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Alternative micropulses and fractional Brownian motion

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Abstract

We showed in an earlier paper (1995a) that negatively correlated fractional Brownian motion (FBM) can be generated as a fractal sum of one kind of micropulses (FSM). That is, FBM of exponent $H < \frac{1}{2}$ is the limit (in the sense of finite-dimensional distributions) of a certain sequence of processes obtained as sums of rectangular pulses. We now show that more general pulses yield a wide range of FBMs: either negatively (as before) or positively ($H > \frac{1}{2}$) correlated. We begin with triangular (conical and semi-conical) pulses. To transform them into micropulses, the base angle is made to decrease to zero, while the number of pulses, determined by a Poisson random measure, is made to increase to infinity. Then we extend our results to more general pulse shapes.

Keywords: Fractal sums of pulses; Fractal sums of micropulses; Fractional Brownian motion; Poisson random measure; Self-similarity; Self-affinity; Stationarity of increments

1. Introduction

This paper shows how fractional Brownian motion can be obtained as a limit of processes defined as sums of pulses generated by a Poisson random measure. In the limit, the pulses become infinitely numerous and infinitely small, so we call them *micro*pulses.

When we think about the process difference $X(t) - X(0)$, we visualize it as the sum of heights of all pulses alive at time t minus the sum of heights of all pulses alive at time 0. The sums at t and at 0 may diverge, however, if considered separately, and we wish to set $X(0) = 0$. Therefore, we define $X(t)$ as the sum of changes in the pulse amplitude between times 0 and t.

Section 2 investigates triangular pulses of two different shapes, or "templates". The right triangle (semiconical) pulse starts with a jump (a discontinuity) and then decays linearly. The isosceles triangle (conical) pulse increases linearly to a point and then

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decreases with the same rate. We also allow for pulses with negative amplitude. The location of the pulse in time, its width (the base of a triangle) as well as its height are all random and governed by an appropriate Poisson measure. The analogous construction for "up-and-down" (rectangular) pulses was already considered in Cioczek-Georges and Mandelbrot (1995a), to be referred as *Micropulses* (1995). There we transform pulses into micropulses by letting the *height* of the pulse go to zero. Here, we let the *tangent of the base angle* go to zero.

Section 3 extends our construction to more general pulse shapes. Negatively correlated FBMs with $0 < H < \frac{1}{2}$ were examined in *Micropulses* (1995). The present discussion compares the generalization with the results of this paper. We also include several examples of less regular pulse shapes.

Rectangular or triangular pulses that yield stable self-affine stochastic processes with stationary increments are described in Cioczek-Georges et al. (1995) and Cioczek-Georges and Mandelbrot (1995b). See also Mandelbrot (1995a) or (1995b) for more general discussions on fractal sums of pulses.

2. FBM with $0 < H < 1$ as a sum of triangular micropulses

Let us consider pulses of the two triangular shapes described above and introduce the pulse address space $A = R \times R \times R_+$, where $R_+ = (0, \infty)$. Each semi-conical or conical pulse is represented in A by a point with coordinates x, τ and w, corresponding, respectively, to the tangent of the base angle of the triangle, the time of birth and the width (duration) of a pulse. A pulse peaks at τ in the case of right-triangular pulses and a pulse peaks at $\tau + w/2$ for isosceles triangles. Now we introduce a scale factor $\varepsilon > 0$, that will eventually be made to decrease to 0, and consider only pulses of the base angle tangent rescaled to εx and unchanged τ and w. Thus, the amplitude at time t of a "rescaled" pulse with coordinates x, τ and w equals

$$\varepsilon x(w - t + \tau) I[\tau \leqslant t \leqslant \tau + w]$$

for semi-conical pulses, and

$$\varepsilon x(w/2 - |t - \tau - w/2|) I[\tau \leqslant t \leqslant \tau + w]$$

for conical pulses.

The number of pulses with given coordinates is determined by a Poisson random measure. Let $\mathscr{A} \equiv \mathscr{B}(A)$ be the Borel σ-field on A. For each $\varepsilon > 0$, we consider a Poisson random measure N_ε on $(A, \mathscr{A})$ with mean n_ε given by

$$n_\varepsilon(\mathrm{d}x, \mathrm{d}\tau, \mathrm{d}w) = \varepsilon^{-2} w^{-\delta-1} F(\mathrm{d}x)\,\mathrm{d}\tau\,\mathrm{d}w,$$

for $(x, \tau, w) \in A$, where $1 < \delta < 3$ and F is the distribution of a random variable X with finite second moment. In the simplest situation, when X is a non-zero constant, the shapes of both triangles defining pulses are fixed, i.e. pulses differ only by τ and w, but have the same base angle determined by ε.

Adding up pulses corresponds to the integration of the difference in the pulse amplitude with respect to the Poisson measure. Let us define two functions; for the semi-conical pulses,

$$X_{1\varepsilon}(t) := \int_A \varepsilon x g_1(t,\tau,w) N_\varepsilon(\mathrm{d}x, \mathrm{d}\tau, \mathrm{d}w), \quad t \geqslant 0; \tag{2.1}$$

and for the conical pulses,

$$X_{2\varepsilon}(t) := \int_A \varepsilon x g_2(t,\tau,w) N_\varepsilon(\mathrm{d}x, \mathrm{d}\tau, \mathrm{d}w), \quad t \geqslant 0, \tag{2.2}$$

where g_1 and g_2 equal, respectively,

$$g_1(t,\tau,w) = (w - t + \tau) I[\tau \leqslant t \leqslant \tau + w] - (w + \tau) I[\tau \leqslant 0 \leqslant \tau + w], \tag{2.3}$$

$$\begin{aligned} g_2(t,\tau,w) = {} & (w/2 - |t - \tau - w/2|) I[\tau \leqslant t \leqslant \tau + w] \\ & - (w/2 - |\tau + w/2|) I[\tau \leqslant 0 \leqslant \tau + w]. \end{aligned} \tag{2.4}$$

We want to investigate what happens when $\varepsilon \to 0$. Do finite-dimensional distributions of $\{X_{i\varepsilon}(t), t \geqslant 0\}$ converge, and if they do, what is the limiting process for $i = 1, 2$? We shall conclude that the finite-dimensional distributions of $\{X_{i\varepsilon}(t), t \geqslant 0\}$ converge to those of fractional Brownian motion (FBM) $\{B_H(t), t \geqslant 0\}$ with the self-affinity exponent $H = (3 - \delta)/2$ (for definition of FBM look at Mandelbrot and VanNess, 1968). First, however, let us determine when the integrals in (2.1) and (2.2) are well-defined random variables, i.e. whether these integrals converge and in what sense.

In general, stochastic integrals with respect to a Poisson measure are defined in the sense of a.s. convergence. They exist for (non-random) functions which are integrable with respect to the Poisson mean (intensity) measure (cf. Resnick, 1987, p. 127). This implies that (2.2) is well-defined for $1 < \delta < 2$ since, in this case, $\int_A |\varepsilon x g_2(t,\tau,w)| \varepsilon^{-2} w^{-\delta-1} F(\mathrm{d}x)\,\mathrm{d}\tau\,\mathrm{d}w < \infty$. The situation is more delicate for $2 \leqslant \delta < 3$. Here we define Poisson integrals in the sense of some *conditional* a.s. convergence, which basically means the integrals are a.s. limits of finite *ordinary* Poisson integrals. The condition sufficient for such a convergence is the integrands' L^2-integrability with respect to mean n_ε. We proceed as follows.

Let $A_n = R \times R \times (2^{-n}, 2^{-n+1}]$, $n = 1, 2, \ldots$, and $A_0 = R \times R \times (1, \infty)$, be a partition of the address space A. Note that $x g_i I[A_n] \in L^1(n_\varepsilon)$ if $\delta > 1$ (this condition is required for $i = 2$ to ensure the finiteness of the integral over A_0) and, moreover,

$$\int_{A_n} \varepsilon x g_i(t,\tau,w) \varepsilon^{-2} w^{-\delta-1} F(\mathrm{d}x)\,\mathrm{d}\tau\,\mathrm{d}w = 0, \quad n = 0, 1, 2, \ldots,$$

Hence, random variables $\int_{A_n} \varepsilon x g_i(t,\tau,w) N_\varepsilon(\mathrm{d}x, \mathrm{d}\tau, \mathrm{d}w)$, $n = 0, 1, \ldots$, for $i = 1, 2$, are well-defined and they have zero expectations. Their characteristic functions at $\xi \in R$

equal[1]

$$E\exp\left(\mathrm{i}\xi\int_{A_n}\varepsilon x g_i(t,\tau,w)N_\varepsilon(\mathrm{d}x,\mathrm{d}\tau,\mathrm{d}w)\right).$$

$$=\exp\left(\int_{A_n}(\mathrm{e}^{\mathrm{i}\xi\varepsilon x g_i(t,\tau,w)}-1)n_\varepsilon(\mathrm{d}x,\mathrm{d}\tau,\mathrm{d}w)\right)$$

$$=\exp\left(\int_{A_n}(\mathrm{e}^{\mathrm{i}\xi\varepsilon x g_i(t,\tau,w)}-1-\mathrm{i}\xi\varepsilon x g_i(t,\tau,w))n_\varepsilon(\mathrm{d}x,\mathrm{d}\tau,\mathrm{d}w)\right). \tag{2.5}$$

We claim that the series

$$\sum_{n=0}^{\infty}\int_{A_n}\varepsilon x g_i(t,\tau,w)N_\varepsilon(\mathrm{d}x,\mathrm{d}\tau,\mathrm{d}w) \tag{2.6}$$

converges a.s. Since the Poisson measure N_ε is independently scattered the terms are independent random variables and it is enough to show convergence in distribution. Note that the logarithm of the characteristic function in (2.5) is bounded by

$$\frac{1}{2}\int_{A_n}\xi^2x^2g_i^2(t,\tau,w)w^{-\delta-1}F(\mathrm{d}x)\,\mathrm{d}\tau\,\mathrm{d}w.$$

Summing over n we get

$$\frac{1}{2}\xi^2\,EX^2\int_{R\times R_+}g_i^2(t,\tau,w)w^{-\delta-1}\,\mathrm{d}\tau\,\mathrm{d}w. \tag{2.7}$$

The above integrals are finite if $2<\delta<3$ for $i=1$ and if $1<\delta<3$ for $i=2$. Thus, under the same conditions the series (2.6) converges in distribution and a.s.

We have proven the conditional a.s. convergence of integrals in (2.1) and (2.2). As the Poisson integrals are, in fact, infinite sums, their conditional convergence may be understood in this context as an order imposed on summing heights of the triangular pulses. To ensure convergence, we first look at the pulses with the largest width w and sum the differences of their heights between points t and 0. Then we proceed to the shorter pulses in descending order.

We summarize our findings in the following theorem.

Theorem 2.1. *The processes* $\{X_{1\varepsilon}(t),\ t\geqslant 0\}$ *and* $\{X_{2\varepsilon}(t),\ t\geqslant 0\}$ *are well-defined for* $2<\delta<3$ *and* $1<\delta<3$, *respectively, with the a.s. convergence of integrals being unconditional for* $1<\delta<2$ *and conditional for* $2\leqslant\delta<3$.

To prove the convergence in the sense of finite-dimensional distributions of $\{X_{i\varepsilon}(t),\ t\geqslant 0\}$, $i=1,2$, to the appropriate FBM, consider the characteristic function of

[1] As noted by our referee, these integrals can also be interpreted as integrals with respect to "centered" (or compensated) Poisson measure $\tilde{N}_\varepsilon=N_\varepsilon-EN_\varepsilon=N_\varepsilon-n_\varepsilon$.

$\sum_{k=1}^{n} \xi_k X_{i\varepsilon}(t_k)$, $t_k \geqslant 0$, $\xi_k \in R$, $k = 1, 2, \ldots, n$, $n \in N$, and notice that it equals

$$\exp\left\{\int_A \left[e^{i\sum_{k=1}^{n} \xi_k \varepsilon x g_i(t_k,\tau,w)} - 1 - i\sum_{k=1}^{n} \xi_k \varepsilon x g_i(t_k,\tau,w) \right] n_\varepsilon(\mathrm{d}x, \mathrm{d}\tau, \mathrm{d}w)\right\}.$$

The arguments showing the convergence in distribution of the series (2.6) (in particular, a bound analogous to (2.7)) can easily be changed to prove that the above characteristic function approaches that of $\sum_{k=1}^{n} \xi_k X(t_k)$, where $(X(t_1), X(t_2), \ldots, X(t_k))$ is a Gaussian vector with the covariance matrix

$$\begin{aligned} Cov(X(t_k), X(t_j)) &= EX^2 \int_{R\times R_+} g_i(t_k,\tau,w) g_i(t_j,\tau,w) w^{-\delta-1}\,\mathrm{d}\tau\,\mathrm{d}w \\ &= \frac{1}{2} EX^2 \left\{ \int_{R\times R_+} g_i^2(t_k,\tau,w) w^{-\delta-1}\,\mathrm{d}\tau\,\mathrm{d}w \right. \\ &\quad + \int_{R\times R_1} g_i^2(t_j,\tau,w) w^{-\delta-1}\,\mathrm{d}\tau\,\mathrm{d}w \\ &\quad \left. - \int_{R\times R_+} (g_i(t_k,\tau,w) - g_i(t_j,\tau,w))^2 w^{-\delta-1}\,\mathrm{d}\tau\,\mathrm{d}w \right\}. \end{aligned}$$

The change of variables $\tau/t \to \tau$, $w/t \to w$, shows directly that the first two integrals are proportional (up to the same constant) to $t_k^{3-\delta}$ and $t_j^{3-\delta}$, respectively. Similarly, using a simple translation in τ, followed by the scalings $\tau/|t_k - t_j| \to \tau$ and $w/|t_k - t_j| \to w$, we can show that the third integral is proportional to $|t_k - t_j|^{3-\delta}$. More precisely, if one puts

$$C_i := \int_{R\times R_+} g_i^2(1,\tau,w) w^{-\delta-1}\,\mathrm{d}\tau\,\mathrm{d}w,$$

then

$$Cov(X(t_k), X(t_j)) = C_i \frac{EX^2}{2} (t_k^{3-\delta} + t_j^{3-\delta} - |t_k - t_j|^{3-\delta}),$$

which is the covariance of FBM with the self-affinity exponent $H = (3-\delta)/2$. We have proven

Theorem 2.2. *As $\varepsilon \to 0$, the finite-dimensional distributions of $\{X_{i\varepsilon}(t),\ t \geqslant 0\}$, $i = 1, 2$, converge to those of FBM with exponent $H = (3-\delta)/2$ and variance $EB_H^2(1) = C_i EX^2$ for $i = 1, 2$, respectively.*

Remark 2.1. The main difference between $\{X_{1\varepsilon}(t),\ t \geqslant 0\}$ and $\{X_{2\varepsilon}(t),\ t \geqslant 0\}$ is as follows. The process $\{X_{1\varepsilon}(t),\ t \geqslant 0\}$ always approaches negatively correlated FBM with $0 < H < \frac{1}{2}$. In contrast, $\{X_{2\varepsilon}(t),\ t \geqslant 0\}$ may also converge to the positively correlated FBM with $\frac{1}{2} < H < 1$ as well as to the ordinary Brownian motion with $H = \frac{1}{2}$.

3. Extensions to more general micropulse templates

Note that functions g_i, $i = 1, 2$, given by (2.3) and (2.4) can be written in the following form:

$$g_i(t, \tau, w) = w\left[f_i\left(\frac{t-\tau}{w}\right) - f_i\left(\frac{-\tau}{w}\right)\right],$$

where

$$f_1(y) = (1-y)I[0 \leqslant y \leqslant 1],$$

and

$$f_2(y) = (1/2 - |y - 1/2|)I[0 \leqslant y \leqslant 1].$$

Hence, f_i's are the "generic" triangles, or "templates", used to determine the pulse shapes. This representation suggests a generalization by replacing f_i, $i = 1, 2$, by any function f supported in the interval $[0, 1]$. The amplitude of an f-shaped pulse, starting at τ and ending at $\tau + w$, depends also on the angle εx and equals

$$\varepsilon x w f\left(\frac{t-\tau}{w}\right).$$

at time t. As before, we define

$$X_\varepsilon(t) := \int_A \varepsilon x w\left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right] N_\varepsilon(\mathrm{d}x, \mathrm{d}\tau, \mathrm{d}w), \quad t \geqslant 0.$$

If $\delta > 1$ and

$$\int_{R \times R_+} \left[f\left(\frac{1-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right]^2 w^{1-\delta}\,\mathrm{d}\tau\,\mathrm{d}w < \infty, \tag{3.1}$$

$X_\varepsilon(t)$ is well-defined in the sense of conditional a.s. convergence described in Section 2. Indeed, under condition (3.1) integrals

$$\int_{A_n} xw\left|f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right| w^{-\delta-1} F(\mathrm{d}x)\,\mathrm{d}\tau\,\mathrm{d}w$$

are also finite for $n = 1, 2, \ldots$, and for $n = 0$ if $\delta > 1$, since in this case L^2-integrability implies L^1-integrability. Moreover, condition (3.1) (analogously to (2.7)) implies that the series of respective independent zero mean random variables converges a.s. to $X_\varepsilon(t)$.

The proof of Theorem 2.2 extends to the present case and yields

Theorem 3.1. *As $\varepsilon \to 0$, the finite-dimensional distributions of $\{X_\varepsilon(t),\ t \geqslant 0\}$, with f having support in $[0, 1]$ and satisfying (3.1) for some $1 < \delta < 3$, converge to those of FBM $\{B_H(t),\ t \geqslant 0\}$ with variance*

$$\operatorname{Var} B_H(t) = EX^2 \int_{R \times R_+} \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right)\right]^2 w^{1-\delta}\,\mathrm{d}\tau\,\mathrm{d}w.$$

Clearly, the above integral is proportional to $t^{3-\delta}$ and the self-affinity exponent equals $H = (3 - \delta)/2$.

It remains to investigate what type of functions f satisfy condition (3.1). *Micropulses* (1995) considered functions which are Hölder continuous in $[0, 1]$ with an exponent α (cf. condition (5.4) with $\theta = \delta - 2$). It turns out that if $\alpha > (3 - \delta)/2$ then (3.1) holds (cf. Proposition 5.2. of *Micropulses* (1995)). This assertion, however, required that $2 < \delta < 3$ $(0 < \theta < 1)$, hence, does not allow to obtain positively correlated $(1 < \delta < 2)$ FBM via templates which are Hölder continuous. Now we are ready to make a stronger statement which improves the results (Proposition 5.2) of *Micropulses* (1995).

Proposition 3.1. *Let a function f, with the support in $[0, 1]$, be Hölder continuous in $[0, 1]$ with an exponent $\alpha > 0$, i.e.,*

$$|f(x) - f(y)| \leqslant M|x - y|^{\alpha}$$

for some $M > 0$ and any $x, y \in [0, 1]$. Then (3.1) *holds for* $\max(2, 3 - 2\alpha) < \delta < 3$. *If in addition $f(0) = f(1) = 0$, then* (3.1) *holds for* $3 - 2\alpha < \delta < 3$.

Proof. Both parts of the theorem easily follow from Hölder continuity and boundedness of f. In particular, in the proof of the first part use the boundedness of f to evaluate the integral in (3.1) over the regions $\{(\tau, w)\colon 0 < (1 - \tau)/w < 1, \ -\tau/w < 0\}$ and $\{(\tau, w)\colon 1 < (1 - \tau)/w, 0 < -\tau/w < 1\}$. In the proof of the second part, first note that $f((1 - \tau)/w) = f((1 - \tau)/w) - f(0)$ and $-f(-\tau/w) = f(1) - f(-\tau/w)$ in the respective regions and then use Hölder continuity. □

Remark 3.1. The two triangular shapes in Section 2 are both Hölder continuous with $\alpha = 1$, therefore, they clearly pinpoint the difference indicated in Proposition 3.1. To obtain positively correlated FBM ($H > \frac{1}{2}$) or even ordinary Brownian motion ($H = \frac{1}{2}$) we must add micropulses which do not have jumps at their starting or ending points, i.e., which are continuous on the whole line. In addition, to obtain an FBM with a given H (which determines the smoothness of the sample paths of this FBM) we require the pulse to be smooth enough, namely we require $\alpha > H$. It is the exponent δ in the intensity of w, however, which ultimately determines the value $H = (3 - \delta)/2$.

Remark 3.2. Although Proposition 3.1 can be used in place of Proposition 5.2 (i) of *Micropulses* (1995), we must underline that the two papers take entirely different paths towards FBM. Indeed, the process $\{X_\varepsilon(t),\ t \geqslant 0\}$ of this paper and the process $\{\tilde{X}_\varepsilon(t),\ t \geqslant 0\}$ of *Micropulses* (1995) have different finite-dimensional distributions (compare e.g. their characteristic functions). Their existence, however, is guaranteed by the same condition (3.1), both have zero means and the second moments are off by a factor of one-half (again, a consequence of (3.1)). The process $\{\tilde{X}_\varepsilon(t),\ t \geqslant 0\}$ was a natural generalization of a process obtained as a sum of simple “up-and-down” micropulses which can never lead to positively correlated FBM (cf. discussion in Section 2 of *Micropulses* (1995)).

Remark 3.3. It is also possible to generalize part (ii) of Proposition 5.2 in *Micropulses* (1995) by allowing the template f to possess first-order discontinuities. More precisely, suppose that f has a finite number of jumps in $[0, 1]$ and otherwise (i.e. in the open intervals between jumps) is Hölder continuous with an exponent α; then (3.1) holds if $\max(3 - 2\alpha, 2) < \delta < 3$.

Examples illustrating the above extension and remarks. The simplest templates are, of course, Lipschitz functions, i.e., having Hölder exponent $\alpha = 1$. Hence, for instance, all differentiable functions with bounded derivative can be used to construct FBM with any $0 < H < 1$. The self-affinity exponent H is then determined solely by the power δ. There is, however, a large collection of other, more irregular shapes which satisfy Proposition 3.1.

1. A rescaled part of the graph of the typical FBM sample path with the self-affinity exponent $\eta > (3 - \delta)/2$. This template was already used in *Micropulses* (1995). It this part of the graph starts and ends at 0 (as in Fig. 1, adapted from Voss 1988), we can allow for $1 < \delta < 3$, i.e. $0 < H = (3 - \delta)/2 < 1$; otherwise we have only$0 < H < \frac{1}{2}$.

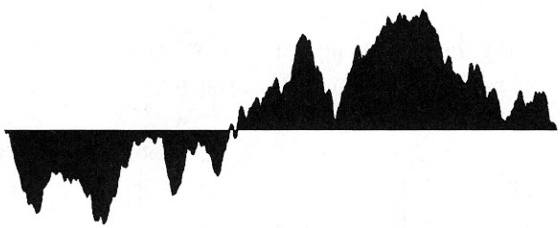

Fig. 1. Sample FBM graph with $\eta = 0.8$ and equal initial and final values.

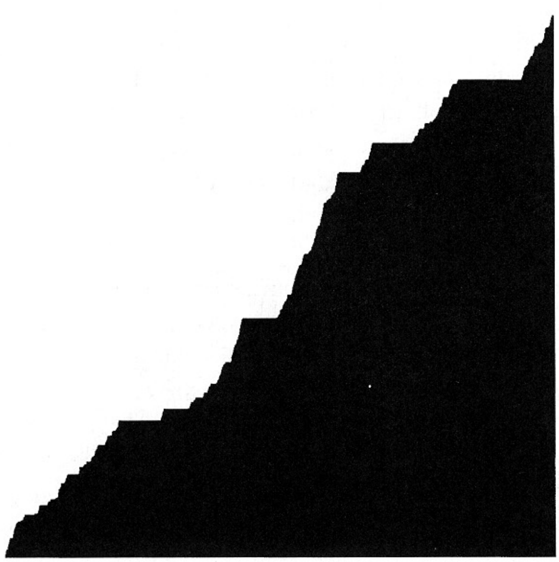

Fig. 2a. Lévy staircase with $\alpha = 0.9$.

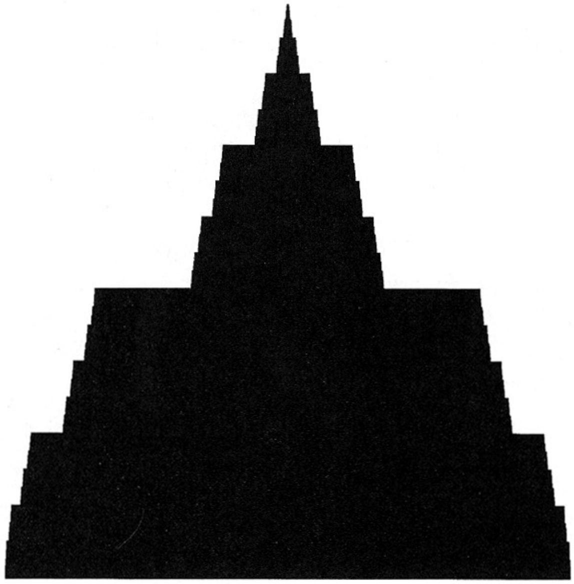

Fig. 2b. Cantor "pyramid".

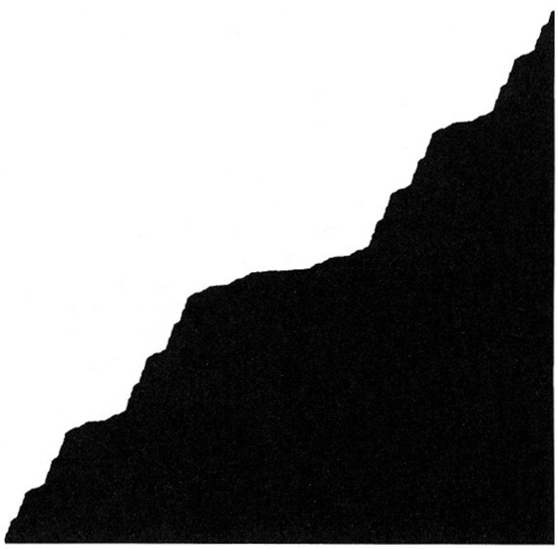

Fig. 3. Trinomial multifractal staircase.

2. Cantor or Lévy staircases, i.e. the singular Cantor distribution function or the inverse of the sample path graph of the Lévy α-stable motion, $0 < \alpha < 1$ (cf. Mandelbrot, 1982, pp. 286–7, 371, and Fig. 2a). Cantor and Lévy staircases are Hölder continuous with maximum exponent $\log 2/\log 3$ ($> \frac{1}{2}$) and α, respectively. Hence, they yield FBM with $H < \frac{1}{2}$ or $H < \min(\frac{1}{2}, \alpha)$, unless we use two "staircases" to make

a "pyramid" (as in Fig. 2b) in which case $H < \log 2/\log 3$ or $H < \alpha$, respectively. (Of course, both functions are constant almost everywhere. Thus, the Hölder exponent is determined by the points of increase.)

3. Multifractal staircases, that is, graphs of cumulative multifractal measures on $[0, 1]$ (for the theory of multifractals, see, for example, Evertsz and Mandelbrot, 1992). A simple example of trinomial (multinomial base 3) measure is presented in Fig. 3, adapted from Mandelbrot (1975). The originality of multifractal theory is that the Hölder exponent is not defined on an interval (as above) but is a local quantity. Its variation is described by a function $f(\alpha)$, called multifractal (or Hölder) spectrum; the usual (interval) Hölder exponent is $\alpha_{\min} = \inf\{\alpha: f(\alpha) > 0\}$. In that case it is enough to require the exponent H of the constructed FBM to satisfy $H < \alpha_{\min}$. In addition, because of the discontinuity at 1, we need $H < \frac{1}{2}$.

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