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The Canopy and Shortest Path in a Self-Contacting Fractal Tree

This article concerns the fractal trees that are obtained recursively by symmetric binary branching. A trunk of length 1 divides into two branches of length r, each of which makes an angle $\theta > 0°$ with the linear extension of the trunk. Each branch then divides by the same rule. Some basic information on such trees is found in Chapter 16 of [FGN], on which this article elaborates.

It is well known that the branch tips of these trees can take any dimension satisfying $0 < D \leq 2$. Moreover, when $1 < D < 2$, it is possible for different branches to have tips, but no other points, in common. These trees, to be called "self-contacting," include points one cannot access from infinity, except by crossing a composite curve called the "hull." In the interesting cases, the hull includes a fractal called the "canopy."

For $\theta < 90°$, the canopy can be characterized in another way: as the shortest path along the branch tips from the upper left corner to the upper right corner. The self-contacting branch tips screen from infinity some other branch tips, thus providing shortcuts between parts of the tree, effectively jumping over the screened regions.

For $\theta > 90°$, the canopy is disconnected because of additional screening by branch segments. The shortest path along branch tips remains a variant of the Koch curve, so the shortest path and canopy no longer coincide. The fractal dimensions of the canopy, shortest path, and the set of branch tips are compared in the range $0° < \theta < 180°$.

For certain ranges of θ, the canopy, shortest path, and the set of branch tips are Koch curves. Consequently, the constructions presented here provide alternative ways to draw Koch curves.

In addition, the angles $\theta = 90°$ and $\theta = 135°$ mark profound topological discontinuities in the canopy and shortest path. Consequently, we think of these as *topological critical points.*

The structure of self-avoiding and self-contacting trees (with their canopies and shortest paths) is instructive and entertaining. It seems to make the subtle distinction between denumerable and nondenumerable infinity concrete and near-palpable.

A Classification of Binary Trees

In the preceding construction, each branch is determined by a finite number of choices of the form "bear left" or "bear right," so each branch defines, in obvious fashion, an "address" that is a finite sequence of letters L and R. Therefore, the branches are denumerable. For $r \geq 1$, the outcome of this construction is easily seen to be unbounded. For example, $LRLRLR... = (LR)^\infty$ defines a sequence in which every R branch is vertical and every L branch makes an angle θ with the vertical. Thus, the total vertical extent of this branch sequence is

$$1 + r\cos(\theta) + r^2 + r^3\cos(\theta) + r^4 + \cdots,$$

diverging for $r \geq 1$. However, if $r < 1$, a limit tree is reached after an infinite number of branchings; it depends on θ and will be denoted by $\mathcal{T}$. Each branch tip defines an address that is an infinite sequence of L and R. A tip's address is the same as an infinite sequence of 0 and 2, hence, in turn, the same as a point in the classic ternary Cantor set.

Many geometrical properties of these trees can be deduced from the positions of branch tips. Denote by $A_1A_2A_3\ldots$ the address of a branch tip and by d_n the number of R's minus the number of L's in $A_1A_2A_3\ldots A_n$. Placing the base of the trunk at the origin, this branch tip is located at the point with coordinates

$$x = r\sin(d_1\theta) + r^2\sin(d_2\theta) + r^3\sin(d_3\theta) + \cdots, \tag{1}$$
$$y = 1 + r\cos(d_1\theta) + r^2\cos(d_2\theta) + r^3\cos(d_3\theta) + \cdots.$$

When the address is eventually periodic, closed expressions for the coordinates can be found by summing the appropriate geometric series. For example, the branch tip with address $(LR)^\infty$, a point of maximal height of the tree for $0° < \theta \leq 135°$, has y coordinate

$$\frac{1 + r\cos(\theta)}{1 - r^2}.$$

To generate pictures of the set of branch tips, the standard method already used in [FGN] is now [B] referred to as "iterated function systems" (IFS). The two functions required are

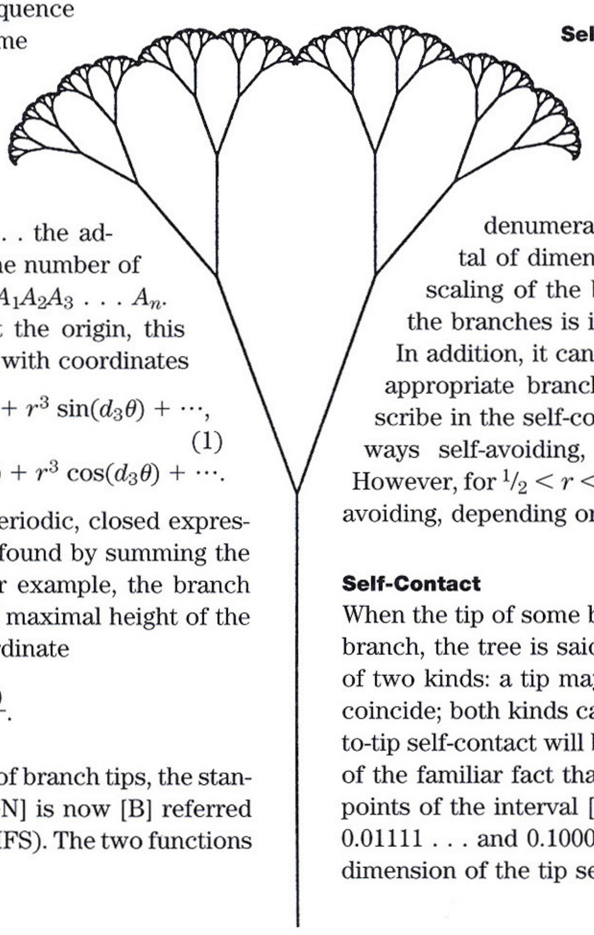

Figure 1. The self-contacting $\theta = 20°$ tree.

$$B_R(x,y) = (xr\cos(\theta) - yr\sin(\theta), xr\sin(\theta) + yr\cos(\theta)) + (0,1)$$

$$B_L(x,y) = (xr\cos(-\theta) - yr\sin(-\theta) + yr\cos(-\theta)) + (0,1).$$

The *tip set* is the set of limit points of all finite compositions of B_R and B_L applied to (0,1).

To include the trunk (and all the branches), for $\theta \leq 135°$ add a third function

$$\mathrm{Tr}(x,y) = (0,sy),$$

where

$$s = \frac{1 - r^2}{1 + r\cos(\theta)}.$$

Here, s is the reciprocal of the height of the tree, hence the vertical scaling factor of the trunk. Note that all the IFS transformations must be contractions, so the trunk can be generated with $\mathrm{Tr}(x,y)$ only so long as $s < 1$; that is, for $\theta < 135°$. However, B_R and B_L will generate the set of branch tips for all θ. For $\theta > 135°$, replace Tr with two functions

$$\mathrm{Tr}_1(x,y) = (0,y/2),$$
$$\mathrm{Tr}_2(x,y) = (0,y/2) + (0,1/2).$$

The wide availability of IFS software makes this area accessible to computer experiments.

Self-Avoidance

When $\mathcal{T}$ has no double point (i.e., no loop), it is said to be *self-avoiding.* If so, the branch tips are distinct points and, like the points in a Cantor set, are nondenumerable. They form a self-similar fractal of dimension $D = \log(2)/\log(1/r)$. That the scaling of the branch tips is identical to that of the branches is illustrated by the IFS formulation. In addition, it can be derived from the addresses of appropriate branch tips, using the method we describe in the self-contacting case. For $r \leq 1/2$, $\mathcal{T}$ is always self-avoiding, regardless of the value of θ. However, for $1/2 < r < 1$, the tree may or may not be self-avoiding, depending on θ.

Self-Contact

When the tip of some branch also belongs to some other branch, the tree is said to *self-contact.* Self-contacts are of two kinds: a tip may lie on a branch or two tips may coincide; both kinds can be found on the same tree. Tip-to-tip self-contact will be seen to involve a generalization of the familiar fact that in binary representations of the points of the interval [0, 1], the points corresponding to 0.01111 . . . and 0.10000 . . . are identical. Here, too, the dimension of the tip set is $\log(2)/\log(1/r)$.

Analysis of Tip-to-Tip Self-Contact

Note that the self-similarity and left–right symmetry of the tree imply that self-avoidance is guaranteed if none of the branches attached to the first left branch intersects the linear extension, $\mathscr{L}$, of the trunk. "First" tip-to-tip self-contact occurs where the rightmost branch tip of the left half of the tree lies on $\mathscr{L}$. How can this happen? Not surprisingly, the answer depends on θ. For $0° < \theta \leq 90°$, to find the appropriate branch tip, note that the branch LR is a vertical line segment of length r^2 and is at a horizontal distance $r \sin(\theta)$ from $\mathscr{L}$. The rightmost branch tip of this side of the tree is realized as the limit of a sequence of branches, each attached to its predecessor in the sequence; this tip lies on $\mathscr{L}$ if the lengths of the horizontal displacements of these branches sum to 0. Suppose N is the smallest integer for which $N\theta \geq 90°$. The desired sequence is described by the address $LR^{N+1}(LR)^{\infty}$; that is, after the initial LR, the branches turn right until the first horizontal or negative slope branch. After that, they alternate bearing left and right. Combining all like terms, we see the branch tip lies on $\mathscr{L}$ if its x coordinate is 0; that is, if

$$r \sin(-\theta) + r^3 \sin(\theta) + \cdots + r^N \sin((N-2)\theta) + \frac{r^{N+1}}{1-r^2} \sin((N-1)\theta) + \frac{r^{N+2}}{1-r^2} \sin(N\theta) = 0. \quad (2)$$

We shall analyze several explicit cases in a moment. Figures 1 through 8 show examples at interesting angles throughout the range $0° < \theta < 180°$.

Figure 9 shows, as a function of θ, the critical contraction ratio r that ensures self-contact. For $0° < \theta < 90°$, this graph is obtained by solving Eq. (2) for the appropriate N. For $\theta > 90°$, different branch tips must be used.

For $90° < \theta \leq 135°$, the relevant branch tip is $L^3(RL)^{\infty}$. Figures 5 and 6 give examples; for tip-to-tip self-contact in this θ range, r and θ are related by

$$r \sin(-\theta) + \frac{r^2}{1-r^2} \sin(-2\theta) + \frac{r^3}{1-r^2} \sin(-3\theta) = 0.$$

This equation can be solved explicitly for r:

$$r = \frac{-\cos(\theta) - \sqrt{2 - 3\cos^2(\theta)}}{4\cos^2(\theta) - 2}. \quad (3)$$

In this range, the minimum r value is $\sqrt{3/8}$, occurring at $\theta = \arccos(-1/\sqrt{6})$. Examining Figure 9 reveals that for given $r < \sqrt{3/8}$, tip-to-tip self-contact occurs at only two values of θ, whereas for given $r > \sqrt{3/8}$, tip-to-tip self-contact occurs at four values of θ.

For $135° < \theta \leq 180°$, the relevant branch tip is $L^2(RL)^{\infty}$. Figures 7 and 8 give examples; for tip-to-tip self-contact in this θ range, r and θ are related by

$$\frac{r}{1-r^2} \sin(-\theta) + \frac{r^2}{1-r^2} \sin(-2\theta) = 0.$$

Again, this equation can be solved for r:

$$r = -\frac{1}{2\cos(\theta)}.$$

The corresponding equations for $\theta < 90°$ do not admit such clean solutions.

Self-Overlap

When two branches of $\mathscr{T}$ intersect beyond tips contacting tips, the tree is said to *self-overlap*. This case is not of concern in this article, but it deserves a digression, because it supports interesting computer experiments. For $r = 1/\sqrt{2}$, the formal dimension is $D = 2$, suggestive of the plane-filling property of Peano curves. This property is observed for two values of θ, for which $\mathscr{T}$ self-contacts tip to branch and fills a portion of the plane. One of these values is $\theta = 90°$, when it is well known that the tree fills a rectangle, as is easily checked by hand and noted in the caption of Plate 155 of [FGN]. However, this is not all. The definitions in this article do not exclude the θ satisfying $\theta > 90°$, for which branches droop down the trunk instead of rising. It is easily checked by hand that for $\theta = 90° + 45° = 135°$, the tree $\mathscr{T}$ fills a right isosceles triangle. For other values of θ, there is a massive self-overlap.

For $1/\sqrt{2} < r < 1$, the formal expression for the dimension $D = \log(2)/\log(1/r)$ satisfies $D > 2$. This inequality expresses nicely the fact that the tree cannot avoid massive self-overlap and (irrespective of the choice of θ) covers a portion of the plane.

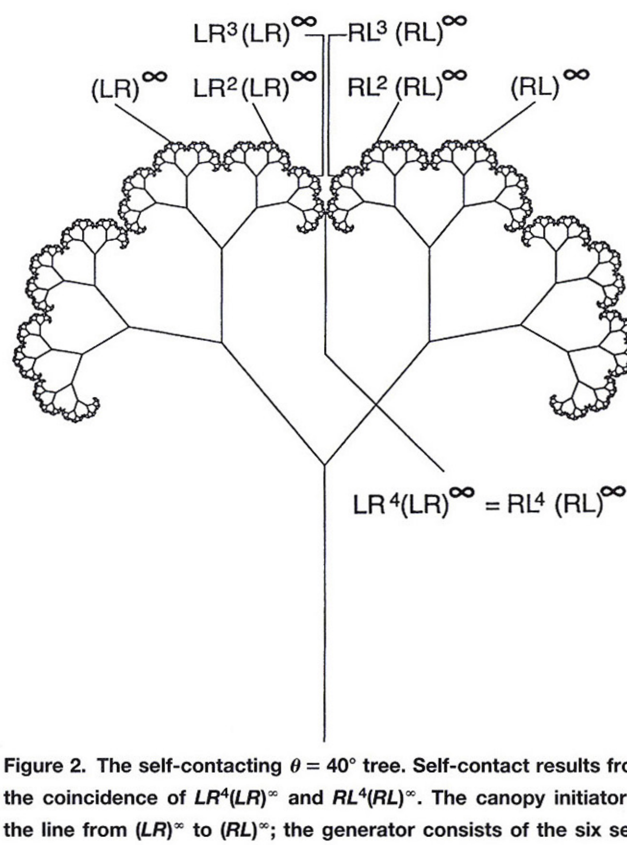

Figure 2. The self-contacting $\theta = 40°$ tree. Self-contact results from the coincidence of $LR^4(LR)^{\infty}$ and $RL^4(RL)^{\infty}$. The canopy initiator is the line from $(LR)^{\infty}$ to $(RL)^{\infty}$; the generator consists of the six segments connecting the branch tips $(LR)^{\infty}$, $LR^2(LR)^{\infty}$, $LR^3(LR)^{\infty}$, $LR^4(LR)^{\infty} = RL^4(RL)^{\infty}$, $RL^3(RL)^{\infty}$, $RL^2(RL)^{\infty}$, and $(RL)^{\infty}$. The corresponding binary fractions are $(LR)^{\infty} \to 1/3$, $LR^2(LR)^{\infty} \to 5/12$, $LR^3(LR)^{\omega} \to 41/96$, $LR^4(LR)^{\infty} \to 83/192$, $RL^4(RL)^{\infty} \to 25/48$, $RL^3(RL)^{\infty} \to 13/24$, $RL^2(RL)^{\infty} \to 7/12$, and $(RL)^{\infty} \to 2/3$.

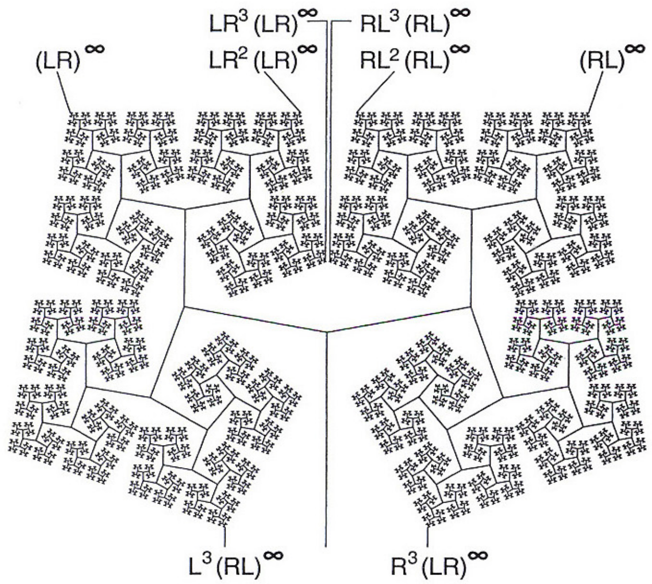

Figure 3. The self-contacting $\theta = 80°$ tree. This rendering presents only 13 branchings, so the self-contacting nature is not apparent. Here, the canopy generator consists of four segments connecting the branch tips $(LR)^\infty$, $LR^2(LR)^\infty$, $LR^3(LR)^\infty = RL^3(RL)^\infty$, $RL^2(RL)^\infty$, and $(RL)^\infty$. For comparison with the $\theta = 90°$ tree, the branch tips $L^3(RL)^\infty$ and $R^3(LR)^\infty$ are indicated.

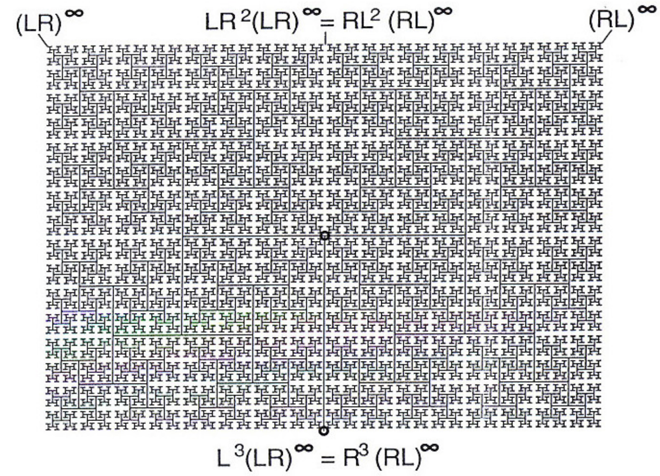

Figure 4. The self-contacting $\theta = 90°$ tree. In the infinitely branched limit, this tree fills the rectangle. The circle in the middle indicates the branch tips $LR^3(LR)^\infty = L^4(RL)^\infty = R^4(LR)^\infty = RL^3(RL)^\infty$; the bottom circle indicates $L^3(RL)^\infty = R^3(LR)^\infty$. As θ crosses the topological critical value $\theta = 90°$, the topology induced by the contacts between tips changes throughout. All the links present for $\theta < 90°$ are broken and new links established.

Definition of the Canopy of a Self-Contacting Fractal Tree

Following [FGN], p. 242, one defines the tree's *hull* or *outer boundary*, as the set of points that can be reached from far away by following a curve that does not intersect the tree. The *hull* is an intrinsically interesting concept and has been investigated for many fractal sets. In a self-avoiding tree, this notion is without interest, because the hull is identical to the whole tree with its tips. More interesting are the self-contacting trees that are illustrated in Figures 1 through 8 of this article. (They are adapted from Plate 155 of [FGN].) These figures show that the boundary includes two very different components whose structure depends on the sign of $\theta - 90°$.

One component is made of straight intervals that are reached by approaching the tree "from below." In Figures 1 through 3, for which $\theta < 90°$, these intervals join in two broken lines that start in the root, move up by fanning to the right and to the left, and end in spirals. Each straight portion in these broken lines is a full branch of the tree. Moreover, these broken lines are of finite length, therefore of dimension 1. The other part of the boundary is a fractal curve $\mathcal{C}$ made of points that can be reached by approaching the tree "from above." The curve $\mathcal{C}$ was considered in [FGN], p. 153, and called the *canopy* of the tree. Loosely speaking, $\mathcal{C}$ is made of branch tips that have somehow "coalesced"; in a moment, this idea will be made precise and we shall then evaluate the fractal dimension, $\delta_{\mathcal{C}}$, of the canopy. Because $\mathcal{C}$ is a curve, we have $\delta_{\mathcal{C}} \geq 1$, and since $\mathcal{C}$ is a subset of $\mathcal{T}$, we have $\delta_{\mathcal{C}} \leq D$.

However, Figures 5 through 8, for which $\theta > 90°$, exemplify a totally different situation. The straight intervals are not full branches but portions of branches, and the canopy can, at best, be defined as a dust of points: much of what looks like a canopy is screened from infinity by other portions of branches.

Extrapolation

The significance of the sign of $\theta - 90°$ is confirmed if the tree is suitably extrapolated. The idea is to imagine that what is drawn in the figures is not a free-standing tree but only a branch in an infinite tree. Alternatively, one can "zoom in" a small portion of a tree drawn in the figures, with the constraint that this portion touches the boundary.

When $\theta < 90°$, the extrapolation yields a simplified boundary: the portion made of branches vanishes and one is left with a piece of fractal canopy. When $\theta > 90°$, to the

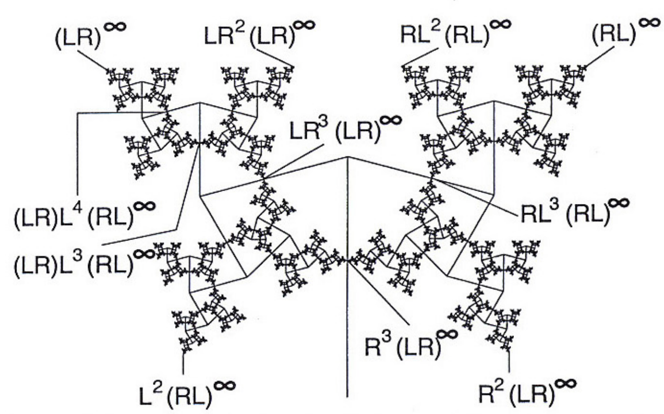

Figure 5. The self-contacting $\theta = 105°$ tree. Both addresses of the indicated double points are $L^3(RL)^\infty = R^3(LR)^\infty$, $LR^3(LR)^\infty = L^4(RL)^\infty$, $RL^3(RL)^\infty = R^4(LR)^\infty$, $(LR)L^3(RL)^\infty = (LR)R^3(LR)^\infty$, and $(LR)L^4(RL)^\infty = (LR)LR^3(LR)^\infty$.

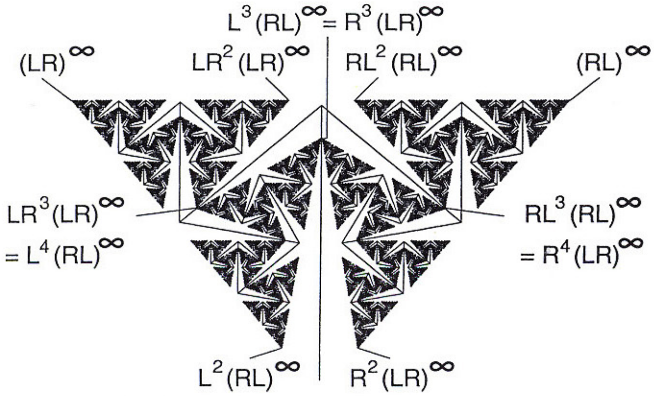

Figure 6. The self-contacting $\theta = 130°$ tree. Left to right: the top labels indicate the branch tips $(LR)^\infty$, $LR^2(LR)^\infty$, $RL^2(RL)^\infty$, and $(RL)^\infty$; the label on the trunk indicates the double point $L^3(RL)^\infty = R^3(LR)^\infty$; the label below and to the left indicates the double point $LR^3(LR)^\infty = L^4(RL)^\infty$; the lowest labels indicate $L^2(RL)^\infty$ and $R^2(LR)^\infty$.

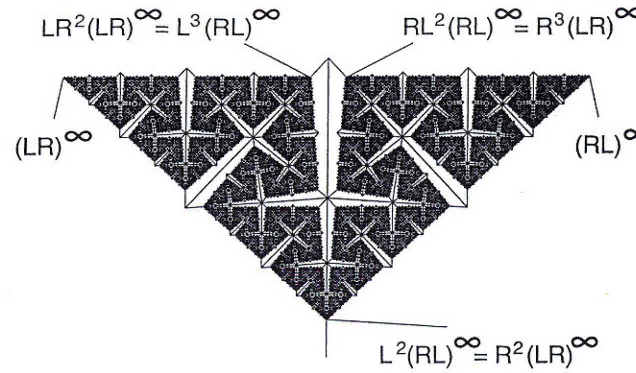

Figure 7. The self-contacting $\theta = 137°$ tree. The $\theta = 135°$ tree (not drawn) fills a triangle. As θ increases and crosses the topological critical value $\theta = 135°$, the topology induced by the contacts between tips changes throughout, just as when θ crosses the other topological critical point, $\theta = 90°$. For this value $\theta = 137°$, the shortest path nearly coincides with the Cesàro curve illustrated on Plate 65 of [FGN] as a boundary between white and black.

contrary, the extrapolated boundary continues to include parts of the branches.

The Structure of the Canopy When $\theta < 90°$

Each piece of $\mathscr{C}$ contains both single and double points. For example, Figure 3 shows clearly that the tips $(LR)^\infty$ and $(RL)^\infty$ cannot be reached in any other way; they are single points. Representing L by 0 and R by 1, we can "parse" each sequence of 0 and 1 as a binary fraction. Then, these two single point tips are represented by 0.010101 . . . = 1/3 and 0.101010 . . . = 2/3. By contrast, for $45° \leq \theta < 90°$, the tips $LR^3(LR)^\infty$ and $RL^3(RL)^\infty$ coincide and form a double point. These tips are represented by the binary fractions 0.0111010101 . . . = 1/2 − 1/24 and 0.1000101010 . . . = 1/2 + 1/24; they define an excluded subinterval of [0, 1] centered on the midpoint. Every point of that excluded subinterval corresponds to a tip that falls within $\mathscr{T}$ and not on its boundary; therefore, it does not belong to the canopy $\mathscr{C}$. The excluded subintervals can be ordered by decreasing size; therefore, they are denumerable. The same is true of the canopy double points that correspond to those intervals.

The Koch-like Structure of the Canopy and Its Fractal Dimension When $\theta < 90°$

The question was posed in [FGN] and had seemed difficult, but after sufficiently large numbers of actual trees had been plotted and examined with sufficient attention, the answers became self-evident. They are elementary but tedious, because they depend on the value of θ. Figure 9 shows that depending on the value of r, self-contact occurs for either two, three, or four threshold values of θ.

As a function of θ, the dimension of the canopy turns out to have negative discontinuities for $\theta = 135°$ and for all angles of the form $\theta = 90°/k$, $k > 1$, and to be right-continuous in the intervals between these critical angles. See Figure 10, which also includes the dimension D of the tip set, and the dimension $\delta_{\mathscr{S}}$ of the shortest path.

Inspection of the top part of the canopy, the part between $(LR)^\infty$ and $(RL)^\infty$, shows that the canopy is a Koch curve. The self-similarity of the canopy allows us to use this top to compute the canopy dimension. Although the numerical value of the initiator length, l, does not matter (units can be chosen to make it 1), we do need an expression for it in terms of r and θ to derive the scalings of the generators. The left and right ends of the initiator are the branch tips $(LR)^\infty$ and $(RL)^\infty$. Thus, l is the difference of the x coordinates of these tips. From Eqs. (1), we see that

$$l = \sin(\theta)(r + r^3 + \cdots) - \sin(-\theta)(r + r^3 + \cdots) = \frac{2r\sin(\theta)}{1 - r^2}.$$

As to the generator of $\mathscr{C}$, it depends discontinuously on θ. Let us examine a few cases.

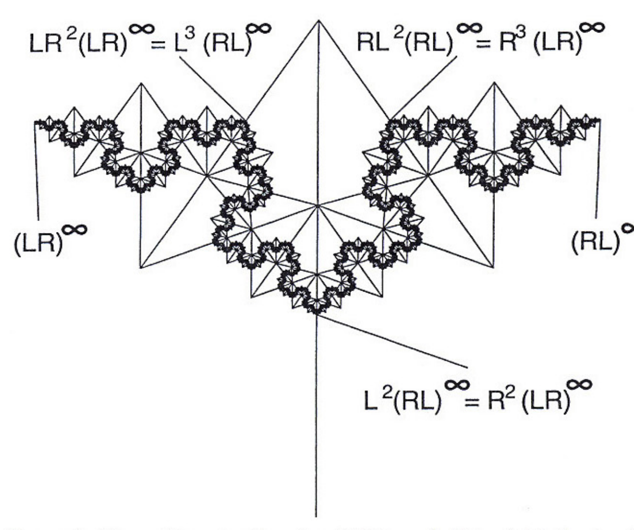

Figure 8. The self-contacting $\theta = 145°$ tree. Left to right: the top labels indicate the branch tips $(LR)^\infty$, $LR^2(LR)^\infty = L^3(RL)^\infty$, and $(RL)^\infty$; the other (lower) label indicates $L^2(RL)^\infty = R^2(LR)^\infty$. The shortest path is the classical triadic Koch curve.

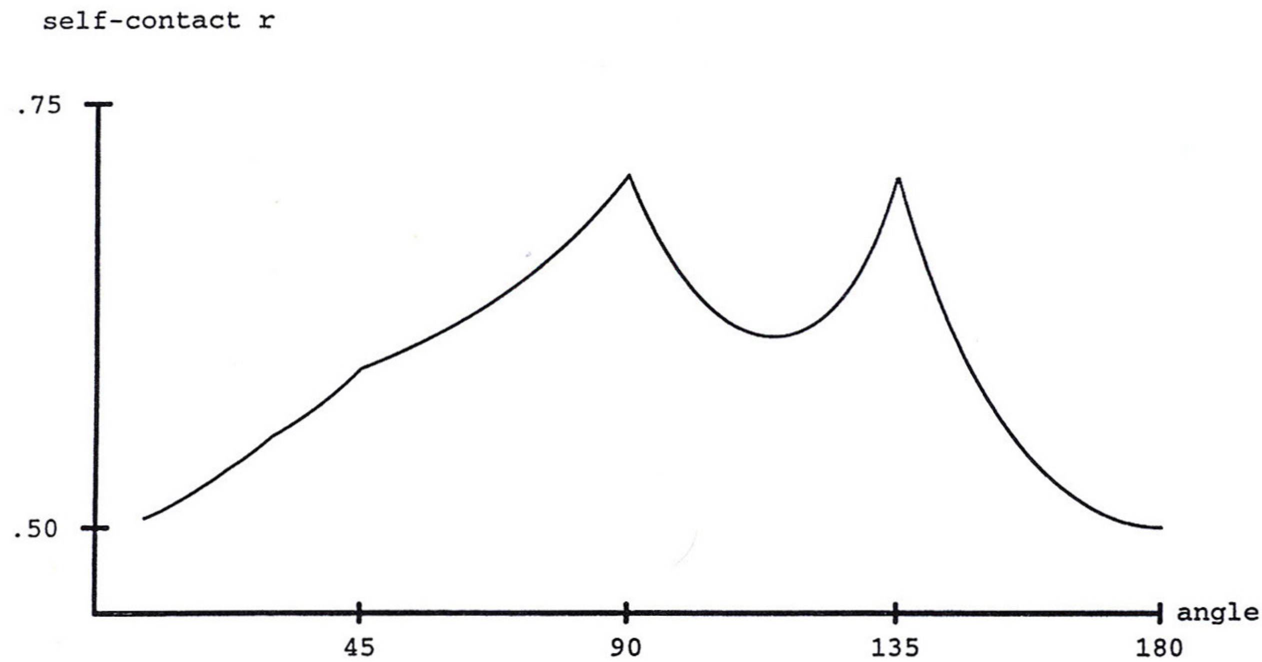

Figure 9. The relation between θ in degrees and the critical contraction ratio r that ensures self-contact. Note that for r less than $\sqrt{3/8}$, self-contact occurs at two θ values. For larger r (except $r = \sqrt{2}$), self-contact occurs at four θ values.

The Range $45° \leq \theta < 90°$

Here, the mininial self-contact sequence occurs when the branch tip $LR^3(LR)^\infty$ coincides with $RL^3(RL)^\infty$; that is, from Eq. (2), we see θ and r are related by

$$r\sin(-\theta) + \frac{r^3}{1-r^2}\sin(\theta) + \frac{r^4}{1-r^2}\sin(2\theta) = 0.$$

By inspection, the structure of the canopy is clearest to the eye when θ is close to 90°. Therefore, let us begin by comparing a plane-filling tree (Fig. 4) and one that is close to filling (Fig. 3). As observed, the top of Figure 4 is simply an interval of dimension $\delta_{\mathscr{C}} = 1$.

Just below $\theta = 90°$, the canopy "opens up" discontinuously, as seen in Figure 3, but its structure remains clearly

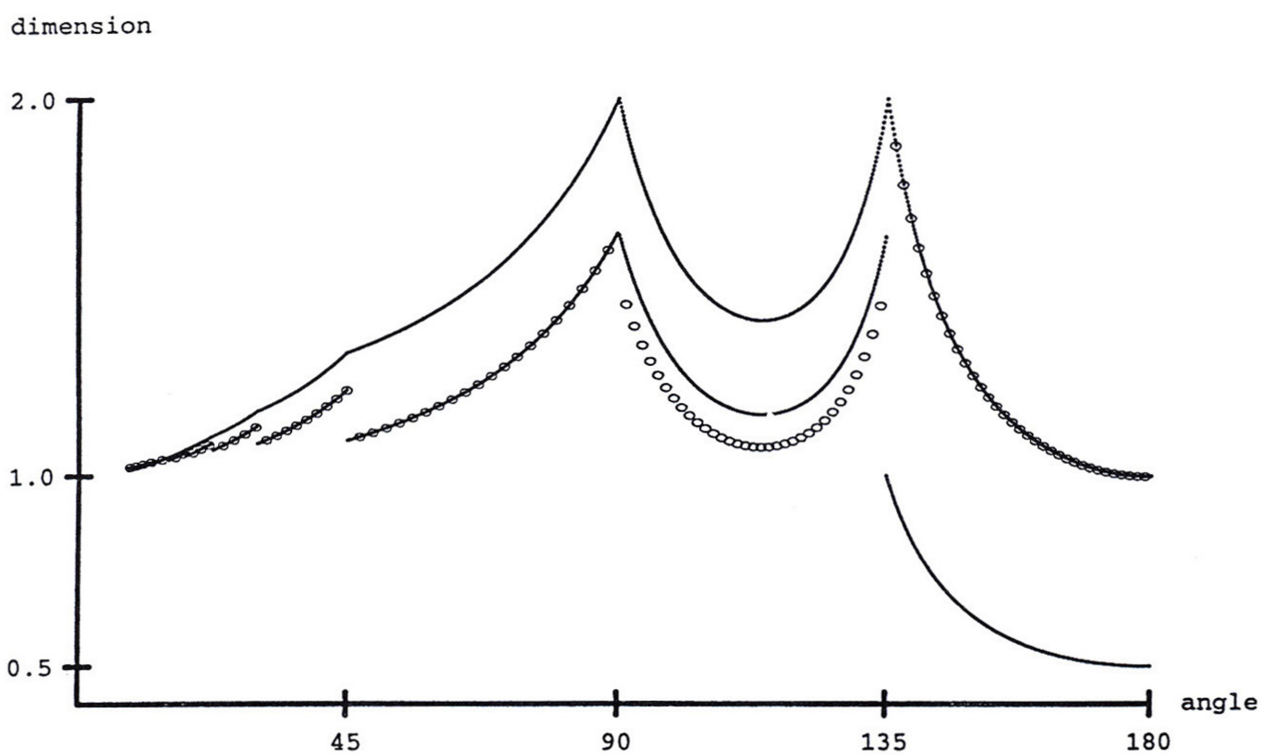

Figure 10. The dimension D of the tip set (top, continuous curve); the dimension $\delta_{\mathscr{C}}$ of the canopy (bottom, broken curve); and the dimension δ_S of the shortest path (small circles). Note $\delta_{\mathscr{S}} = \delta_{\mathscr{C}}$ for $\theta < 90°$, and $\delta_{\mathscr{S}} = D$ for $\theta > 135°$.

recognizable. It is a Koch curve whose generator is made up of four intervals. The endpoints of the left interval are the branch tips $(LR)^\infty$ and $LR^2(LR)^\infty$; those of the next right interval are $LR^2(LR)^\infty$ and $LR^3(LR)^\infty$. From the coordinates of these points, and then by bilateral symmetry of the tree, and using Eqs. (1), we see the generator intervals have lengths

$$r_1 = lr^2, \qquad r_2 = lr^3, \qquad r_3 = lr^3, \quad \text{and} \quad r_4 = lr^2.$$

The dimension $\delta_{\mathscr{C}}$ of the canopy is known to be the (unique) solution of the Moran generating equation, namely (taking $l = 1$) the equation

$$\sum r_j^\delta = 1.$$

Here, the Moran equation takes the form

$$2r^{2\delta} + 2r^{3\delta} = 1.$$

It is convenient, in this article, to write $r^\delta = \lambda$, with λ the solution of the third-order equation

$$\lambda^2 + \lambda^3 = {}^1\!/_2.$$

By contrast, in the plane-filling Figure 4, one could have said that λ was the solution of the equation $2\lambda^2 = 1$, which yields $\lambda^2 = r^{2\delta} = 1/2$, hence, $\delta_{\mathscr{C}} = 1$. (For $\theta = 90°$, minimal self-contact requires $r = 1/\sqrt{2}$.) Therefore, $\delta_{\mathscr{C}}$ jumps discontinuously from the value $\delta_{\mathscr{C}} = 1$ that it takes when $D = 2$, to a larger value it takes when $D = 2 - \epsilon$.

The Range $30° \leq \theta < 45°$

As θ continually decreases to 45°, the canopy reaches a second discontinuity because not only do the branch tips $LR^3(LR)^\infty$ and $RL^3(RL)^\infty$ coincide but also $LR^4(LR)^\infty$ and $RL^4(RL)^\infty$ coincide (and, of course, so do many other branch tips between these). As θ decreases below 45°, $LR^4(LR)^\infty$ and $RL^4(LR)^\infty$ coincide at a smaller r value than do $LR^3(LR)^\infty$ and $RL^3(RL)^\infty$, so minimal self-contact occurs when $LR^4(LR)^\infty$ coincides with $RL^4(RL)^\infty$. See Figure 2. Thus, from Eqs. (2), we see θ and r are related by

$$r\sin(-\theta) + r^3\sin(\theta) + \frac{r^4}{1-r^2}\sin(2\theta) + \frac{r^5}{1-r^2}\sin(3\theta) = 0.$$

Passing 45° adds two intervals to the generator, each of length given by the distance between the branch tips $LR^3(LR)^\infty$ and $LR^4(LR)^\infty$; that is, the generator is a broken line of six intervals of lengths

$$r_1 = lr^2, \qquad r_2 = lr^3, \qquad r_3 = lr^4, \qquad r_4 = lr^4,$$
$$r_5 = lr^3, \quad \text{and} \quad r_6 = lr^2.$$

The dimension $\delta_{\mathscr{C}}$ is now the solution of the Moran generating equation

$$\sum_{j=1}^{6} r_j^\delta = 2r^{2\delta} + 2r^{3\delta} + 2r^{4\delta} = 1,$$

and $\lambda = r^\delta$ is the solution of the fourth-order equation

$$\lambda^2 + \lambda^3 + \lambda^4 = {}^1\!/_2.$$

Again, $\delta_{\mathscr{C}}$ jumps discontinuously from the value it takes for $\theta = 45°$ to a larger value it takes for $\theta = 45° - \epsilon$.

The General Range $90°/N \leq \theta < 90°/(N-1)$

Figure 1 shows a typical example in this range. The generalization is obvious. In this range, minimal self-contact occurs when r and θ are related by Eq. (2) and λ is the solution of the equation

$$\lambda^2 + \lambda^3 + \cdots + \lambda^{N+1} = {}^1\!/_2$$

Each passage through a value of the form $\theta = 90°/N$ is accompanied by a jump up in the value of $\delta_{\mathscr{C}}$.

The Limit $N \to \infty$ and $D \to 1$

The Moran generating equation becomes

$$2(r^{2\delta} + r^{3\delta} + \cdots) = \frac{2r^{2\delta}}{1 - r^\delta} = 1.$$

Its real positive solution is $r^\delta = 1/2$. At the same time as $D \to 1$, one has $r \to 1/2$; hence, the preceding equation yields $\delta_{\mathscr{C}} \to 1$, which is where we started for $\theta = 90°$. This result was to be expected. We had known all along that the passage to the limit $N \to \infty$ makes the canopy become smaller and smaller. Moreover, we noted already that because $\mathscr{C}$ is a curve and a subset of $\mathscr{T}$, one has $1 \leq \delta_{\mathscr{C}} \leq D$. When $\theta \to 0$, $D \to 1$ and the canopy necessarily becomes smoother and smoother.

As D decreases from 2 to 1, θ decreases from 90° to 0°, and we see that, as announced, $\delta_{\mathscr{C}}$ has negative discontinuities for θ of the form $\theta = 90°/N$ and is right-continuous and increasing in the intervals between these discontinuities. See Figure 10. Knowing r as a function of θ in each of the ranges $90°/N \leq \theta < 90°/(N-1)$, we obtain $\delta_{\mathscr{C}}$ by solving the corresponding Moran equation.

The Value $\theta = 90°$

Here minimal self-contact requires $r = 1/\sqrt{2}$. See Figure 4. To see the gap between the upper left and right sides of the tree close, note that the branch tip $LR^2(LR)^\infty$ coincides with $RL^2(RL)^\infty$, $LR^3(LR)^\infty$ coincides with $RL^3(RL)^\infty$, and every branch tip between $LR^2(LR)^\infty$ and $LR^3(LR)^\infty$ with x coordinate 0 coincides with the corresponding tip on the right side of the tree. A similar argument shows the gap between the lower left and right sides of the tree closes. Here, $L^3(RL)^\infty$ coincides with $R^3(LR)^\infty$, and $L^4(RL)^\infty$ coincides with $R^4(LR)^\infty$. In addition, from the IFS formulation, it is easy to see that the set of branch tips is a filled-in rectangle. Consequently, the canopy consists of the edges of this rectangle and thus has dimension $\delta_{\mathscr{C}} = 1$.

The Cantor-like Structure of the Canopy and Its Fractal Dimension when $\theta > 90°$

Unlike the situation when $\theta < 90°$, for minimal self-contact now there are only two relevant branch tips, instead of one for each interval $(90°/N, 90°/(N-1))$. For $90° < \theta < 135°$, minimal self-contact is obtained by requiring that the branch tips $L^3(RL)^\infty$ and $R^3(LR)^\infty$ coincide; for $135° < \theta < 180°$, we require that $L^2(RL)^\infty$ and $R^2(LR)^\infty$ coincide.

Also, unlike the situation when $\theta < 90°$, portions of branches screen from infinity some of the branch tips that might visually appear to belong to the canopy. Figures 5 through 8 illustrate this situation. The effect of this screening is to totally disconnect the canopy. Instead of a variant on the Koch curve, it is now a variant on the Cantor set.

The Range $90° < \theta < 135°$

As seen in Figures 5 and 6, the canopy generator consists of four segments. The two on the left have corners $(LR)^\infty$ and $LR^2(LR)^\infty$, and $LR^2(LR^\infty$ and $LR^3(LR)^\infty = L^4(RL)^\infty$. [The segment with corners $LR^3(LR)^\infty$ and $L^3(RL)^\infty = R^3(LR)^\infty$ is screened from infinity by branch segments.] The first of these generator segments has length $r_0 = lr^2$; the second has length $r_1 = lr^3$. Again, the tree is symmetric; hence, the Moran generator equation becomes

$$2r^{2\delta} + 2r^{3\delta} = 1.$$

So, $r^\delta \approx 0.565198\ldots$, the real root of $\lambda^2 + \lambda^3 = 1/2$. Figures 5 and 6 show the 105° and 130° trees, respectively.

The Value $\theta = 135°$

As in the $\theta = 90°$ case, here self-contact requires $r = 1/\sqrt{2}$. To see the gap between the lower left and right sides of the tree close, note that the branch tip $L^3(RL)^\infty$ coincides with $R^3(LR)^\infty$, $L^2(RL)^\infty$ coincides with $R^2(LR)^\infty$, and every branch tip between $L^3(RL)^\infty$ and $L^2(RL)^\infty$ with x coordinate 0 coincides with the corresponding tip on the right side of the tree. In addition, from the IFS formulation, it is easy to see that the set of branch tips is a filled-in triangle. As in the $\theta = 90°$ case, $\delta = 1$.

The Range $\theta > 135°$

As θ increases from 135° to 180°, the branch tips form a family of Koch curves, with the generator consisting of four segments, the two on the left with corners $(LR)^\infty$ and $LR^2(LR)^\infty = L^3(RL)^\infty$, and $L^3(RL)^\infty$ and $L^2(RL)^\infty$. All four segments have length $r_0 = r^2 l$, where r^2 varies from 1/2 when $\theta = 135°$ to 1/4 when $\theta = 180°$. Hence, the Moran equation for the dimension of the branch tips is simply

$$4r^{2\delta} = 1 \quad \text{or} \quad r^\delta = 1/2.$$

Figures 7 and 8 show trees in this range. Here, branch segments completely screen the Koch curve generator with corners $LR^2(LR)^\infty$ and $L^2(RL)^\infty = R^2(LR)^\infty$, and the generator with corners $R^2(LR)^\infty$ and $R^3(LR)^\infty = RL^2(RL)^\infty$. The remaining generators for the canopy have corners $(LR)^\infty$ and $LR^2(LR)^\infty$, and $RL^2(RL)^\infty$ and $(RL)^\infty$. These segments have length $r_0 = r^2 l$, so the Moran equation is

$$2r^{2\delta} = 1 \quad \text{or} \quad r^\delta = 1/\sqrt{2}.$$

The Critical Value $\theta = 180°$

Here, the tree collapses to its trunk.

Canopy Dimension as a Function of θ

Figure 10, a summary of the calculations presented here, is a graph of the canopy dimension $\delta_{\mathscr{C}}$ and the tip set dimension D, as a function of the branching angle θ. The dimension $\delta_{\mathscr{S}}$ of the shortest path, discussed in the next section, is also shown.

The discontinuities of $\delta_{\mathscr{C}}$ at the critical points $\theta = 90°$ and $\theta = 135°$ deserve special mention. Comparing Figures 3 and 5 shows why $\delta_{\mathscr{C}}$ approaches the same value as $\theta \to 90°^-$ and as $\theta \to 90°^+$ (not the value of $\delta_{\mathscr{C}}$ at $\theta = 90°$). For both $\theta \to 90°^-$ and $\theta \to 90°^+$, the top of the canopy is generated by two copies of the portions between the branch tips $(LR)^\infty$ and $LR^2(LR)^\infty$, and between $LR^2(LR)^\infty$ and $LR^3(LR)^\infty$. The distances between these pairs of tips approach the same limits as $\theta \to 90°$.

Figures 6 and 7 illustrate the discontinuity at $\theta = 135°$. For $\theta < 135°$, exemplified by Figure 6, the generator of the top of the canopy consists of four pieces. For $\theta > 135°$, exemplified by Figure 7, segments of branches shield two of these four pieces. Thus, the discontinuity at $\theta = 135°$ represents a change in the number of generators, as do the discontinuities at $\theta = 90°/N$, $N > 1$. Only the $\theta = 90°$ discontinuity does not involve a change in the number of generators.

Shortest-Path Dimension as a Function of θ

As mentioned earlier, for $\theta \leq 90°$, the shortest path along the branch tips from $(LR)^\infty$ to $(RL)^\infty$ coincides with the portion of the canopy between these points. For $\theta > 90°$, the canopy disconnects into a Cantor set and, thus, cannot be the shortest path. For $\theta > 135°$, the tip set becomes a Koch–Cesàro curve and is the shortest path from $(LR)^\infty$ to $(RL)^\infty$. [Note there is a shorter path in the tree, following those branch segments that screen portions of the tip set and that lie above the line through $(LR)^\infty$ and $(RL)^\infty$. See Figure 8.] At $\theta = 135°$, the shortest path is a line segment.

The range $90° < \theta < 135°$ is more interesting. The shortest path must pass through the double points

$$L^3(RL)^\infty = R^3(LR)^\infty, \qquad L^4(RL)^\infty = LR^3(LR)^\infty$$

[and the corresponding point $R^4(LR)^\infty = RL^3(RL)^\infty$ on the right half; for simplicity, these corresponding points on the right half will no longer be mentioned],

$$(LR)L^3(RL)^\infty = (LR)R^3(LR)^\infty,$$
$$(LR)L^4(RL)^\infty = (LR)LR^3(LR)^\infty, \ldots,$$
$$(LR)^n L^3(RL)^\infty = (LR)^n R^3(LR)^\infty,$$
$$(LR)^n L^4(RL)^\infty = (LR)^n LR^3(LR)^\infty, \ldots.$$

Note that as $n \to \infty$, these points approach $(LR)^\infty$, as expected.

The line segments connecting these points in order, together with those connecting the corresponding points on the right side, form the generator of the shortest path. Formulas (1) and (2) could be used to find the lengths of these segments, and so determine their scalings, but a much simpler approach is to use the IFS construction of the tip set.

First, B_L takes the tip set to that part determined by $(LR)^\infty$, $LR^2(LR)^\infty$, $LR^3(LR)^\infty$, $L^3(RL)^\infty$, and $L^2(RL)^\infty$; B_R

takes the tip set to the corresponding part on the right. Letting A_1 denote the segment from $L^2(RL)^\infty$ to $L^3(RL)^\infty$ and A_2 the segment from $R^2(LR)^\infty$ to $R^3(LR)^\infty = L^3(RL)^\infty$, we see that

the segment from $L^3(RL)^\infty$ to $L^4(RL)^\infty$ is $B_L(A_1)$
the segment from $L^4(RL)^\infty$ to $(LR)L^3(RL)^\infty$ is $B_LB_R(A_2)$
the segment from $(LR)L^3(RL)^\infty$ to $(LR)L^4(RL)^\infty$ is $B_LB_RB_L(A_1)$, . . .

That is, these generator segments have lengths r (length of A_1), r^2 (length of A_2), r^3 (length of A_1), ... The initiator has length $2r\sin(\theta)/(1 - r^2)$, and from their endpoints, we compute directly that A_1 and A_2 have length $2r^3\sin(\theta)/(1 - r^2)$. Taking units so the initiator length is 1, the generator for the shortest path consists of two segments of length r^3, two of length r^4, two of length r^5, and so on. The Moran equation is

$$2(r^3)^d + 2(r^4)^d + 2(r^5)^d + \cdots = 1$$

or

$$\frac{(r^d)^3}{1 - r^d} = {}^1\!/_2.$$

Thus, $r^d \approx 0.589755$, so $d \approx \log(0.589755)/\log(r)$, where r is expressed in terms of θ by Eq. (3).

The Mix Map

As mentioned earlier, in the range $135° < \theta < 180°$, the tip set becomes a Koch–Cesàro curve. In this range, the double points have addresses

$$L^2(RL)^\infty = R^2(LR)^\infty$$

and

$$SL^2(RL)^\infty = SR^2(LR)^\infty,$$

where S is any finite string of L and R. Figure 8 makes this clear: note the locations of $L^2(RL)^\infty = R^2(LR)^\infty$, $L^3(RL)^\infty = LR^2(LR)^\infty$, and $RL^2(RL)^\infty = R^3(LR)^\infty$, for example.

Parsing these strings in the usual way, we have

$$L^2(RL)^\infty = 00(10)^\infty \text{ and } R^2(LR)^\infty = 11(01)^\infty$$

and so on. Noting the addresses of these double points, we define the *mix map* as the function on binary sequences taking the addresses of double points to sequences corresponding to the same real number; that is, the mix map is defined by

$$m(b_1b_2b_3b_4\ldots) = b_1\bar{b}_2b_3\bar{b}_4\ldots,$$

where $\bar{b}_i = 1 - b_i$.

To see the claimed identifications, first note that $L^2(RL)^\infty$ coincides with $R^2(LR)^\infty$, and

$$m(00(10^\infty)) = 01(11)^\infty = 1/4 + 1/8 + 1/16 + \cdots,$$
$$m(11(01)^\infty) = 10(00)^\infty = 1/2,$$

and the binary sequences corresponding to $SL^2(RL)^\infty$ and $SR^2(LR)^\infty$ are sent by m to

$$m(S)01(11)^\infty \quad \text{and} \quad m(S)10(00)^\infty$$

if S contains an even number of elements, and

$$m(S)10(00)^\infty \quad \text{and} \quad m(S)01(11)^\infty$$

if S contains an odd number of elements. Here, $m(S)$ denotes the obvious restriction of m to a finite sequence. Regardless of the number of elements in their common initial string, sequences corresponding to double points are sent by m to equal real numbers.

As a function on binary sequences, the mix map is clearly self-inverse. Viewing it as a map on the unit unterval requires more care: equivalent binary expansions do not get sent to the same numbers. For example, $m(0(11)^\infty) = 0(01)^\infty \rightarrow 1/6$, whereas $m(1(00)^\infty) = 1(10)^\infty = \rightarrow 5/6$. Consequently, to view m as a map $[0, 1] \rightarrow [0, 1]$, we must adopt some convention—no terminal strings of all 0, for example—about sequences representing points in the domain. Figure 11 is a graph of the mix map.

To emphasize the sizes of the jumps, in Figure 11 we have drawn a line segment between points of the form $(x,m(x))$ and $(y,m(y))$, where $x = s1(00)^\infty$ and $y = s0(11)^\infty$, s any finite string of 0 and 1; that is, the binary strings x and y correspond to the same real number.

Even without adding the line segments, the graph has dimension 1. This can be seen by an argument analogous to that showing that the product of two Cantor sets, each of dimension 1/2, has dimension 1.

The longest line segment connects $m(0(11)^\infty) = 0(01)^\infty \rightarrow 1/6$ and $m(1(00)^\infty) \rightarrow 5/6$, so has length 2/3. The second longest line segments connect $m(11(00)^\infty) \rightarrow 5/12$ to $m(01(00)^\infty) \rightarrow 1/12$, and $m(11(00)^\infty) \rightarrow 7/12$ to $m(10(11)^\infty) \rightarrow 11/12$; both have length 1/3. Continuing in this fashion, we see that the first variation of m is

$$\frac{2}{3} + 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{6} + \cdots + 2^n \cdot \frac{1}{3 \cdot 2^{n-1}} + \cdots,$$

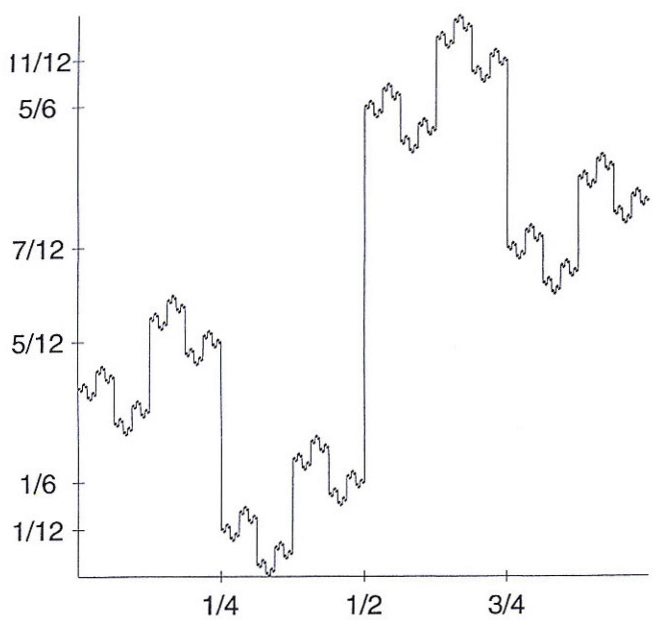

Figure 11. The graph of the mix map, with jumps connected with line segments

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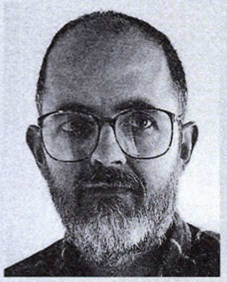

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Michael Frame started as a topologist, with a Ph.D. from Tulane University under Terry Lawson. Since then, he has been attracted to the relations between topology and fractals. He has spent five semesters at Yale working with Benoit Mandelbrot. There, he introduced two courses on fractal geometry, one for science and engineering students and one for humanities students. As text for the latter course, he wrote *Chaos Under Control*, with David Peak.

which diverges. The qth variation is

$$\left(\frac{2}{3}\right)^q + 2 \cdot \left(\frac{1}{3}\right)^q + 4 \cdot \left(\frac{1}{6}\right)^q + \cdots + 2^n \cdot \left(\frac{1}{3 \cdot 2^{n-1}}\right)^q + \cdots$$

Because $2^n \cdot (3 \cdot 2^{n-1})^{-q} = (2/3^q) \cdot (2 \cdot 2^{-q})^{n-1}$, the qth variation converges for $q > 1$.

Finally, by using different bases or modifying different length substrings, we can produce many other functions with different patterns of changes. For example, $m_1(b_1b_2b_3 \ldots) = b_1b_2\bar{b}_3b_4b_5\bar{b}_6 \ldots$ is another self-inverse function, whereas $m_2(b_1b_2b_3 \ldots) = b_2b_3b_1b_5b_6b_4 \ldots$ satisfies $m_2^3 = \mathit{identity}$. This general theme can lead in other directions. For instance, by flipping values at increasingly widely separated positions, we produce functions that are extraordinarily mixing in the large, but less mixing in the small. However, this has (apparently) digressed from our study of trees, and so we leave it for another time.

Animating the Changes

A clear understanding of the morphology change of self-contacting trees as θ increases from 0° to 180° is easily communicated through animation. Several efforts in this direction can be found at http://www.union.edu/PUBLIC/MTHDEPT/research/fractaltrees/

Acknowledgment

We thank the referee, whose thoughtful comments led to several extensions of the original version.

Further Reading

Besides [FGN], Chap. 16, trees are discussed in [PJS], [L], and many other sources. The construction of trees as L-systems is described in [PL].

Iterated function systems are used in [FGN], Chap. 20, and developed in [B] (which introduced the currently accepted term); see also [H].

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