953).

9. Conjectured generalization of the result in Section 8

The condition $EW = 1$ is sufficient in order for $X_n(t)$ to converge almost surely.

10. Positive weights: definitions of the exponents q_{crit} and β

When $F(0) = 0$ and $EW = 1$, let

$$q_{crit} = \max\{1, \sup [q{:}EW^q < b^{q-1}]\} \quad \text{and} \quad D = 1 - EW \log_b W.$$

When $F(0) = 0$, let

$$\overline{\tau}(q) = \log_b[EW^q / b^{q-1}] = \log_b EW^q - (q-1).$$

{P.S. 1996. Today, $\overline{\tau}(q)$ is usually denoted by $-\tau(q)$ }. This function $\overline{\tau}(q)$ is convex, and $\overline{\tau}(1) = \log EW = 0$, so that the quantity q_{crit} is the larger of 1 and of the second zero of $\overline{\tau}(q)$. Formally, $D = -\overline{\tau}'(1)$.

When $F(0) = 0$ and $\overline{\tau}(0+) > 0$, let β be the value of q for which the straight line adjoining 0 to $[\beta, \overline{\tau}(\beta)[$ has no other points in common with the graph of $\overline{\tau}(q)$.

Proposition concerning q_{crit}

The condition $q_{crit} = \infty$ holds if and only if $W < b$.

Remark. We will see that, as long as $q_{crit} = \infty$, the X_∞ are regular meaning that, $E^q_\infty < \infty$ for all q. If $1 < q_{crit} < \infty$, the X_∞ are irregular. When $D < 0$, the X_∞ are degenerate. The case $D = 0$ remains to be studied.

11. Proposition concerning q_{crit} and the convergence of moments

Let $F(0) = 0$, $EW = 1$ and $q_{crit} > 2$. Then $EX^q_\infty(t) = \lim_{n\to\infty} EX^q_n(t)$ for all $q <$ the largest integer $< q_{crit}$.

Proof. Again this follows directly from the classical theorem on the convergence of martingales. (P.S. 1996. A misprint in the original was corrected}.

12. Conjectures concerning the generalization of Section 11

(A) Let $F(0) = 0$ and $EW = 1$. In order for $X_\infty(t)$ to be non-degenerate, and for the equation of Section 4 to have a non-degenerate solution with $A = 1$, it is sufficient that $q_{crit} > 1$.

(B) Furthermore, in this case, $EX_\infty^q(t) = \lim_{n\to\infty} EX_n^q(t)$ holds for *all* $q < q_{crit}$.

13. Proposition concerning divergent moments

Let $F(0) = 0$, $EW = 1$ and $q_{crit} > 1$. If $X_\infty(t)$ is nondegenerate, $EX_\infty^q(t) = \infty$ for all $q > q_{crit}$.

Proof. It suffices to take t in the form b^{-n}, and then to limit the study to $t = 1$. Then, for all $q > 1$, Section 4 gives $EX_\infty^q(1) > b^{1-q}EW^qEX_\infty^q(1)$. If $q > q_{crit}$, this requires either $EX_\infty^q(1) = 0$, which is excluded, or $EX_\infty^q(1) = \infty$, which is thereby proven.

14. Remark on related multiplicative measures

When $F(0) = 0$, $X'_\infty(t)$ generalizes the Besicovitch measure. This is the non-random singular measure {P.S. Today, it is mostly called multinomial.} one obtains when the weights $W(i_1)$ are not random, but imposed in advance, and satisfy

$$W(i_1, i_2, ..., i_n) = W(i_n).$$

It is convenient to assume that the attainable values of W are all different, the probability p_j of each value w_j being $1/b$, with $\sum_{j=0}^{b-1} w_j/b = 1$. The Besicovitch measure rules the distribution of numbers for which the "decimals" in base b have the probabilities $\pi_j = p_j w_j$.

To generalize, we shall proceed in several stages. First, while keeping the $W(i_1)$ fixed, let their sequence follow a randomly chosen permutation. Next, allow the values of the $W(i_1)$ to vary – and in particular let the number of possible values vary – while imposing on them the following sequence of "conservation relations:"

$$\sum_{i_1=0}^{b-1} W(i_1) = b, \quad \sum_{i_2=0}^{b-1} W(i_1, i_2) = b \quad \text{for all } i$$
$$\sum_{i_3=0}^{b-1} W(i_1, i_2, i_3) = b \quad \text{for all pairs } (i_1, i_2) \quad \ldots$$

The resulting measure – which has been considered by Yaglom (see M 1974f{N15}) – can be called "microcanonical." Finally, generate the W independently, thereby obtaining $X'_\infty(t)$. The possibility of degeneracy does not appear until this last step. However, when t is generalized to be multidimensional, degeneracy may already appear in one-dimensional sections.

15. A weak limit law yielding a box dimension

Let $F(0) = 0$ and $EW = 1$. For all $\varepsilon > 0$, there exists an $n_0(\varepsilon) > 1$ such that, for all integer $n > n_0(\varepsilon)$, one can write $X_n(t) = Y_n(t) + Z_n(t)$. In this representation, $Y_n(t)$ is very small, that is to say $EY_n(1) < \varepsilon$. As to $Z_n(t)$, it varies only on a small portion of the intervals of the form $kb^{-n} < t \leq (k+1)b^{-n}$, that is, a number whose expectation is much less than $(1-\varepsilon)b^{n(D+\varepsilon)}$.

Proof. It is postponed until Section 19.

16. Corollary: a condition for degeneracy

Let $F(0) = 0$, $EW = 1$ and $D < 0$. In this case, $X_\infty(t) = 0$ almost surely, and the equation of Section 4, with $A = 1$, has no non-degenerate solution that could be constructed by the method of Section 1.

Proof. If $D < 0$, and for n sufficiently large, $(1-\varepsilon)^{-1} b^{n(D+\varepsilon)} < \sqrt{\varepsilon}$. From this it follows that $\Pr\{Z_n(1) > 0\} < \sqrt{\varepsilon}$. Moreover, $\Pr\{Y_n(1) \geq \sqrt{\varepsilon}\} \leq \sqrt{\varepsilon}$. It follows that $\lim_{n\to\infty} \Pr\{X_n(1) > 0\} = 0$.

17. Conjecture

Let $F(0) = 0$, $EW = 1$ and $D < 0$. Then the equation of Section 4, with $A = 1$, has $X = 0$ as its only solution.

{P.S. 1998: See the P.S. at the end of section 4}.

18. Conjectured strong limit law concerning Hausdorff dimension

Let $F(0) = 1$, $EW = 1$ and $D > 0$.

(A) Define $N(a,b,t,n)$ as the number of weights W in the sequence $W(i_1)$, $W(i_1, i_2)$, $W(i_1, i_2, \ldots, i_n)$ that satisfy $a \le W \le b$. In a sense that remains to be specified, the domain of variation of $X_\infty(t)$ is characterized by $\lim_{n\to\infty} n^{-1} N(a,b,t,n) = \int_a^b fw\, dF(w)$.

(B) The Hausdorff dimension of said domain of variation is almost surely equal to D.

Remark. In the case when W has b distinct possible values w_j with the probabilities b^{-1}, the clause (A) above resembles the strong classical law for the probabilities $\pi_j = w_j/b$, and the dimension in clause (B) becomes $-\sum \pi_j \log_b \pi_j$, which is formally identical to the dimension of the Besicovitch measure (see Billingsley 1967).

19. Proof of the " box dimension" weak law stated in Section 15

The main idea of the proof is easily expressed in the finite case where $\Pr\{W = w_j\} = p_j$, with $\sum p_j = 1$ and $EW = 1$, hence $\sum \pi_j = 1$, with $\pi_j = p_j w_j$. Denoting by $n\,\psi_j$ the number of times that w_j appears in the product that defines $X'_n(t)$, one has

$$X'_n(t) = \prod w_j^{n\,\psi_j}.$$

The hypothesis $EX'_n(t) = (EW)^n = 1$ yields

$$\sum n!\left(\prod (n\,\psi_j)!\right)^{-1} \prod w_j^{n\,\psi_j} p_j^{n\,\psi_j} = 1.$$

When the π_j are interpreted as probabilities, this last equality is simply the multinomial expansion of $(\sum \pi_j)^n = 1$. A theorem (Billingsley 1967) that is used in some proofs of the weak law of large numbers. It states that, given $\varepsilon > 0$, there exists a $n_0(\varepsilon)$ so that for $n > n_0(\varepsilon)$, the terms of the expansion of $(\sum \pi_j)^n$ can be classified as follows:

- in the first class, $2|\psi_j - \pi_j| < n^{-1/2}\sqrt{\pi_j(1-\pi_j)}$ holds for all i',
- the sum of all the terms in the second class is $< \varepsilon$.

A fortiori, the first class satisfies

$$\left|\sum (\psi_j - \pi_j) \log_b \pi_j\right| < \sum (\psi_j - \pi_j)|\log_b \pi_j| < n^{-½} \sum |\log_b \pi_j| \sqrt{\pi_j(1-\pi_j)}\,.$$

For large enough n, this last quantity is $< \varepsilon$. It follows that the values of $X'_n(t)$ in the second class are contained between $b^{n(H-\varepsilon)}$ and $b^{n(H+\varepsilon)}$, with

$$H = \sum \pi_j \log_b w_j = EW \log_b W.$$

It is easily established that $H > 0$. The probability of each of these $X'_n(t)$ is contained between $(1-\varepsilon)b^{-n(H+\varepsilon)}$ and $(1-\varepsilon)b^{-n(H-\varepsilon)}$. Finally, noting that $]0, 1]$ divides into b^n equal intervals on which $X'_n(t)$ is constant, the number of those intervals for which $X'_n(t)$ is of the second class is at most

$$(1-\varepsilon)b^{n(1-H+\varepsilon)} = (1-\varepsilon)b^{n(D+\varepsilon)}.$$

The case where W is not bounded is treated through bounded approximations. In the case where $\log W$ is Gaussian, the direct verification is easy.

20. The case $EW = 0$: choice of A_n to insure convergence in distribution

Only some formal results are available. When $EW^2 < \infty$, one can insure that $A_n = 1$ by normalizing W so that $EW^2 = b$. Define

$$\tilde{q}_{crit} = \max\{1, \ \sup\,[q : EW^q < b^{q-1}]\},$$

where q is an even integer > 2. Two cases must be distinguished.

From $\tilde{q}_{crit} = \infty$, which implies $|W| < b$, it follows that $0 < \lim_{n\to\infty} EX_n^q(t) < \infty$ when q is an even integer, and $\lim_{n\to\infty} EX_n^q(t) = 0$ when q is an odd integer. One may conjecture that $X_n \to X_\infty$, with $EX_n^q(t) \to EX_\infty^q(t)$, and that the limit X_∞ is a symmetric r.v..

When $\tilde{q}_{crit} < \infty$, the moment of order $q < \tilde{q}_{crit}$ converges either to a limit that is both > 0 and $< \infty$, or to a limit that is identically 0. The moments of even order $q > \tilde{q}_{crit}$ converge to infinity; the moments of odd order $q > \tilde{q}_{crit}$ can either tend to infinity or oscillate while moving away from 0, which raises problems. In the former case, one can conjecture that $X_n \to X_\infty$, just as above.

21.The probability distribution of X'

{P.S. 1998. This translation is limited to the middle part of Section 21 of the French original, namely to the part reproduced photographically in

Chapter N2. See the *Comments on Sections 10 and 21* in the *Annotations* appended to this chapter.} We use an inequality in Chernoff 1952 that requires $E(\log W) < \infty$, but allows $EW = \infty$. Writing $-\log_b A_n = \alpha n$ and neglecting complicating slowly ranging factors that do not affect the present argument, the Chernoff inequality takes the form

$$\Pr\{\log [X'_n(kb^{-n})b^{-n}] \geq -\alpha n \log b\} \sim b^{-nC(\alpha)},$$

where

$$-C(\alpha) = \text{Inf}\{\alpha q + \log_b E(W/b)^q\} = -1 + \text{Inf}[\overline{\tau}(q) + q\alpha].$$

22. {P.S. 1998. See the *Comment on Section 10* in the *Annotations* appended to this chapter.}

23. Hyperbolically distributed weights

Let $F(0) = 0$ and $\Pr\{W > w\} = w^{-\gamma}L(w)$, where $L(w)$ is a slowly varying function for $w \to \infty$. When $\gamma > 1$, we have either $\alpha > 1$ with $\alpha \leq \gamma$, or $\beta < 1$ with $\beta \leq 1$. When $\gamma < 1$, we have $\beta < 1$ and $\beta \leq \gamma$. It was not unexpected that X_∞ is at least "as irregular" as W. But it was surprising that W_∞ could be strictly more irregular, or that X_∞ could be irregular when W is regular. An example where X_∞ and W are of precisely the same level of irregularity ($\alpha = \gamma$) occurs when $L(w)(\log w)^2$ tends very rapidly (as $w \to \infty$) towards a sufficiently small limit. {P.S. 1998. This section was translated without being thought through.}

24. Conjecture concerning the case when $F(0) = 0$ but $\overline{\tau}(0+) > 0$

The behavior of X goes beyond the above theorems and conjectures. An example is when X_n is ruled by the birth and death process of Section 2. When $p < 1/b$, one has $X_n(1)/p^n \to 0$ a.s., and there can be no sequence A_n such that $Y_n \to Y_\infty$, with a non degenerate Y_∞. It is conjectured that this conclusion holds whenever $F(0) = 0$ and $\overline{\tau}(0+) < 0$.

25. Final remark: all the features of Y_∞ investigated above are determined by the geometry of the graph of $\overline{\tau}(q)$

When $D > 0$, the moments of $X_\infty(t)$ and the dimension of its set of concentration are ruled by different features of $\phi(q)$ or of W. The same holds for the covariance of $X'_\infty(t)$, which can be shown to be ruled by $\overline{\tau}(2)$.

&&&&&&&&&&&& ANNOTATIONS &&&&&&&&&&&&

The French original is made up of two papers, presented to the Académie de Sciences on November 19, 1973 by Szolem Mandelbrojt.

Editorial changes. Section titles were overly terse in the original. To help orient the reader, many were made more explicit without special mention to that effect. Other cases where the original is not followed closely is marked as {P.S. 1998}. The notation was updated: the original denotes the present b by C, the present q by h, the present $\overline{\tau}(q)$ as $f(h)$, the present q_{crit} by α, the present α by T, and the present b^{-C} by e^{-Q}. This change of base eliminates typographical clutter and brings out the co-dimension function $C(\alpha) = 1 - f(\alpha)$, be discussed in a *Comment* directed to the Legendre transform.

How this paper came to be written. M 1972j{N13} and M 1974f{N14} include and/or imply a host of conjectures of a purely mathematical nature. For example, they involve an informal box dimension for the set on which an arbitrarily large proportion of a total multiplicative measure concentrates. (The meaning of this box dimension was not fully clarified until M 1995k). It was natural to go further and conjecture that these quantities were also the dimensions in the sense of Hausdorff-Besicovitch, but the proofs turned out to be elusive.

As described in an *Annotation* to Chapter N13, I brought the matter up with Jean-Pierre Kahane; he and his then student Jacques Peyrière soon confirmed several of my conjectures relative to the base-bound case. They also advised me to restate my work for the mathematicians. This is how the present chapter came to be written.

Literature. The most important follow-up articles are Peyrière 1974, Kahane 1974, Kahane & Peyrière 1976{N17}, Ben Nasr 1987, Durrett & Liggett 1983, and Guivarc'h 1987, 1990. When referred to jointly, the last three are denoted by DLG.

Generalization to the case when the weights W_g in the recursion of Section 3 are not independent. Dependence between weights affects neither the evaluation of $\tau(q)$ nor the derivation of $f(x)$ by using the Cramer theory of large derivation. Durrett and Liggett 1983 and Ben Nasr

1987 show that the same theorems continue to hold. This generalization is important in M 1991k, 1995k, which deal with sample estimates of $\tau(q)$.

***The tail behavior* $\Pr\{X_\infty > x\} = x^{-q_{crit}}$.** This behavior, conjectured in M 1974f{N15}, follows directly from Sections 11 to 13. Proofs of my conjecture were provided by DLG and Guivarc'h.

Scientific comment on the quantity β defined in Section 10. Let $f(\alpha) > 0$ for some αs but $f(\alpha) < 0$ in the interval from $\alpha^*_{\min,}$ to $\alpha^*_{\max}$, where $\alpha^*_{\min} > \alpha_{\min}$ and $\alpha^*_{\min} \leq \alpha_{\max}$. If so, β is defined and is identical to $\alpha^*_{\min}$

Alternative and equivalent definitions of $\alpha^*_{\min}$ and $\alpha^*_{\max}$, involve the narrowest "fan" that is contained between two half lines that start at the point of coordinates $q = 0$ and $\overline{\tau}(q) = \tau(q) = 0$ and contains the graph of $\overline{\tau}(q)$. The quantities $\alpha^*_{\min}$ and $\alpha^*_{\max}$ are the slopes of the half-lines that bound this narrowest fan.

M 1991k and M 1995k vindicated (belatedly) the usefulness of the quantity β by showing that the quantities $\alpha^*_{\min}$ and $\alpha^*_{\max}$ play an important role on their own terms. Indeed, for certain purposes, the only part of $f(\alpha)$ that matters corresponds to positive fs. Correspondingly, the part $\overline{\tau}(q)$ that matters is between the abscissas $q^*_{\max}$ and $q^*_{\min}$, where the graph of $\overline{\tau}(q)$ touches the half lines of slopes $\alpha^*_{\min} = \beta$ and $\alpha^*_{\max}$.

However, those roles of β were not known to me in 1974 and the role attributed to it in the French original of M 1974c was incorrect. In addition, the discussion of β was very confused, and to translate it fairly would be difficult and pointless. Therefore, Section 22 and large portions of Section 21 were omitted in this translation.

Scientific comments on Sections 4 and 17. The need to distinguish between two forms of cascade: direct (interpolative) and inverse (extrapolative). Lévy's semi-stable distributions. DLG closed the issue raised at the end of Section 4 and showed the conjecture in Section 17 to be incorrect. When $F(0) = 0$, $EW = 1$ and $D < 0$ – and also under milder restrictions – the functional equation introduced in Section 4 has additional solutions. These solutions have infinite expectations, and are *not* obtained by the measure-generating multiplicative scheme in Section 1. The functional equation of Section 4 continues to interest mathematicians.

Recall that my multiplicative scheme generates a measure proceeds to increasingly *small* eddies. Thus, it can be called an "interpolative" or "direct" cascade that *roughens* a uniform measure. To the contrary, the remaining solutions of the functional equation, as obtained by DLG, are obtained by an extrapolative inverse cascade. This cascade reinterprets the fundamental recursion rule of Section 3 as being a *smoothing* multiplicative scheme that proceeds to increasingly *large* eddies.

The inverse cascade. It is worth repeating here a few lines from M 1984e, Section 3.2.2. This and the preceding chapters show that in studying fractal measures relevant "to noise and turbulence, it is not only inevitable but essential to introduce a process of renormalization somewhat analogous to Lévy's semi-stability. And the somewhat analogous (though different) indeterminacy and complication are present in the resulting random variables, and are concretely very important. The key ingredient in this more general renormalization is to replace ordinary addition by randomly weighted addition. The weights are a semi-infinite array of independent identically distributed r.v. with row index n and column index i, namely $W(n, i)$. Now we start with $X(n_1, 1) \equiv 1$, and the first step of renormalization is to form the array $X^*(n_2, 2) = \sum W(n_1, 1)X(n_1, 1)$ with the sum carried over the indexes n_1 of the form n, followed by an integer between 0 and $b-1$. Of the many classes of W that have been examined, the simplest, and only class characterized by $W \geq 0$, and $\langle W \rangle = 1$ was studied [before 1984]. The proper second step in renormalization is then $X(n_2, 2) = b^{-E}X^*(n_2, 2)$, and $\langle X(n_k, k) \rangle \equiv 1$.

The first object of study is, then, the fixed-point random variable $X = \lim_{k \to \infty} X(n_h, k)$ that is invariant under renormalization.

The second object is to interpolate $X(n_k, k)$ into a random function $X(n_1, k)$, and to compare the contributions to $X(n_1, k)$ from the addends $X(n_1, k) = X(n_1, k) - X(n_n - 1, k)$ that originate in the little cubes."

On the unstable solutions discovered by DLG. The contrast between "direct" and "inverse" cascades is familiar in the study of turbulence. In statistical physics, my physicist co-authors near-always build up large structures from atoms, while mathematicians prefer an interpolative "direct" cascade. Thus, the description reproduced in the preceding paragraphs took it for granted that the inverse and the direct cascades are completely equivalent. Such is not the case in the present instance.

The existence of inverse cascade fixed points means that, given the W, one can "load" the $X(n, 1)$ in such a way that the $X(n, k)$ have the same distribution for all k. Moreover, the tail of this distribution satisfies $\Pr\{X(n, 1) > \alpha\} \sim \alpha^{-q_{crit}}$. Finally, if, and only if, the $X(n, 1)$ have essentially the same tail behavior as the fixed point, the distribution of $X(n, k)$ converges to the fixed point distribution. This tail behavior characterizes the fixed point's domain of attraction.

An important feature of the new fixed points of the smoothing transformation found by DLG is that they are unstable. When the distribution being smoothed fails to be very specifically matched to the smoothing operation, its smoothed form ceases to converge to a non trivial limit. Unstable solutions may well be of no interest in physics.

Scientific comment on Section 20 and the generalization of the multiplicative processes to multipliers that may be negative. This Section gives very formal results on an important topic that is only now coming into its own. It corresponds to the multifractal functions sketched in Chapter N1, which oscillate up and down. I hope to develop my preliminary findings further and present them in a suitable forthcoming occasion.

Scientific comment on Section 21 and the Legendre transform. Section 21 was the first statement of the thermodynamical formalism of $\tau(q)$ and $f(\alpha)$. As mentioned in Chapter N2, Frisch & Parisi 1985 and Halsey et al 1986 made this formalism familiar to many scientists. The present $\tau(q)$ is my old $-\overline{\tau}(q)$, and the present $f(\alpha)$ is my old $1-C(\alpha)$. That is (but – once again – in 1974 I did *not* state it in these terms), $C(\alpha)$ is simply the fractal co-dimension corresponding to the dimension $f(\alpha)$.

My first encounter with Chernoff 1952 and the Cramèr theory of large deviations was in a paper on coding, M 1955t, Section 4.2; the mathematics behind codes and measures is often the same.

The Cramèr theory naturally leads to the Legendre transform Inf $[\overline{\tau}(h)+h\alpha]$ but this use of the Cramèr theory in the study of the multifractals did not become fully understood until many years after this paper. The valuable core of the original Section 21 was placed among statements that concern the quantity β defined in Section 10, were not thought through, and were not translated.

Let me end by restating an important remark implicit in Chapter N2, Section 1.7: the Cramèr theory does not restrict the range of values of $C(\alpha)$. For some αs, this function may well satisfy $C(\alpha)>1$, leading to $f(\alpha)<0$. This remark has led to extensive discussion in M 1989c, 1989ge, 1990r, 1991k, 1995k and other papers to be collected in M1998L.