EUROCONFERENCE

FIFTH PARIS COSMOLOGY COLLOQUIUM

FUNDAMENTAL PROBLEMS IN CLASSICAL, QUANTUM AND STRING COSMOLOGY

within the framework of the International School of Astrophysics
'DANIEL CHALONGE'

and sponsored by the Commission of European Communities
TMR Programme

OBSERVATOIRE DE PARIS, 1-6 JUNE 1998

Editors

H. J. DE VEGA

LPTHE, Université de Paris VI

N. SÁNCHEZ

DEMIRM, Observatoire de Paris

FRACTAL LACUNARITY AND SCENARIOS FOR THE NEAR-ISOTROPIC DISTRIBUTION OF GALAXIES

B.B. MANDELBROT
Yale University
New Haven CT 06520-8283, USA

Abstract. The author proposed fractal scenarios for the distribution of galaxies in 1975 and 1977; they were expanded in 1982 and are being developed further. The claim is that galaxies have a scale-invariant fractal distribution with a dimension well below 3. Fractality is the only input that is needed to account for the observed clustering combined with voids and walls. Unfortunately, the 1975-1977 fractal scenarios have inacceptable features: they present too many holes ("lacunas") to agree with the approximate isotropy of the sky. However, better chosen fractals eliminate this defect. Indeed, this paper's words and pictures show that, in a subtle new way that deserves further study, the impression of isotropy can be approached arbitrarily closely by a fractal. In fact, there are many ways of "tuning" a fractal's lacunarity. Several of the sections improve on statements already made in 1982 and other earlier publications. Section 3 is the most novel in this paper: it extends our understanding of lacunarity via a generalization of Lévy flights to be called "cyclic stutter flights."

The paper concludes with two important questions: a) an empirical one that concerns antipodal correlations among galaxies and b) a theoretical one that postulates a link between fractality and the Laplace or other equations of gravitation.

THIS PAPER'S ILLUSTRATIONS DEMONSTRATE that a random fractal's perceived texture can be made closer than is generally expected to being "homogeneous" or "isotropic." Therefore, the perceived smooth texture of galaxy maps is not an argument against the thesis that the large-scale structure of the universe is fractal over a very broad range of distances.

More precisely, the shapes that created most scientists' intuition of fractality are notorious for being very uneven: they are "lacunar," in the sense of being full of holes. This paper's point is that this familiar high lacunarity *is not* an invariable characteristic of fractals. Quite to the contrary, without destroying fractality or changing the fractal dimension, a geometric pattern can – in a sense – come arbitrarily close to homogeneity or isotropy, hence closer to representing the galaxy data. In particular, Section 3 of this paper is the first publication concerning the properties and possible use of a new family of fractals, namely,"cyclic stutter dusts."

Because of new material and important background issues, this paper eventually expanded into a presentation of fractal lacunarity that may interest mathematicians and scientists not directly concerned with galaxies. It must be emphasized that many basic issues remain wide open.

There will be many references to my books [1] [2] [3] but this paper is largely self-contained. The discussion of lacunarity is both more quantitative than in my books and more richly illustrated.

Section 6 raises a challenging and possibly important empirical question. Section 7 sketches the author's general scenario concerning the solutions of partial differential equations: it postulates a dynamical origin of fractality that may apply to the distribution of galaxies.

Editorial comment. A shorter paper with nearly the same title [4] appears in the Proceedings of a NATO School held in Erice (Sicily) in September 1997. That earlier text was perceived as too concise to be useful, and new results immediately made it incomplete. I am grateful to this book's editors for suggesting that this paper should combine, within a more user-friendly and broader presentation, the bulk of [4] and substantial additional material. An even more complete treatment is planned for [5].

1. Four basic scenarios for the spatial distribution of galaxies, especially from the viewpoint of isotropy

1.1. REASONS FOR DISTINGUISHING MORE THAN TWO SEPARATE SCENARIOS

When discussing the large scale distribution of galaxies, [1], [2] and [3] drew a contrast between two scenarios, namely, homogeneity and fractality. See also [6]. Unfortunately, this contrast proved over-simplified, insofar as it led to diverse misunderstandings. In response, Section 1 proposes that the number of different scenarios deserves to be viewed as not equal to two, but at least three; moreover the third scenario, fractality, may deserve to be split further. The reasons are subtle and are best understood after the fact, rather than before. However, the better-informed readers may welcome the introductory sketch to be presented in this Section 1.1. Other readers will prefer to skim it at first reading and return to it later.

Historical sequence of scenarios for the distribution of galaxies: hierarchy, homogeneity and fractality. Those scenarios can be viewed "dialectically" as involving a thesis, an antithesis and a synthesis. Newton is reputed to have thought of a uniform infinite universe, but feared it would be unstable.

The first scenario to find strong advocates appeared shortly after Newton: it postulated an infinite universe of galaxies that forms a hierarchical structure. Ironically, this scenario was put forward well before galaxies, clusters and superclusters were actually identified as astronomical bodies. It came from the likes of I. Kant and was adopted by science-fiction writers and only late by a few physicists or astronomers, like Charlier. Early on, the sole motivation was that a hierarchy avoids the Olbers paradox. Much later, de Vaucouleurs [7] built upon early data of Carpenter and obtained an empirical power-law form for the mass-radius relation; he could not account for it, except by a hierarchy.

The next scenario was not fully formulated until Einstein: it states that the distribution of galaxies is homogeneous. This came about as a necessary consequence of the cosmological principle and an important corollary is scale invariance with respect to overall (Hubble) expansion.

Even those who ignore or despise the "dialectic" terminology may find it useful to consider "hierarchies" as a non-physical "thesis" and "homogeneity" as an anti-thesis primarily based on a theory. A dialectic argument becomes full when completed by a "synthesis." The synthetic scenario I proposed in the 1960s and 1970s asserts that the distribution of galaxies can be approximated by a suitable random fractal dust. The notion that this scenario is best viewed as a synthesis did not occur to me until long after the fact.

Compared to hierarchical structures, fractality contradicts neither the motivating desire to solve the Olbers paradox nor the Carpenter - de Vaucouleurs power-law form for the mass-radius relation.

Compared to homogeneity, fractality is motivated by a theoretical argument that is in very similar style, but broader on an essential point. This argument does contradict the cosmological principle, but only by generalizing it to take a less demanding form I proposed, namely, the *conditional cosmological principle.* The key fact is that scaling invariance obviously holds for the homogeneous distribution, but in addition is shared by shapes that are highly non-homogeneous, namely, the fractals. (This invariance leads to one of several broad reasons why geometric fractals are widespread and fundamental in physics: they accompany the key analytic ideas of *renormalization* and *scaling.*)

The novelty brought by the fractal synthesis is that an approximate hierarchical appearance need no longer be deliberately built-in as an *input* in the model. Instead, it is obtained as an essential *output.* This feature can be viewed as the major consequence of fractality and justifies - after the fact - the occasional recourse to strict hierarchies as "cartoons" of properly fractal models.

This paper shall only consider those fractals, called self-similar, which are invariant under similarities; a self-similar structure is one that can be decomposed into small parts, each of reduced-scale copy of the big part. (The graph of a function, for example of a temperature in terms of time, cannot be self-similar. Such a graph is called fractal when it is self-affine, that is, invariant under transformations called "affinities", which are more general than similarities.)

Let us review the reason for the preceding complications. It resides in the fact that the logical contrast between homogeneity and fractality is very asymmetric. Homogeneity is uniquely defined, while fractality incorporates a broad range of possibilities. The first non-hierarchical fractal scenarios for the distribution of galaxies [1][2] proved clearly inacceptable because of their *extreme non-isotropy.* Some critics reacted to this freature by concluding that the broad notion of fractal remains inacceptable, even when one moves on beyond hierarchical structures. The alternative path that I took consisted in seeking to preserve fractality and scaling under more suitable implementations.

Altogether, many misunderstandings are avoided if the non-hierarchical fractal scenarios are split into two kinds. A) Pedagogical non-hierarchical random fractals that do not claim to be realistic but are readily implemented graphically. In their absence, fractality would most probably have failed to attract any attention. They help understanding and provide strong foundations for generalizations. B) Suitable fractals. [3] went to great lengths to disclaim the pedagogical scenarios as realistic and moved on to subtler and more "proper" scenarios that involve low lacunarity. When (as often happens) I am criticized for the resulting complexity, my response has not changed since [3] came out in 1982: I did what I could, asking more questions that I could answer, and kept inviting the critics to do better.

Once again, this distinction should disappear with the misundertandings that provoked it.

To summarize, as things stand today there are good reasons to distinguish four pure scenarios and a multitude of hybrids. Two of the pure scenarios are classical and my books can be said to have added two more scenarios.

A misunderstanding and a metaphor concerning fractality itself and other concepts in their historical development. We discussed a confusion between two roles of hierarchies. It is worth discussing it further in a broader context via a metaphor, that is, a comparison that is useful but must not be taken textually. The reader knows that the algebraic notion of group arose about 1830 as a generalization of the classical elementary operations of addition, multiplication and rotation. Does it follow that primitive humans who counted using pebbles on the beach were already practicing group theory? To say so might be correct on a narrow legalistic sense, but would not only be anachronistic but useless, in fact, thoroughly misleading. One who invokes group theory ordinarily thinks of a framework that is broader than addition, multiplication and rotation. To be fair and precise, various structures that were recognized as groups after that notion was developed, may on occasion be usefully called "proto-groups." Furthermore, when pondering properties of general groups, it is often useful to see what they become in the case of addition or rotation invoked as "cartoons." (Analogous comments could be made about real versus complex numbers.)

Similarly, the hierarchical structures exemplified by Cantor dusts and Peano or Sierpinski curves arose as special tools of the nascent point-set topology. However, the actual drawing of any topological object is necessarily an object in Euclidean space. As a result, in addition to the properties that motivated their introduction, those structures have a host of non-topological properties. Above all, they were drawn as self-similar. This property was devoid of significance until fractal geometry made it of intrinsic interest. Today, those "proto-fractal" structures provide useful "cartoons" of more significant fractals.

1.2. THE STANDARD HOMOGENEITY SCENARIO FOR THE LARGE-SCALE DISTRIBUTION OF GALAXIES

The overwhelming majority, considered by Newton but formulated by Einstein [8] assumes a homogeneous distribution, except for "local" disturbances that must be specified separately and are not highly significant from the large-scale viewpoint. Their quantitative features include the "local" correlation function. Their "qualitative" features involve many aspects of texture, clusters, voids, filaments, walls and the like.

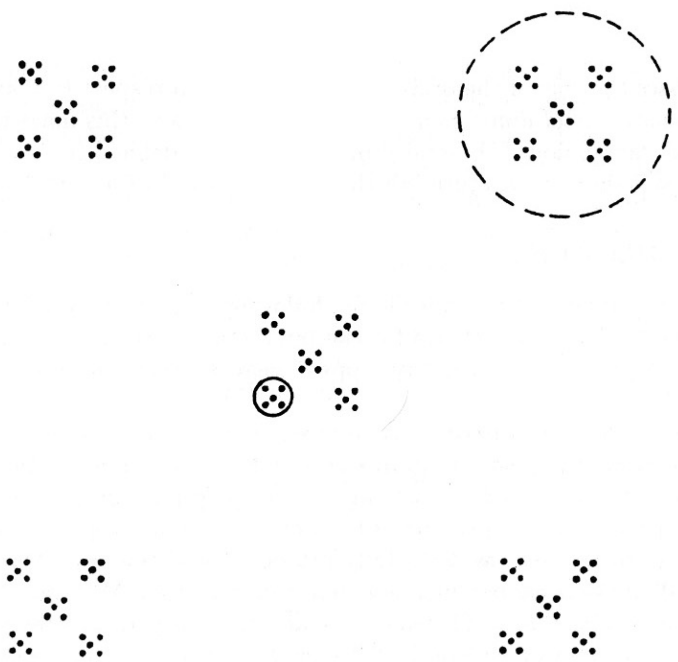

Figure 1: A simple proto-fractal: an elementary strict hierarchy.

1.3. STRICT NON RANDOM HIERARCHY, THE ANOMALOUS MASS-RADIUS RELATION, AND THE MASS AND SIMILARITY DEFINITIONS OF FRACTAL DIMENSION

An ancient alternative to homogeneity postulates that the distribution of galaxies involves a scale-invariant hierarchy of clearly separate and physically meaningful levels of structure, together with a well-defined "center". This view did not originate with astronomers, but philosophers and science-fiction writers, as documented in [3]. It was phrased in terms of stars and came about well before the actual discovery of galaxies and clustering.

The hierarchy shown in Figure 1 is drawn in the plane rather than the space. It is self-explanatory, but an additional comment is found at the end of Section 2.5, and extensive elaborations are found in [3], pp. 95-96.

The anomalous mass-radius relation at the center. Consider the functional dependence upon R of the mass $M(R)$ that is contained within a sphere of radius R whose center coincides with the center of the universe. Each time R is multiplied by some factor $1/r$, $M(R)$ is multiplied by a factor N. Therefore, writing $D = \log N/\log(1/r)$, one finds that $M(R)$ is of the order of R^D.

In Figure 1, N and $1/r$ are equal, therefore $D = 1$. The average density $M(R)/\pi R^2$ is of the order of R^{-1}, therefore tends to 0 as $R \to \infty$.

Classical geometry tells us that the formal relation "$M(R)$ of the order of R^D" also holds in a homogenous distribution, where the exponent D is the dimension of space and takes the value $D = 3$. Thus the exponent fractal D

generalizes the role of the ordinary dimension with respect to mass. Hence it is called *mass fractal dimension.* When $D < 3$ in space, this mass-radius relation is called "anomalous." Fractal dimension can be defined in many other ways, but for self-similar structures all the definitions yield the same value.

1.4. PROJECTIONS

The key issue in this paper is the following very controversial one. In order to fit the evidence, a scenario for the large scale distribution of galaxies must, after it is projected on the sky, appear near isotropic, that is, near invariant rotationally.

The homogenous distribution implies, as immediate corollary, the strongest possible form of isotropy, both in space and in projections on the sky.

By contrast, hierarchical sets and *all other* fractals occupy a negligible portion of space; strictly speaking, all are of zero volume and are non-isotropic in space. As to projections, especially projections on the sky, their effect is that fractal dimension cannot increase, but can decrease. More notably, project a D-dimensional set in 3-dimensional space on to a plane or the sky. The projection is at most of dimension 2. Therefore, if $D > 2$, projection necessarily decreases dimension; if $D < 2$, projection *may* preserve dimension; and $D = 2$ is a borderline case. The latest data of Pietronero et al [9] suggest $D = 2$ for galaxies, therefore this borderline case deserves particular attention. This is best done in the plane, where the borderline dimension is $D = 1$. The great wealth of possibilities is illustrated by five properties of Figure 1, which we now proceed to sketch.

Projections of the elementary hierarchy in Figure 1, when it is infinitely interpolated but not extrapolated. First consider the parallel projections along the directions $\theta = 0, \theta = 45°$ and $\theta = 90°$. Many points hide behind one another, and the projections are Cantor dusts made of $N = 3$ parts each of which is identical to the whole reduced in the ratio $r = 1/5$. The formula $D = \log N / \log(1/r)$ for the similarity fractal dimension yields $\log 3/\log 5$. The fact that this value is < 1 expresses a decrease of dimension under projection. One can weight each point in the projection by the number of points in space leading to that projection. This procedure defines a multifractal measure, an important topic but beyond the scope of this paper.

Next, a parallel projection along the direction $\theta = \tan^{-1}(1/3)$ behaves very differently: as is easily seen, a projection along this angle fills an interval, no point being hidden behind any other point. Imagine that the smallest square including Figure 1 is of side 1, and that the total mass in this square is 1, hence the density is 1. Then the $\theta = \tan^{-1}(1/3)$ projection is of length $6/5$ and its density is $5/6$. Similarly, the $\theta = \text{cotan}^{-1}(1/3)$ projection is of length 4 and density $1/4$.

Next, project Figure 1 on the "sky," which in the plane is the circle of radius 1 with the same center as the hierarchy. In each of four directions, there is, once again, exact superposition between small clusters nearby and large super or super-super-clusters far away; these "hide" one another, and yield the same "projection supercluster of angular width $2\tan^{-1}(1/5) \sim 23°$." Those superclusters are separated by large empty arcs of the circular "sky", adding up to a very nonhomogeneous projection and a loss of dimension.

Next, modify Figure 1 as follows: rotate the middle fifth by a random angle $< \pi r/2$, next rotate the middle fifth of the middle fifth, then rotate the middle fifth of the middle fifth of the middle fifth (approximated on Figure 1 by a point) etc. This process carried over several hierarchical levels will insure near-isotropy, but only on a sky with the same origin as the construction itself.

The case $D > 1$. Figure 1 is readily changed to one of dimension > 1. The corresponding projections are non-isotropic in the absence of random rotation and less extensive random rotations suffice to insure approximate isotropy.

1.5. TWO DISTINCT PEDAGOGICAL FRACTALS: LÉVY DUSTS AND ROUND-TREMA DUSTS, THE OBSERVED NEAR-ISOTROPY OF THE SKY AND THE CONCEPT OF FRACTAL LACUNARITY

Fractal constructions featured and extensively illustrated in [1] [2] [3] affected greatly the scientists' perception of what a random fractal is, therefore deserve to be called pedagogical fractals. They will be studied in detail in Sections 3 and 4, respectively, but deserve to be sketched here.

The more widely known was given a "nickname", Lévy dust, in [2]. It is created by running (both forward and backward in time) a random walk such that the steps' direction is isotropic and their length U is random and follows the scaling probability distribution $Pr\{U > u\} = u^{-D}$ with $0 < D < 2$. This distribution is assumed to hold for $0 < u < \infty$, implying that $Pr\{U > 0\} = \infty$. This divergence is startling but reveals itself to be acceptable upon further consideration because the resulting infinity of infinitesimal steps adds up to a finite total contribution. All concerns about rigor vanish if a cutoff $u > \varepsilon$ is imposed, and the value of ε disappears from the final results.

The Lévy dust has now become widely known and helps identify what is meant by "pedagogical fractals." The Lévy dust's correlations, both in space and in projection on the sky were obtained in [10] and the derivation is reproduced in [8].

The second pedagogical fractal is the "round trema dust," "trema" being the Greek word for hole. It will be extensively discussed in section 4 and its description is best withheld until then.

Projections on the sky of a Lévy dust or a round trema dust. They are grossly non-isotropic. Figure 2 shows what happens for the Lévy dust with $D = 1.23$. The same is true for most of the other widely known fractals.

This last property, when added to the nonisotropy of the hierarchical model, led to the following widely-held opinion.

Widespread converse belief: the observed isotropy provides sufficient reason to demand homogeneity and exclude fractality.

In fact, this sparseness is NOT an unavoidable consequence of fractality, and the bulk of this paper describes scenarios that suffice to establish that fractality can approximate isotropy as closely as the empirical evidence may

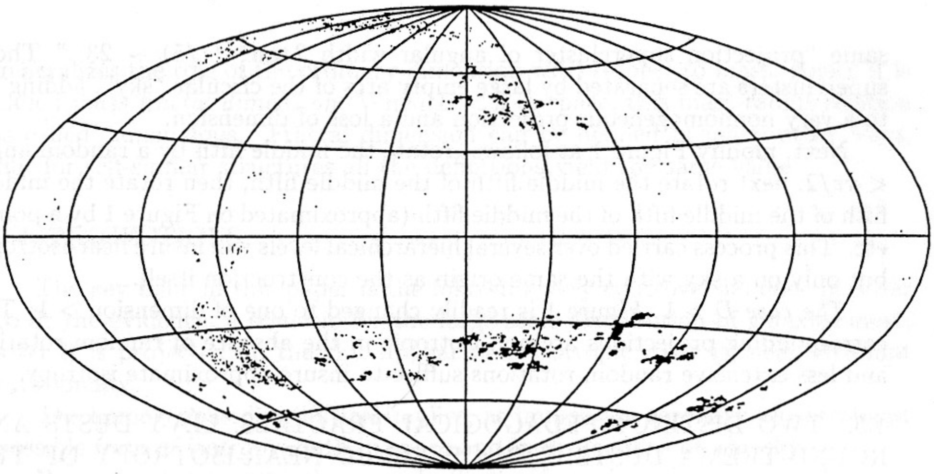

Figure 2: Projection of a three-dimensional "pedagogical fractal" on the sky. This fractal is a Lévy dust drawn by a Lévy flight. Its extreme sparseness and non-isotropy make it clearly inappropriate as a scenario of the distribution of galaxies.

require. These scenarios are meant to provide alternative proofs of existence and do not claim to be realistic.

This approach will necessarily be indirect, because it encounters a profound mathematical difficulty. The general study of the notion of "closeness" is part of topology, which lists innumerable alternative definitions. In the present context, however, no standard definition is applicable. Stated differently, no concept from the presently available mathematics is able to describe the notion that is required here, namely, that a homogenous distribution can be "approximated" by fractal distributions.

"Lacuna" is the Latin word for hole. Since non-isotropy is manifested by the presence of holes, Chapters 34 and 35 of [3] proposed the term "lacunarity" to describe the propensity of a set to include holes. In advance of an appropriate quantitative measure of lacunarity, the fact that lacunarity is tunable could not have been proven analytically. But I demonstrated its validity constructively by exhibiting several examples of distinct sequences of succesive approximations, each involving a special construction that leads to explicit pictures. Many years passed until the appearance and properties of those illustrations suggested, not one, but several possible measures of lacunarity. None of these definitions were (or, in my opinion, could have been) guessed in advance.

This indirect procedure did accomplish what was intended. To summarize this paper, it will go beyond [3] and elaborate on the following surprising fact.

Contrary to widespread opinion, fractality and the absence of conspicuous holes are not contradictory properties. " Proper" fractals can be characterized as made of those cases where these two properties coexist. Additional further evidence and testing will narrow down the meaning of "proper."

1.6. PROPERLY FRACTAL SCENARIOS FOR THE DISTRIBUTION OF GALAXIES

We come at last to the relatively new synthesis of hierarchy and homogeneity to be discussed in this paper. It must be presented and labeled with great caution, as the "properly fractal scenario." Once again, [1] [2] [3] call it simply "fractal scenario" (or "fractal model," but the word model seems to have a different flavor in comology.) However, the word "fractal" covers a very broad range of shapes that are unsuitable for galaxies. This versatility contributes powerfully to the broad usefulness of the concept of fractal, but in the present context it created two sorts of misunderstanding. Each led to a major pedagogical complications.

The first complication is that both classical constructions, namely homogeneity and a strict hierarchy, are special cases of fractality, but both are completely "atypical." A second source of complication is so fundamental to this paper that it deserves to be discussed separately in the following subsection.

1.7. A CONTRIVED SEQUENCE OF INCREASINGLY ISOTROPIC HIERARCHICAL STRUCTURES OF UNCHANGING DIMENSION

To begin, isotropy is grossly violated in the hierarchical structure at the top line of Figure 3. However, without having to change the fractal dimension of a hierarchical universe, it is easy to change its construction to decrease lacunarity at will. The remainder of Figure 3 illustrates this possibility.

To continue, isotropy is also grossly violated in the "Lévy dusts" and the "round trema dusts" of Section 1.5. However, those "pedagogical fractals" are thoroughly understood, yet of transparent simplicity. Hence, not only are they unavoidable illustrations of some basic properties of the fractals, but each provides an excellent point of departure for the construction of "proper" fractal scenarios, as will be seen, respectively, in sections 3 and 4.

1.8. HYBRID SCENARIOS; THE KEY QUESTION IS WHETHER OR NOT THE DATA SHOW A WELL-DEFINED AND FINITE CROSSOVER TO HOMOGENEITY

Let us interrupt to mention an issue that is very important to this paper's topic but not to this paper itself. As already mentioned, the mainstream exemplified in [8] represents the galaxies' overall distribution through a "hybrid" scenario that grafts local perturbations on homogeneity in the large-scale range. In this mental picture, fractality gained acceptance in the mainstream as a possible representation of the local perturbations. Observe that such hybrids are not scale-invariant.

The large scale is often defined as lying beyond a small cross-over scale r_0, typically $4Mpc$. Some time after this hybrid was proposed, the discovery of the voids convinced many observers, myself among them, that the distance to the crossover (if there is one) must far exceed $4Mpc$. (As a matter of fact, the very notion of r_0 is simply the result of inapplicable statistical tests.)

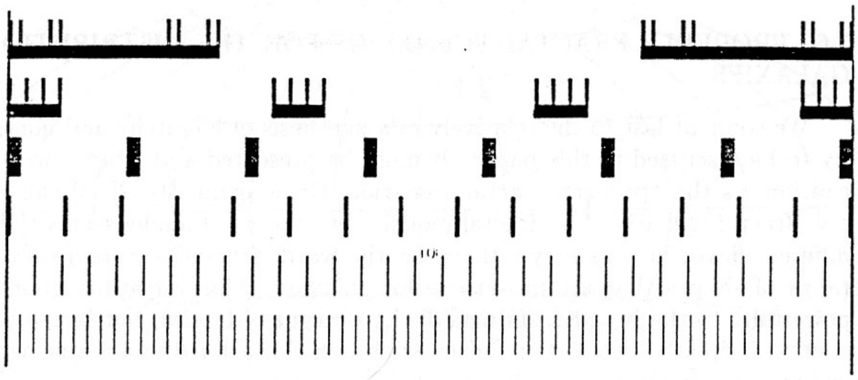

Figure 3: A stack of Cantor dusts on the interval [0, 1]. The k-th line from the top shows the construction of a Cantor dust made of $N = 2^k$ parts, each of which is identical to the whole reduced in the ratio $r = 4^{-k}$. Those parts are equally spaced. Therefore, $D = log\, N/log\,(1/r) = 1/2$ on every line. However, as k increases, those dusts are perceived as being increasingly "close" to translational invariance, that is, to the one-dimensional counterpart of rotational invariance on the sky. This notion of closeness is non-classical and continues to be somewhat elusive.

The task of determining whether or not the hybrid picture is justified with a meaningful and finite crossover was faced brilliantly by Luciano Pietronero of Rome, and his team. Their work [9] interprets fractality as extending to the very limits of observation. It is described elsewhere in these *Proceedings* and I have nothing significant to contribute. Little in this presentation is affected by the existence of a crossover or its precise value.

2. Diverse observations and beliefs, and the extent to which the conflicting scenarios agree with them

The reader may either continue with this section or proceed immediately to sections 3 and 4 for a continuation of the study of isotropy and lacunarity. Section 4 describes very valuable "trema dusts" . Their construction is based on a generating shape of unit area called "template." In the "round trema scenario," the template is simply a circle and the trema set is characterized by big holes and "neutral" lacunarity. For the purpose of galaxies, this lacunarity is too high. However, as we shall see, changing the template greatly modifies the texture by making it smoother at all scales. This is achieved without destroying fractality or even changing the fractal dimension.

The intellectual path being followed presents a bit of paradox.

In this section, each subsection's title will describe a property that an ideal scenario is expected to satisfy. Those properties overlap; each will be symbolized by a letter.

2.1. OBSERVED PROPERTY B: BUNCHING. GALAXIES FORM CLUSTERS AND SUPERCLUSTERS

Since clustering contradicts homogeneity, it is ordinarily set apart as belonging to local disturbances. In the hierarchic structures, clustering is deliberately put in.

The singular feature of fractals is that they are very special on this account. In their case, hierarchical clustering need not be a conscious input; it can be an inevitable consequence. More precisely, every pedagogical and proper fractal is completely scale-free: its definition involves no linear scale whatsoever. For example, the definition of Lévy dusts in Section 1.4 involves the probability distribution $Pr\{U > u\} = u^{-D}$ over $0 < u < \infty$. After the fact, however, the picture of every sample of those random fractals is analyzed by human brains into a largely subject-independent hierarchy of clusters and superclusters. In other words, samples of random fractals exhibit near-universal clustering that has no "trigger" in the generating mechanism.

Are galaxy clusters "real"? The preceeding surprising and (to my mind) fundamental discovery creates a question concerning the actual distribution of galaxies. In the words and gestures of an unidentified attendee at a recent lecture of mine, could it be that galaxy clusters are here (he pointed to his head) and not there (he pointed to the ceiling)?

I think this is a serious question that deserves a serious experimental investigation. To my knowledge, no such question arises in alternative scenarios for clustering. I view it as a very strong "plus" for the fractal scenario.

2.2. OBSERVED PROPERTY V: VOIDS. GALAXIES FORM VOIDS, FILAMENTS, WALLS AND CONFIGURATIONS

The comments in Section 2.1 extend to voids, filaments and walls. At a much earlier lecture, another unidentified attendee inquired how I had arranged for filaments to be present in my simulations. At that time, I did not know about filaments, therefore had made no special arangement. The presence of filaments simply followed from the fractality that was used.

2.3. PROPERTY C: CREATIVITY. PEDAGOGICAL AND PROPER FRACTALS SEEM "CREATIVE," INSOFAR AS THEIR OUTPUT AUTOMATICALLY EXHIBITS TEXTURAL FEATURES THAT WERE NOT A DELIBERATE PART OF THEIR INPUT

Because of its importance, the content of Section 2.1 and 2.2 deserves a restatement. In the homogeneous and hierarchical scenarios, all "output" properties worth noting are immediate logical consequences of the input. This is why local disturbances to the homogeneous scenario must be added using entirely separate free-standing arguments. Starting with a hierarchy, every other feature must also be added separately.

On this account, both pedagogical and proper fractality stand diametrically apart from the two older scenarios. The key underlying assumption consists in scale invariance combined with the replacement of the usual homogeneity by a weakened form that suffices to exclude the hierarchical scenarios. As mentioned in Section 2.1 and 2.2, the surprising key fact is that those assumptions allow innumerable non-quantitative consequences to be drawn.

2.4. PROPERTY D: MATTER HAS A POSITIVE OVERALL DENSITY

Passion and controversy surround property D. It follows immediately from homogeneity, and homogeneity follows from scaling combined with a positive density. To the contrary, all fractal scenarios imply a vanishing overall density. This includes "improper" fractal scenarios like hierarchies and the pedagogical constructions, as well as "proper fractals."

2.5. PROPERTY Ω: THE UNIVERSE HAS NO PRIVILEGED ORIGIN; TWO FORMS OF THE COSMOLOGICAL PRINCIPLES: ABSOLUTE AND CONDITIONAL

A homogeneous universe satisfies an absolute form of the cosmological principle: every point is like every other point.

A hierarchical universe includes a privileged origin, namely, the point around which the hierarchy was constructed. Therefore, the cosmological principle is invalid. A contrary claim is discussed at the end of this subsection.

The pedagogical and proper fractals led [1] [2] [3] to replace of the usual cosmological principle by the following *conditional* form.

"In a frame of reference whose origin is Ω, the distribution of matter is independent of Ω, under the sole condition that Ω must be a material point."

As a consequence, if Ω is *not* a material point and R is fixed, the sphere of radius R centered on Ω is empty with a probability equal to 1.

The basic quantitative consequence predicted by fractality is the "mass-radius" relation that generalizes the relation described in section 1.3. A "properly chosen" sphere of radius R contains a mass of the order of R^D, with $1 < D < 3$. The expression *of the order of* has a technical meaning implying that the ratio $M(R)/R^D$ is not too different from 1.

It remains to explain the words "properly chosen" center. In the hierarchical case, the mass-radius relation requires the sphere of radius R to be centered on the center of the universe. By contrast, random fractals can be constructed so that the mass-radius rule holds under the far weaker condition that the sphere's origin must itself belong to the fractal. Those fractals deserve to be called *fractally* homogeneous.

An extrapolated hierarchical universe and its sequence of privileged origins. The finite picture shown in Figure 1 is meant to be interpolated and extrapolated without end. There is a unique natural interpolation, which consists in replacing each dot by a reduced version of the whole, and so on without end.

But extrapolation is an altogether different matter. Indeed, let us show that there is an infinity of alternative extrapolations. Each is labeled by an infinite "directing sequence" of the integers from 0 to 4; in other words, each is labeled by an identifier G, which is a real number between 0 and 1 that is, written in the base 5. The construction starts with Figure 1 taken to be infnitely interpolated. This figure defines a universe U_0 with privileged origin Ω_0. The five fifths of U_0 are denoted by 0 to 4. Step 1: enlarge U_0 in the ratio of 5, identify the part of this enlargement that is numbered by the first "decimal" of the generating number G and position the enlargement so that this identified part is superposed on U_0. This creates an enlarged universe U_1 with a privileged origin Ω_1. Repeating the same process creates an enlarged universe U_2 with a privileged origin Ω_2, then a sequence of universes U_k with their privileged origins Ω_k. Knowing the universes U_k and U_0 suffices to identify the sequence of origins from Ω_k down to Ω_0. When the identifier G is chosen at random uniformly on $[0, 1]$, $\Omega_k \to \infty$ almost surely as $k \to \infty$ and (for given k) the "coincidence event" $\Omega_k = \Omega_{k-1}$ occurs with the probability $1/5$.

Altogether, such an extrapolated hierarchial universe is no longer cursed with having a single privileged origin, but for the wrong reason: because there is an infinite sequence of privileged origins, characterizing U_0 and a sequence of "layers" surrounding U_0.

2.6. PROPERTY I AND AN INTRODUCTION TO SECTIONS 3 AND 4: AS SEEN FROM EARTH, THE SKY IS NEARLY ISOTROPIC

This second very controversial issue was already mentioned in Section 1. Isotropy follows from homogeneity as an immediate corollary, but hierarchical structures yield a wildly spotty sky. For proper fractals, the situation depends on the value of the dimension D. It also depends on a feature beyond dimension that I introduced and called fractal lacunarity. Sections 3 and 4 describe two distinct ways of controlling lacunarity.

3. Beyond the Lévy dusts: they are too lacunar to represent the distribution of galaxies but newly-introduced "stutter dusts" allow lacunarity to take an arbitrarily low value

This section generalizes Lévy dusts in the spirit of the "fractal sums of pulses," a concept that is sketched in [11] and will be expounded in [5]. The generalization transforms Lévy's independent increments into statistically dependent ones.

A first effect is that the Lévy dusts' excessive lacunarity can be "tuned" down.

A second effect concerns the fractal dimension D of stutter dusts in E-dimensional Euclidean space. For Lévy dusts D cannot exceed 2 and the limit $D \to 2$ is very atypical; it reduces to a continuous curve, namely the path of Brownian motion. In sharp contrast, a shining asset of stutter dusts is as follows. When the "generating template" is cyclic, a stutter dust's fractal dimension is only bounded by E.

3.1. INDEPENDENT INCREMENTAL JUMPS: LEVY FLIGHTS AND LEVY DUSTS RELATION BETWEEN LACUNARITY AND NON-FICKIAN DIFFUSION

The responsibility for coining the nickname, "Lévy flight," weighs heavily on my shoulders, because the flight of a bird or plane is continuous, while a Lévy flight is discontinuous.

In continuous time, Lévy flight $L_\alpha(t)$ is a self-affine process with the exponent $\alpha < 2$. The only consequence that matters here is that $L_\alpha(t)$ diffuses in time like $Ft^{1/\alpha}$. Here F is a random prefactor and the exponent satisfies $1/\alpha > 1/2$. It exceeds the value $1/2$ that characterizes the atypical $\alpha = 2$ limit of Lévy flight, which is Brownian motion. As the dust's dimension α decreases, the Lévy flight's jumps increase, creating increased sparseness. Therefore, the dream of decreasing lacunarity hinges on the possibility of either braking or breaking the big jumps without creating a change in dimension with independent increments the only self-affine jump processes are the Lévy flights, therefore the break-up of big jumps must necessarily involve statistically dependent jumps. The special examples needed here are best described in the plane and preceded by the description of a dead end.

A tempting idea leading to a dead end. One can think of each Lévy jump as a straight arrow with the weight concentrated at one or both ends. Since big holes and high lacunarity are due to long arrows, one may want to fill the holes by spreading mass in a uniform thread between the arrow's endpoints. Refering to [3], Plates 296 to 300, this processing would replace a dust by a broken "contrail" that goes through every point of the dust. However, the resulting threads will be infinitesimally thin and will not fill the holes. An alternative modification of Lévy's construction consists in interrupting each arrow into μ consecutive subarrows of equal length and identical direction. Instead of erasing the contrail arrows (minus the endpoints), one inserts $\mu - 1$ additional points. This artificial processing also leaves the perceived lacunarity unchanged.

3.2. DECREASE OF LACUNARITY ACHIEVED BY "STUTTERING" THAT CREATES STATISTICAL DEPENDENCE BETWEEN THE JUMPS; STUTTER FLIGHTS AND STUTTER DUSTS

Fractal sums of pulses. Isolated points and the replacement of dusts (totally disconnected sets of points,) by continuous curves are both inacceptable. Both are avoided in the fractal sum of pulses (FSP) introduced in [5] [11] [12]. Visualize the set of Lévy jumps in the space (x, y, t), as arrows attached orthogonally to the point t of the time axis, each sticking out in the appropriate direction. The second dead end in Section 3.2 was created by replacing each arrow by μ arrows infinitesimally close to one another.

The thought that suggested the simplest FSP consisted in replacing each arrow by μ repeats (hence the term, "stutter"), separated by equal finite intermissions. During each intermission, other subarrows occur, each "seeding" matter at its endpoints. If, as we do, one wishes the resulting stutter flight to be self-affine, an additional condition must be satisfied. The duration of intermission must be proportional to (arrow length)$^{\alpha}$.

Of special importance is the relation of the stutter flights to Lévy flight. It hinges on another parameter that enters in the construction, namely, the expected value C of the number of jumps (before splitting) that occur during a unit of time and whose length exceeds a unit of jump. The theory of the FSP construction, as sketched in [11] and presented in [5] and [12], is subtle. A basic fact is that if $\alpha < 2$, allowing $C \to \infty$ makes the "stutter flight" concentrate increasingly in large jumps that are nearly independent statistically. As a result, the stutter flight converges to a Lévy flight, following an interesting new form of convergence, called "lateral". But for small C the stutter flight's structure is very different from that of the lateral asymptotics. It is dominated by increments other than the largest ones, namely, by increments that are no longer independent. This is why a decrease in C and/or an increase in μ, lead to a decreased lacunarity.

Fine-tuning the stutter-generating template. The preceeding construction can be called "forward stutter flight," because each arrow is replaced by μ subarrows that all go forward in the same direction. The next step away from Lévy flight is to release the sub-arrows from this special condition and allow their arrangement to be a dilation or a reduction of a more general prescribed pattern called "generating template."

Of particular interest are templates to be called "cyclic." They are defined by the property that the vector sum of the contributing μ subarrows vanishes; examples are a) two subarrows, one forward and one reverse, and b) an array of subarrows that form a regular polygon. Compared to forward generating templates, cyclic templates are far more effective in "braking" the propensity of the Lévy jumps to drift away. An illustration is provided by Figure 4.

A valuable extension of the range of admissible values of the dimension. Cyclic stutter dusts have a remarkable and most valuable property: for them, the dimensional exponent α is freed from the Lévy bound $\alpha < 2$, and becomes unbounded. That is, the rules of dependence in a cyclic stutter process are so finely adjusted that a dust of dimension > 2 can be obtained.

4. Beyond the round trema dusts; their lacunarity, which can be called "neutral" is too large to represent the distribution of galaxies; non-circular trema dusts' lacunarity can be made arbitrarily low

This section, largely reproduced from [4], was written before Section 3, but can be read independently. Both the modifications of Lévy dusts in section 3 and the modifications of the trema dusts to be presented here involve no artificial base and lacunarity can be made to decrease, not increase.

4.1. ROUND TREMA DUSTS IN THE PLANE: CONSTRUCTIONS AND PROPERTIES

It is best to first describe the practical algorithm used in Figures 5A and 6A, and then the theoretical algorithm that underlies and motivates this practical algorithm. The construction is most easily visualized in the plane. "Trema" is

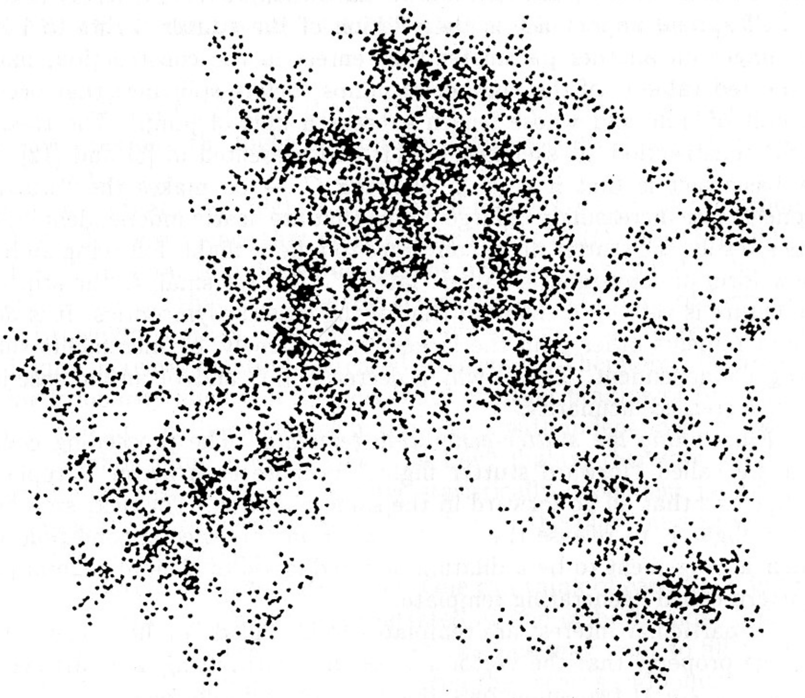

Figure 4: A new fractal dust of dimension $D = 1.26$. Lévy dusts of this dimension ([3], Plates 296 to 300) enormously more lacunar. This variant is a cyclic stutter dust for which the generating template is a low-order approximation of the Cesaro-Peano curve illustrated in [3], Plate 65.

the Greek word for hole. Trema dusts in space will be illustrated in Section 4.3.

Practical algorithm. Calculations are performed in a unit square. One may also interpret them on a square belonging to a periodic tiling of the plane. The first input is a seed which a "random" number generator transforms into a sequence of numbers u_k, with $k \geq 1$. For each $n \geq 1$, one picks the point P_n with the coordinates u_{2n-1} and u_{2n}. The second input is a positive real number that will prove to be a codimension, therefore is denoted by the letter C. (In a terminology more common in the discussion of the correlation between galaxies, this letter is γ.) For each n, one cuts out from the plane a "trema" defined as the circle of center P_n and area $C/2n$. Those circles are allowed to overlap.

Principal result. Taken together, the tremas have a positive probability of covering the unit square. One can show that the probability of covering depends on $C - 2$.

When $C < 2$ the probability of covering is < 1 and the uncovered set is a fractal called "round trema dust" that will be investigated in the remainder of this paper.

When $C > 2$, the probability of covering is 1. In that case the area of the largest trema is > 1, therefore covering might have been expected. However, this argument is at best heuristic and the actual proof is far more delicate.

Proof that the trema dust's fractal dimension is $2 - C$. While the following argument is heuristic, its conclusion can be made rigorous. Given a random point in the unit square, its probability of *not* being covered by the n-th trema is $(1-C/2n)$. Consider the probability $p(\varepsilon)$ of this point's not being covered by the tremas from $n = 1$ to $n = n_{max}$, of which the smallest is of area $C/2n_{max} = \pi\,\varepsilon^2$. This probability $p(\varepsilon)$ is the product $\prod(1 - C/2n)$ taken for $1 \leq n \leq n_{max}$. Replacing $1 - C/2n$ by $exp(-C/2n)$ will introduce errors that amount to a multiplicative prefactor, mostly due to small values of n. Hence, up to this and other multiplicative numerical prefactors,

$$p(\varepsilon) \propto exp[-(C/2)\Sigma(1/n)] \propto exp[-C\,log\sqrt{n_{max}}\,] \propto exp(C\,log\varepsilon) = \varepsilon^C.$$

This probability is also the expected value of the area not covered by tremas. To cover this area with squares of side ε, one needs $N \propto \varepsilon^C\varepsilon^{-2}$ squares. The fractal dimension is given by $log\,N \sim D\log(1/\varepsilon)$, hence $D = 2 - C$, as announced.

The non-specified multiplicative prefactors are easily compensated for. One way, which also corrects for boundary conditions, is to restrict attention to the central portion of the unit square and restrict n to $n > n_{min} > 1$. When C is small, as in Figure 5A, these precautions make little difference.

The rigorous trema construction on the whole plane. The preceding simple algorithm closely approximates an underlying general construction that I introduced in 1972 (see [3], p. 282) on the line and extended to space in [13]. A preliminary step is one that has already been taken in the context of stutter dusts: the origin $\{x, y\}$ and the radius r of a circle to be cut out are used to define an "address point" in an "address half-space" of coordinates x, y and $r > 0$. The construction consists in selecting the address points at random in this half-space, independently of each other, using the following rule.

The number of points in the product domain $[x, x+dx] \times [y, y+dy] \times [r, r+dr]$ is taken to be a Poisson random variable of expectation $(C/\pi)r^{-3}\,dx\,dy\,dr$. It follows that the number of tremas such that $0 < x < 1$, and $0 < y < 1$, and area $> \pi r^2$ is itself a Poisson random variable of expectation $(C/2\pi)r^{-2} = (C/2)/area$.

The practical algorithm stated early in this subsection is obtained as follows: The Poisson variable is replaced by its expectation and the resulting value is identified with the rank n of a trema in order of decreasing areas. In this practical algorithm, the sizes of small tremas are accurate, but the largest tremas are distorted because their random size is replaced by a "typical" value.

4.2. NON CONVEX TREMA DUSTS IN THE PLANE: INCREASINGLY "FRAGMENTED" TEMPLATES INJECT A SELF-SIMILAR SMOOTHING THAT PRESERVES DIMENSION BUT DECREASES LACUNARITY

When C is sufficiently small, the round trema sets are actually not dusts, but connected sets perceived as being the whole plane, minus an infinity of holes or gaps that are near-circular. My search for generalization began with a view concerning human perception. In black-and-white patterns, it seems that the impression of texture combines the respective areas of the two colors and the areas and perimeters of the large single-color domains. This belief suggested that the texture could be changed and made more uniform by increasing the perimeters of the largest visible domains.

To achieve this result the broad recipe is to proceed like in the trema set construction of Section 3.1, except for one change rich in consequences: the circles are replaced by reduced or expanded versions of a more general non-round domain called "template." The template is normalized to be of area 1 and must have a perimeter beyond the value $2\sqrt{\pi}$ that characterizes the circle of unit area. The proof that $p(\varepsilon) = \varepsilon^C$ follows from the distribution of the areas of the tremas, and does not involve their shapes. Therefore, their result remains valid. Therefore, the trema set's dimension remains unchanged.

The simplest templates, all of which were investigated, are as follows: the annulus (bounded by two concentric circles), the union of several non-overlapping annuli, the sieve (a circle with many holes), the "counter-sieve" (to coin a word to denote the holes in a sieve) and the rhombus. Non-isotropic templates are made statistically isotropic by combining reduction with an arbitrary rotation.

Examples are shown in Figures 5A,B,C,D, 6A,B,C,D and 7. In both Figures 5.A, the template is a single circle of unit area. In both Figures 5.B, it is made of 4 circles, each of area 1/4 (namely, a "base" circle, and 3 circles centered at random but without overlap on a "orbit" concentrical to the base circle). In both Figures 5.C, there are 10 circles, each of area 1/10 (namely, a base and 9 circles placed on two orbits). In both Figures 6.D, there are 19 circles, each of area 1/19 (namely, a base and 18 circles placed on three orbits.)

Aside. The fact that fragmentation of the template decreases lacunarity takes time and pratice to be accepted intuitively, and the reader attentive to mathematical subtleties may have identified a delicate point that demands reassurance. It concerns the fact that a simulation necessarily concerns the intersection of a trema set with a finite square. With a probability equal to 1, such a square would fall entirely within a gap of the set, an event that must be excluded by carrying out the simulation carefully. Furthermore, when a template is fragmented, a round trema close enough to the square's boundary creates holes beyond that boundary. Conversely, some large round tremas that lie wholly beyond the boundary will create visible holes within the boundary. These issues have been investigated carefully and their outcome will be described elsewhere.

4.3. TREMA DUSTS IN SPACE

Principle of the construction. The generalization from 2 to 3 dimensions is obvious. The address space is now the product of the 3d space by a half-line, and, the address point density becomes $C(3/4\pi)r^{-4}\, dx\, dy\, dz\, dr$. In the practical algorithm, the volume of the n-th trema is taken to be $C/3n$.

Examples. The two examples in figure 8 show that a suitable choice of template can achieve a fractal that the eye perceives as arbitrarily close to

A B D C

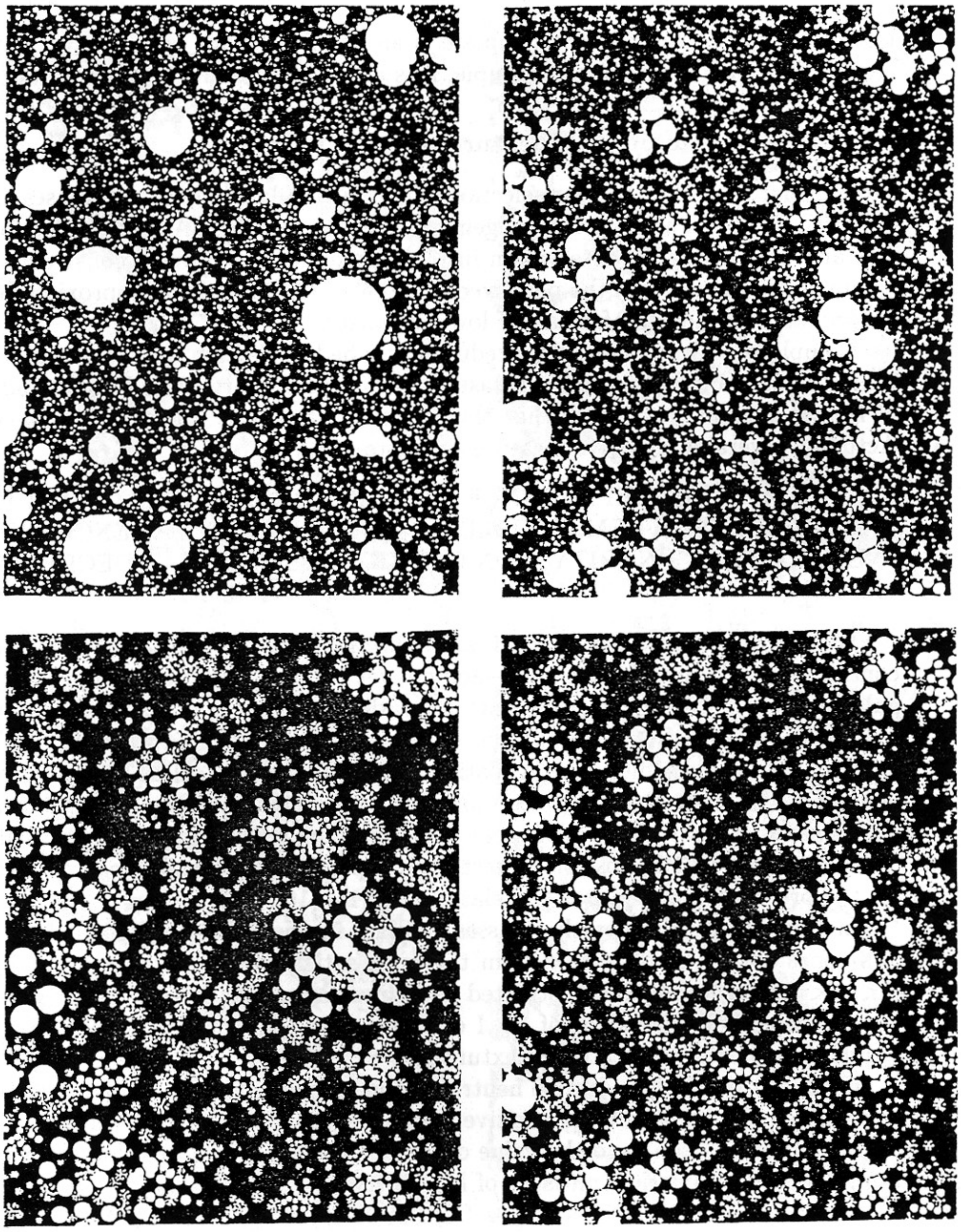

Figure 5: Trema sets on the plane when the template is a circle or a collection of circles, as described in the text. This figure gives a direct visual proof of a basic fact concerning lacunarity: by breaking up the template, one can decrease a trema set's lacunarity as much as one may wish. Here, the codimension C of the tremas is small, hence the dimension $D = 2 - C$ of the trema set is close to 2; the lacunarity decreases clockwise from top left.

isotropy. In one example, the template is an annulus (space between two concentric spheres). In the other example, it is a collection of a few spheres.

5. Alternative numerical measurements of lacunarity

Thus far, the fact that the selection of a non-circular template decreases lacunarity expressed a subjective judgement based on pictures. As seen in Section 3.2, and confirming an observation in [3], this suffices to provide constructive examples that show that the perceived isotropy of the sky can be approximated by properly constructed fractals of low lacunarity.

Completing a roundabout procedure described in Section 1.6, an examination of such examples suggested measures of closeness appropriate for the study of fractal lacunarity. I think that this sufficient stimulant was also necessary. Actually, several alternative measures come to mind.

5.1. ANTIPODAL DEPENDENCE AS PARTIAL MEASURE OF LACUNARITY; ROUND TREMAS YIELD ANTIPODAL INDEPENDENCE AND "NEUTRAL" LACUNARITY; NON CONVEX TREMAS THAT DECREASE LACUNARITY ALSO CREATE POSITIVE GLOBAL CORRELATIONS

We now proceed to describe a first quantitative measure of lacunarity, and continue by making an interesting —and not yet verified— prediction concerning galaxies. It is necessary to begin with a few definitions.

A thin two-directional spatial cone can be viewed as made of two "antipodal" half-cones with the same apex. A spatial distribution will be called "antipodally independent," if everything concerning its intersections with one of those thin half-cones is statistically independent from anything concerning the other half-cone. However, it is impossible to test for independence, while it is easy to test for the presence or absence of correlation [14] [15] [16].

In this spirit, consider the masses that lie in the two above-defined thin half-cones and at a distance r from the apex satisfying $R\mu < r < R$, where $0 < \mu < 1$. Because of the postulated scale invariance, the correlation between those masses is independent of R. I observed that this correlation is a useful (while only partial) indicator of texture.

Lacunarity will be said to be neutral in the case of antipodally uncorrelated sets. When the correlation is positive (resp., negative), lacunarity will be said to be low (resp., high) and the value of the correlation is in a position to serve as one possible numerical measure of lacunarity.

The preceding definitions are useful, indeed immediately applicable. Concerning the Lévy dusts I showed in [10] that in space they are antipodally negatively correlated. This confirms that texture is clumpy to an inacceptable degree, as already mentioned and illustrated in Figure 2. As to the round-trema dusts (e.g., Figures 5A and 6A) it is easy to see that they are antipodally independent, since a round trema can intersect one or the other very thin half-cone, but not both.

A more interesting case is that of a non-convex template (e.g., Figures

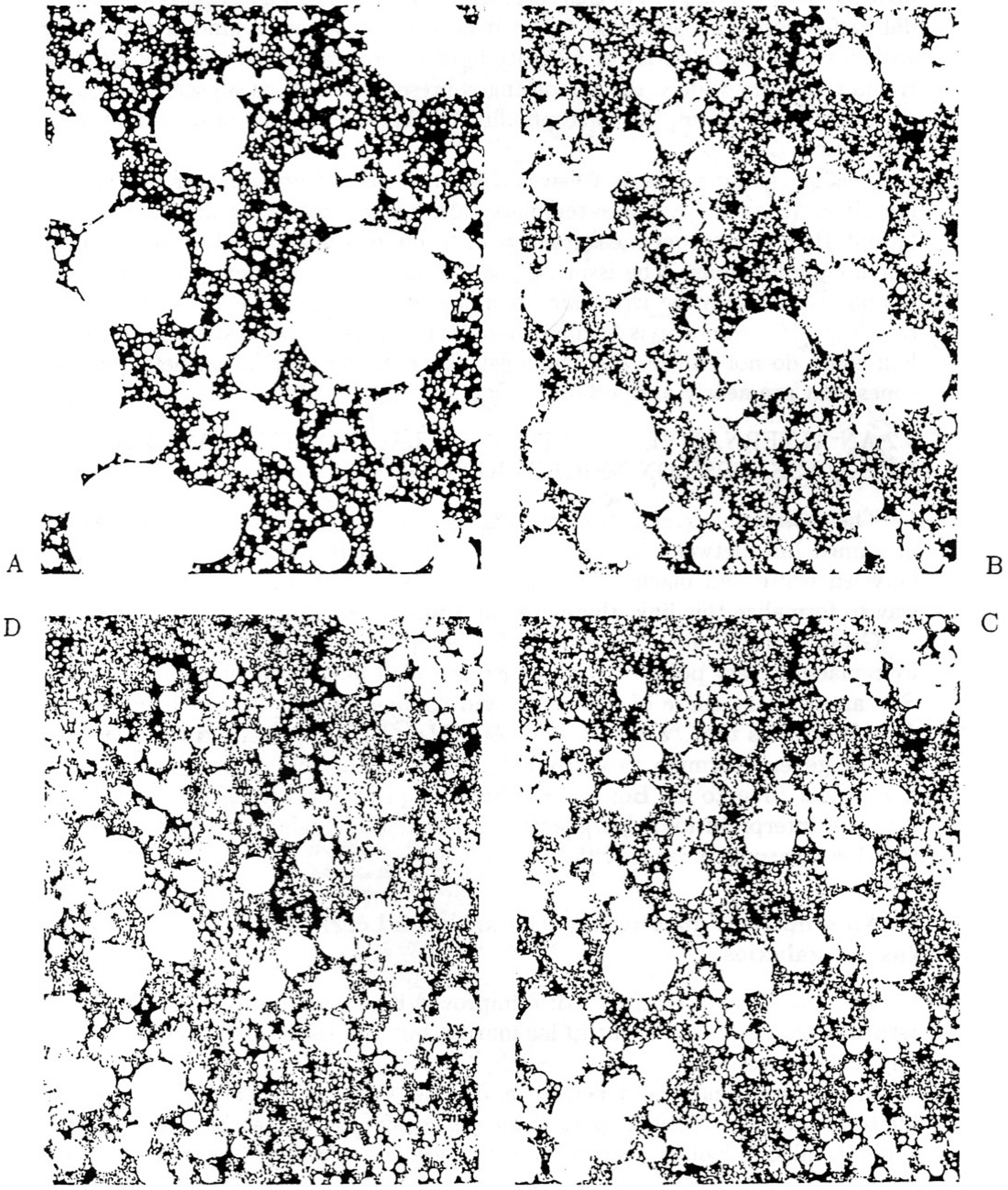

Figure 6: Same as Figure 5, except that now the codimension is larger, hence the dimension is smaller.

5B,C,D, 6B,C,D, and 8.) In this case, the same trema can intersect both antipodal half-cones. This situation, when it occurs, introduces a kind of smoothing with a range of the order of magnitude of the overall trema radius. Because trema radii vary widely, the smoothing is present, overall, at all scales. As a result, as is readily seen, the corresponding trema dust exhibits positive antipodal correlation.

The preceding paragraph leaves aside the case of non-circular but convex templates (Figure 7). Those templates also yield a vanishing antipodal correlation! Hence, a proper measurement of lacunarity unavoidably involves additional complications. The issue only arises in the case of highly non-isotropic templates and may be irrelevant to many applications. Nevertheless, it was raised in [15], where it is argued that, in the general case, strictly antipodal half-cones do not suffice, and one must also consider correlation between half-cones that are nearly antipodal.

5.2 AN ALTERNATIVE DEFINITION OF LACUNARITY, BASED ON MINKOWSKI EPSILON-NEIGHBORHOODS

The motivation stated in section 3.2 should now be restated: it relied on a presumed link between uniformity of texture and the length of the boundary between white and black. The concept of "epsilon-neighborhood" provides a way to formalize this link, therefore our alternative definition of lacunarity.

Given a set S and a radius $\varepsilon > 0$, the epsilon-neighborhood S_ε is defined by replacing every point of S by a disc bounded by a circle of radius ε. When S is an ordinary curve of dimension and codimension $C = D = 1$ and length L, it is obvious that "area $(S_\varepsilon) \sim 2\varepsilon L = 2L\varepsilon^C$." When S is a fractal, one has a more general formula "area $(S_\varepsilon) \sim \Lambda^{-1}\varepsilon^C$." The role of C in this formula is classical (due to G. Bouligand), but the prefactor $1/\Lambda$ remained without concrete interpretation. I argued in [14] that this prefactor is one of several possible measures of lacunarity.

6. An empirical question: Is the antipodal correlation positive in the case of galaxies

The question raised in the title improved itself as soon as isotropy was interpreted as lower-than-neutral lacunarity, but the issue is of obviously wider physical interest.

In the case of neutral dependence, one can say that, when the origin of our bilateral thin pencil cone is part of the set, it acts as a screen between the half-cones it links. When lacunarity is not neutral, the origin fails to act as a screen. Given the long-range nature of gravitational forces, such screening would be surprising, and positive overall dependence would be expected.

In the same spirit and in conclusion, it is interesting to point out a curious form of the contrast between locality and globality. When the universe is viewed as homogenous overall, the very term *local* applied to the disturbances implies that distant portions are statistically independent. To the contrary, every fractal

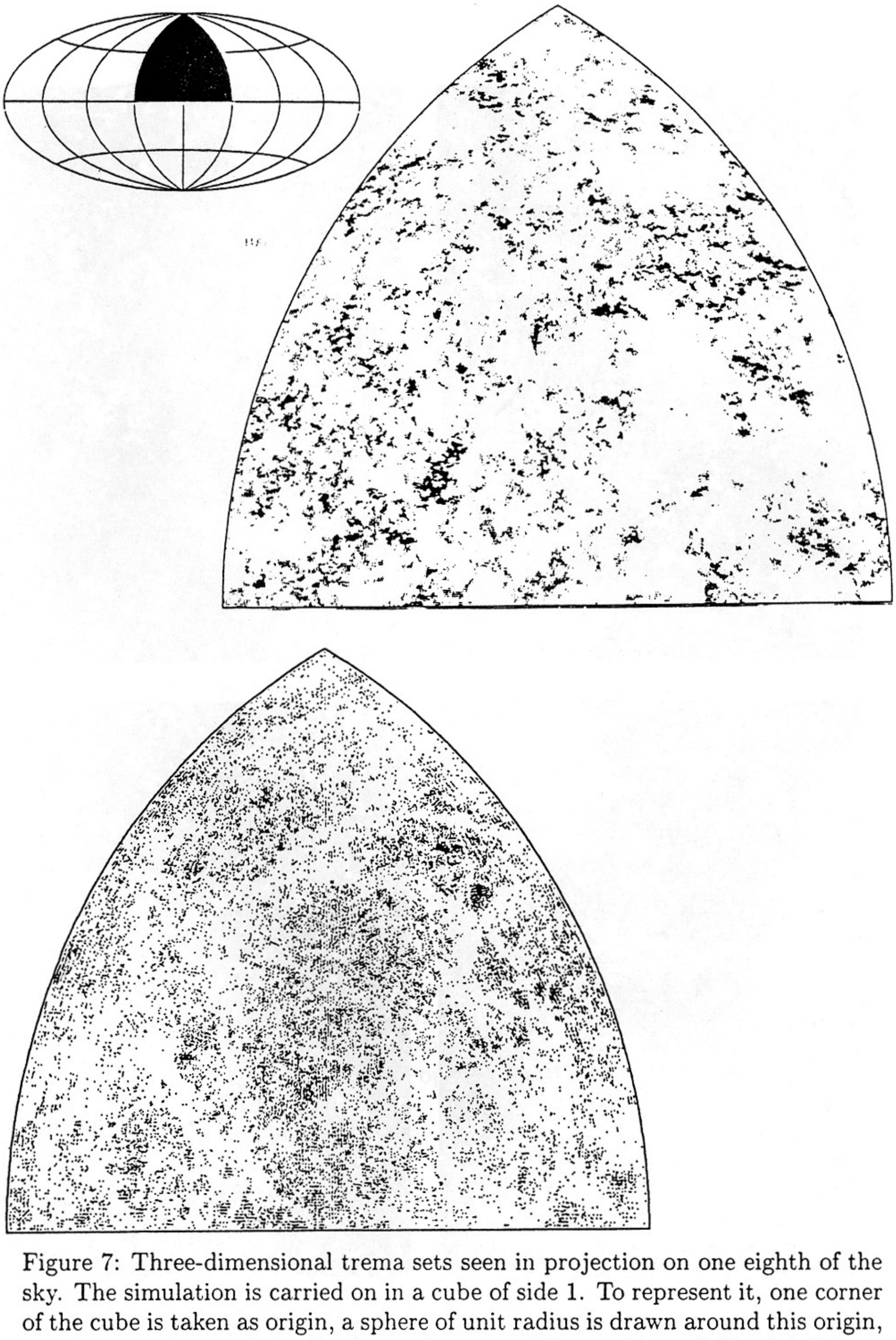

Figure 7: Three-dimensional trema sets seen in projection on one eighth of the sky. The simulation is carried on in a cube of side 1. To represent it, one corner of the cube is taken as origin, a sphere of unit radius is drawn around this origin, and the remainder of the simulation is erased. The position of this eighth is as indicated in a small additional diagram, or its symmetric.

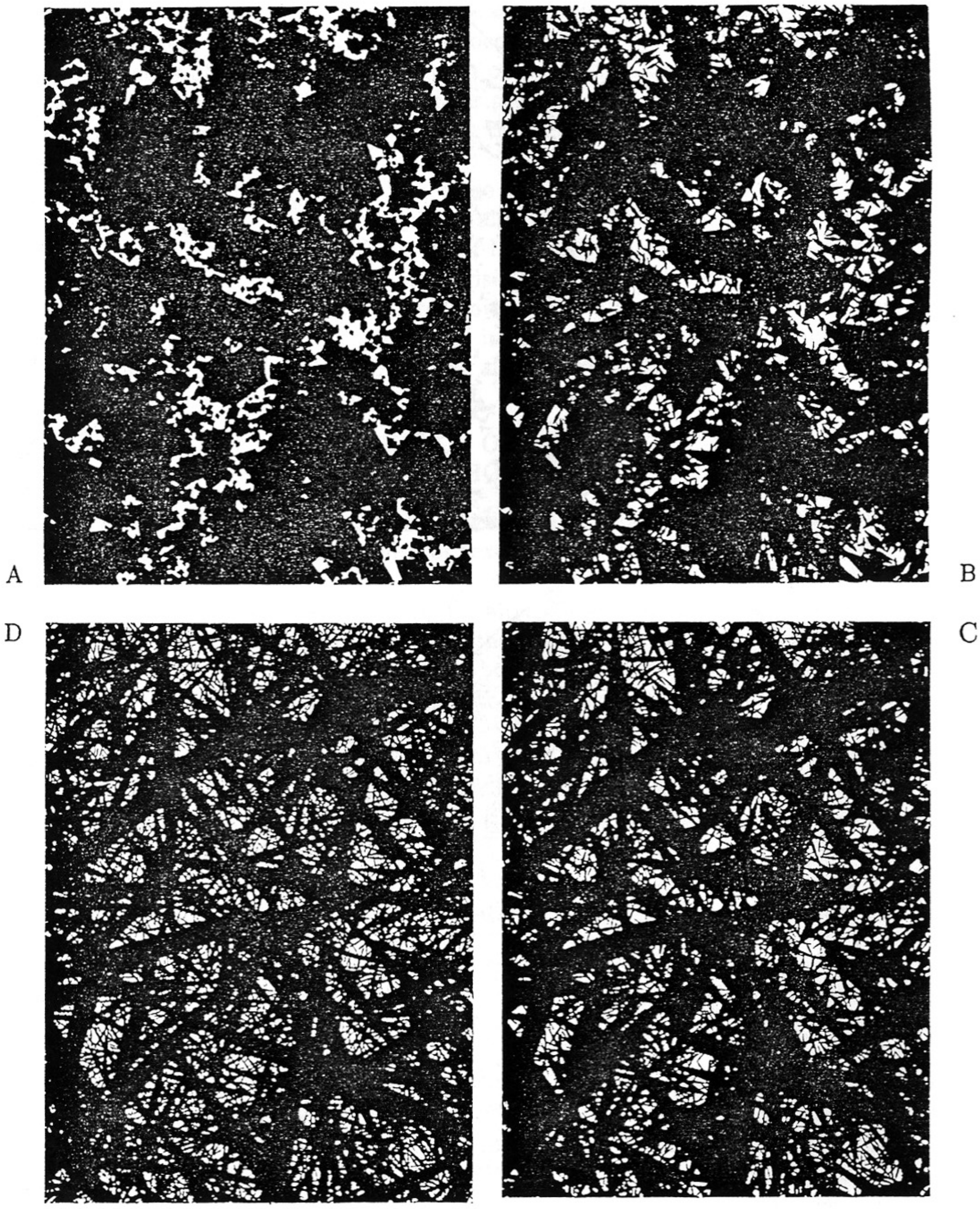

Figure 8: Trema sets on the plane whose template is a unit area rhombus. As the ratio of length to width increases from 7A to 7D, the lacunarity decreases.

universe is non-local, insofar as distant portions are statistically dependent. In statistical physics, this dependence is measured by a correlation exponent, which in this instance is $3 - D$. What is shown by the study of lacunarity is that the concept of non-local dependence is richer than has been thought: It also involves additional features which go beyond D. Neutral lacunarity expresses that, given the value of D, long range dependence is as small as can be. A low lacunarity is a manifestation of long-range dependence stronger than the minimal value compatible with the given D.

7. A very important theoretical issue: the origin of fractality in partial differential equations

How does fractality fit in a view of Nature that is dominated by PDEs that is, partial differential equations? The smoothness of the solutions of those equations seems to flatly contradict fractality. However, Chapter 11 of [3] argued to the contrary: fractality need NOT contradict the basic PDE, but may, to the contrary, describe the long-range properties of their solutions, for example, their moving singularities and boundaries.

The PDE ruling an assembly of classical point masses is the Laplace equation, and those point masses themselves are singularities of the solution. According to the thesis of Chapter 11 of [3], the distribution of those point masses should tend toward fractality. All computer simulations of large assemblies of classical point masses appear to be compatible with this prediction.

Acknowledgements

I owe special thanks to David Layzer and Gérard de Vaucouleurs, who encouraged me in the 1970s to persist in studying the distribution of galaxies, and to Luciano Pietronero for useful continuing discussions over several years, lately joined by Francesco Sylos-Labini.

Credits for the illustrations are as follows: to S.W. Handelman for Figure 2, to L. Siegel for Figure 3, to M. Frame for Figure 4, to L. Halim for Figures 5 and 6. Furthermore — last but not least – credit is due to M. R. Laff for Figures 7 and 8. They were masterpieces of computing when first performed in the late 1970s, and I regret that no opportunity arose to publish them in their youth.

References

[1] Mandelbrot, B.B. (1975) *Les objects fractals: forme, hasard et dimension.* Flammarion, Paris. Fourth (paperback) edition, 1995.

[2] Mandelbrot, B.B. (1977) *Fractals: Form, Chance and Dimension.* W.H. Freeman and Company, San Francisco, CA.

[3] Mandelbrot, B.B. (1982) *The Fractal Geometry of Nature.* W.H. Freeman and Company, New York, NY.

[4] Mandelbrot, B.B. (1998) Fractality, lacunarity, and the near-isotopic distribution of galaxies. *Current Topics in Astrofundamental Physics.* N. Sanchez and A. Zichichi (eds.) Kluwer, Dordrecht and Boston, 585-603.

[5] Mandelbrot, B.B. (1999) *Multifractals, Lacunarity, and Fractal Sums of Pulses* (in preparation). Springer-Verlag, New York.

[6] Mandelbrot, B.B. (1989) The fractal range of the distribution of galaxies: crossover to homogeneity and multifractals, in F. Mardirossian, M. Mezzetti & D. Sciama (eds.), *Large-scale Structure and Motions in the Universe.* Kluwer, Dordrecht, 259-279.

[7] de Vaucouleurs, G. (1970) The case for hierarchical cosmology, *Science*, 167, 1203-1213.

[8] Peebles, P.J.E (1980) *The Large-Scale Structure of the Universe.* Princeton University Press, Princeton NJ.

[9] Sylos Labini, F, Montuori, M and Pietronero, L. (1998) Scale invariance in galaxy clustering. *Physics Reports*, 293, 61-226.

[10] Mandelbrot, B.B. (1975) Sur un modèle décomposable d'Univers hiérarchisé: déduction des corrélations galactiques sur la sphère céleste, *Comptes Rendus* (Paris) 280A, 1551-1554.

[11] Mandelbrot, B.B. (1995) Introduction to fractal sums of pulses. *Lévy Flights and Related Phenomena in Physics.* M.F. Shlesinger, G. Zaslawsky & U. Frisch (eds.) Springer, New York 110-123.

[12] Mandelbrot, B.B. Fractal sums of pulses: self-affine global dependence and lateral limit theorems (to appear).

[13] Mandelbrot, B.B. (1979) Corrélations et texture dans un nouveau modèle d'Univers hiérarchisé, basé sur les ensembles trémas, *Comptes Rendus* (Paris) 288A, 81-83.

[14] Mandelbrot, B.B. (1995) Measures of fractal lacunarity: Minkowski content and alternatives. in C. Bandt, S. Graf and M. Zähle (eds.), *Fractal Geometry and Stochastics*, Birkhäuser, Basel, pp 12-38.

[15] Mandelbrot, B.B. and Stauffer, D. (1994) Antipodal correlations and texture (fractal lacunarity) in critical percolation clusters, *Journal of Physics* A27, L237-L242.

[16] Hovi, J.P., Aharony, A., Stauffer, D., and Mandelbrot, B.B. (1996) Gap independence and lacunarity in percolation clusters. *Phys. Rev. Letters* 77, 877-880.