Local Regularity of Nonsmooth Wavelet Expansions and Application to the Polya Function

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We give criteria of pointwise regularity for expansions on Haar or Schauder basis (or spline-type wavelets) corresponding to large Hölder exponents. As an application, we determine the exact Hölder regularity of the Polya function at every point and show that it is multifractal.   © 1996 Academic Press, Inc.

Suppose that a function $F: \mathbb{R} \to \mathbb{R}$ is known by the explicit knowledge of its coefficients in a given basis. Can one deduce the local or global regularity of F from simple criteria on these coefficients?

Stated as it is, this problem is far too general to have a nontautological answer; however, a natural and common expectation is that a positive answer can hold only if the elements of the basis have at least the corresponding regularity and the regularity is given by a decay condition on the coefficients. Three well-known examples are

• If $F(x) = \sum c_n e^{inx}$ and $\sum |c_n| \, |n|^{\alpha} \leqslant C$, then F is C^{α}.

• Let $\psi_{j,k} = 2^{j/2} \psi(2^j x - k)$ $(j, k \in \mathbb{Z})$ be a wavelet orthonormal basis of $L^2(\mathbb{R})$. If

$$F = \sum C_{j,k} \psi_{j,k} \tag{1}$$

and ψ is C^{γ} for a $\gamma > \alpha$, then F is $C^{\alpha}(\mathbb{R})$ if and only if $|C_{j,k}| \leqslant C2^{-(1/2+\alpha)j}$.

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- Under the same asumptions on the wavelet basis, if

$$|C_{j,k}| \leqslant C 2^{-(1/2+\alpha)j}(1+|2^j x_0 - k|)^\beta \qquad \text{for a} \quad \beta < \alpha, \qquad (2)$$

then F is $C^\alpha(x_0)$.

However, the general expectation suggested by those examples has puzzling exceptions. At the beginning of the study of Fourier series, in 1873, Dubois–Reymond discovered that if F is continuous its Fourier series need not converge uniformly; but in 1909, Haar showed that its Haar series (which is of the form (1), where $\psi = 1_{[0,1/2]} - 1_{[1/2,1]}$) does converge uniformly. Paradoxically a basis composed of discontinuous functions seems more adapted to the representation of continuous functions than the trigonometric basis, which is composed of C^∞ functions.

One side of this paradox was fully investigated later, namely, the pathological regularity properties of Fourier series (see [4 or 10], for instance); but the very puzzling discovery of Haar seems to have had no following. One of our purposes is to understand how regularity criteria can be derived on expansions in the Haar basis. For instance, we will see that for almost every x_0, condition (2) for Haar coefficients implies that F is $C^\delta(x_0)$ for any $\delta < \alpha$, **no matter how large α is**.

Our simple necessary conditions and sufficient conditions of regularity for coefficients in the Haar basis are close to be sharp. They do not depend very much on the particular choice of the Haar basis. The key property is that it is a wavelet-type basis composed of piecewise smooth functions. Thus our criteria will immediately extend to decompositions on the Schauder basis, or "spline" wavelet bases, such as Strömberg or Battle–Lemarié wavelet; see [6 or 9] (the typical case where they cannot be applied is Daubechies compactly supported wavelets). We will actually state most of our results in the case of the Schauder basis because of the following motivation.

In a famous paper [8] published in 1913, Polya defined a "Peano type" function which is a continuous mapping from [0, 1] onto a rectangle triangle (not isoceles). We will compute the coefficients of the Polya function on the Schauder basis. A space-filling function must clearly be very unsmooth. Nonetheless, Lax proved that, if the triangle is flat enough, the Polya function is differentiable on a large set of points (see [5]). In order to recover and improve Lax's result one has to obtain differentiability and higher regularity criteria that bear on the coefficients on the Schauder basis, whose elements are not differentiable; we are thus back to our initial problem. Using the regularity criteria that we will establish in Section 2, we will obtain the Hölder regularity of the Polya function at every point (and the Hölder exponent can be very large at some points if the triangle is flat enough). As a consequence, we will see that the Polya function is multifractal.
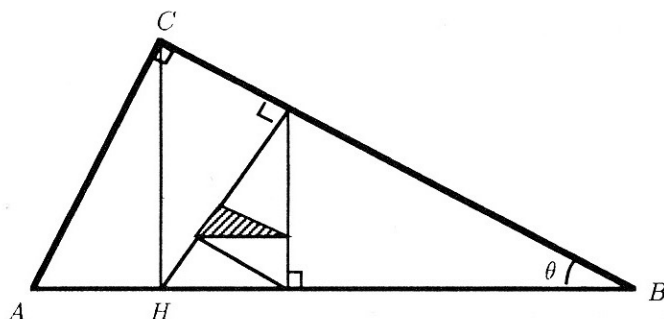
FIG. 1. Points whose binary expansion starts with 0.110110... are mapped into the striped triangle.

The Polya function $F_\theta(t)$ is defined as follows: Let $t \in [0, 1]$ and let $t = 0 \cdot t_1 t_2 \cdots$ be its binary expansion. We now show that each t defines a sequence of embedded triangles (Fig. 1).

The altitude issued from C (where the angle is $\pi/2$) divides the triangle (A, B, C) into two partial triangles similar to the initial one. If $t_1 = 0$, t is mapped in the smaller of these two triangles, and if $t_1 = 1$, t is mapped in the larger. We then iterate by dividing again these rectangles into two pieces by the same rule; the choice is now a function of t_2 and so on. One thus defines step by step the image of any $t \in [0, 1]$. If t is dyadic, one checks easily that the two possible expansions of t give the same result (for instance, $\frac{1}{2}$ is mapped on C) and that the mapping F_θ is onto and continuous. We now recall Lax's remarkable result (see [5]).

THEOREM 1. *Let θ be the smallest angle in the triangle (A, B, C).*

- *If $30° < \theta < 45°$, F_θ is nowhere differentiable.*

- *If $15° < \theta < 30°$, F_θ is not differentiable almost everywhere, but is differentiable on a set which has the power of the continuum.*

- *If $\theta < 15°$, F_θ is almost everywhere differentiable.*

We will sharpen this theorem as follows; we will determine the regularity of F_θ everywhere and thus deduce for each Hölder exponent H, the dimension $f(H)$ of the set where F_θ has this Hölder regularity. This *Hölder spectrum* $f(H)$ will be nonconstant on a whole interval. Thus the Hölder singularities of F_θ are located on a whole collection of sets of different dimensions, and F_θ is truly a "multifractal function." Let us now give some precise definitions.

Let $x_0 \in \mathbb{R}$ and $\alpha > 0$; by definition, a function F is $C^\alpha(x_0)$ if there exists a polynomial P of order at most α such that

$$|f(x) - P(x + x_0)| \leqslant C|(x - x_0)|^\alpha. \tag{3}$$

The Hölder exponent of f at x_0 is

$$\alpha(x_0) = \sup\{\beta : f \in C^{\beta}(x_0)\}.$$

The Hölder spectrum of F is the function $f(\alpha)$ which associates to each α the Hausdorff dimension of the set of points x where $\alpha(x) = \alpha$ (conventionally the dimension of the empty set is $-\infty$).

Let us recall the definition of the Schauder basis. Let

$$\Lambda(x) = \inf(x, 1-x) \qquad \text{if} \quad x \in [0, 1]$$

$$= 0 \qquad\qquad\qquad \text{otherwise.}$$

The Schauder basis is the set of $\Lambda_{j,k} = \Lambda(2^j x - k)$ for $j \geqslant 0$ and $k = 0, ..., 2^j - 1$. The coefficients of a continuous function F defined on $[0, 1]$ (and vanishing at 0 and 1) on this basis are given by

$$C_{j,k} = 2F\left(\frac{k+1/2}{2^j}\right) - F\left(\frac{k}{2^j}\right) - F\left(\frac{k+1}{2^j}\right). \tag{4}$$

The first section exhibits the remarkable expansion of the Polya function on the Schauder basis.

Section 2 gives pointwise regularity criteria bearing on the coefficients of functions on this basis. Such criteria are well known for Hölder coefficients smaller than one (for instance the results of [2] that are recalled in Proposition 1 clearly work for the Schauder basis) but they were not believed to hold for larger exponents; the heuristic reason is that one cannot use a basis to determine regularity exponents higher than the regularity of the basis itself because regularity conditions are usually given by decay conditions of the coefficients on the basis. Let now F by any element of the basis, its coefficients have the best decay condition possible: they all vanish except for one of them; nonetheless F is not smooth, so that no such decay criteria can hold. We will show how this argument can be turned in the case of the Schauder basis, and more generally for Haar or "spline" wavelets.

Section 3 will apply theses criteria in order to compute the regularity of the Polya function everywhere, and derive its Hölder spectrum.

The results given in Section 2 are due to Stéphane Jaffard. Applications to the Polya function are a joint work between Stéphane Jaffard and Benoit Mandelbrot. A more geometric approach to the local regularity of the Polya function was developed by Benoit Mandelbrot. Its starting point is the remark that we can choose a new time variable τ such that equal areas are covered in equal time. With this *intrinsic* time, the *Polya motion* is everywhere $C^{1/2}$ and not smoother. Results concerning the Polya

function itself are obtained by remarking that the change of variable we performed is the primitive of a binomial measure; details can be found in [7].

1. The Schauder Basis Expansion of the Polya Function F_θ

In order to fix ideas, suppose that A is the origin $(0, 0)$ and B is the point $(1, 0)$; let S_0 be the symetry around the bisector of the angle $\widehat{BAC}$ and let S_1 be the symetry around the bissector of $\widehat{ABC}$; let H_0 be the homothety centered at A of ratio $\sin \theta$ and let H_1 be the homothety centered at B of ratio $\cos \theta$. The recursive definition of F_θ shows that

$$\forall x \in [0, \tfrac{1}{2}], \qquad F_\theta(x) = S_0 H_0 F_\theta(2x)$$

$$\forall x \in [\tfrac{1}{2}, 1], \qquad F_\theta(x) = S_1 H_1 F_\theta(2x - 1)$$

because $S_0 H_0$ maps the triangle (ABC) on (ACH) and $S_1 H_1$ maps the triangle (ABC) on (CBH). Define now

$$G_\theta(x) = F_\theta(x) - (x, 0) \qquad \text{if} \quad x \in [0, 1]$$

$$= 0 \qquad \qquad \text{otherwise.}$$

Clearly G_θ is continuous and, except perhaps at 0 and 1, has the same regularity as F_θ.

On the interval $[0, \tfrac{1}{2}]$

$$G_\theta(x) + (x, 0) = S_0 H_0(G_\theta(2x) + (2x, 0))$$

so that

$$G_\theta(x) = \overline{S_0 H_0}(G_\theta(2x)) - (x, 0) + S_0 H_0((2x))$$

where $\overline{S_0 H_0}$ is the linear mapping associated with the affine mapping $S_0 H_0$ (the two mappings coincide because by convention A is the origin, but $S_1 H_1$ and $\overline{S_1 H_1}$ do not coincide).

Since $S_0((2x, 0)) = 2x(\sin^2\theta, \sin\theta \cos\theta)$, we obtain

$$G_\theta(x) = \sin \theta \overline{S_0} G_\theta(2x) + x(-\cos 2\theta, \sin 2\theta). \tag{5}$$

On the interval $[\tfrac{1}{2}, 1]$,

$$G_\theta(x) + (x, 0) = S_1 H_1(G_\theta(2x - 1) + (2x - 1, 0))$$

so that

$$G_\theta(x) = \overline{S_1 H_0}(G_\theta(2x-1)) - (x, 0) + S_1 H_1(2x-1, 0).$$

Since B is a fixed point of $S_1 H_1$ and since the distance from $(2x-1, 0)$ to $(1, 0)$ is $2(1-x)$, we get

$$S_1 H_1(2x-1, 0) = (1, 0) + 2(1-x)(-\cos^2\theta, \sin\theta\cos\theta),$$

so that

$$G_\theta(x) = \cos\theta\overline{S_1}G_\theta(2x-1) + (1-x)(-\cos 2\theta, \sin 2\theta); \qquad (6)$$

(5) and (6) can be rewritten more compactly as

$$\forall x \in [0, 1] \qquad G_\theta(x) = \sin\theta\overline{S_0}G_\theta(2x) + \cos\theta\overline{S_1}G_\theta(2x-1)$$
$$+ \Lambda(x)(-\cos 2\theta, \sin 2\theta). \qquad (7)$$

Let us now deduce the Schauder basis expansion of F_θ. Plugging the definition of G_θ given by (7) in the right-hand side of (7) and iterating, we obtain the everywhere convergent series

$$G_\theta(x) = \sum C_{j, k} \Lambda_{j, k}(x)$$

where $C_{j, k}$ are defined as follows:

$$\text{if} \quad k2^{-j} = 0 \cdot t_1 \cdots t_j,$$
$$C_{j, k} = \prod \sin\theta^{1-t_i} \cos\theta^{t_i} \prod \overline{S_{t_i}}(-\cos 2\theta, \sin 2\theta). \qquad (8)$$

2. POINTWISE REGULARITY CRITERIA ON THE SCHAUDER AND HAAR BASIS COEFFICIENTS

Let us first recall the classical pointwise regularity criteria for smooth orthonormal wavelet bases (see [2]).

PROPOSITION 1. *Let $N > 0$. Suppose that $\psi \in C^N(\mathbb{R})$ and $\forall n \leqslant N$ $|\psi^{(n)}(x)| \leqslant C(1 + |x|)^{-N}$. Let $C_{j, k}$ be the coefficients of a function F on this wavelet basis. For any Hölder exponent $\alpha < N$, we have the following pointwise regularity criteria:*
If $F \in C^\alpha(x_0)$ then

$$|C_{j, k}| \leqslant C2^{-(1/2+\alpha)j}(1 + |2^j x_0 - k|)^\alpha. \qquad (9)$$

Conversely, if there exists $\beta < \alpha$ such that

$$|C_{j,k}| \leqslant C2^{-(1/2+\alpha)j}(1+|2^jx_0-k|)^{\beta}. \tag{10}$$

then $F \in C^{\alpha}(x_0)$.

Condition (10) is the classical two-microlocal condition of J. M. Bony and is denoted $F \in C^{\alpha,-\beta}(x_0)$. This result immediately extends to the Schauder basis as long as $\alpha < 1$. Because of the different normalization we choose for the Schauder basis, the two-microlocal conditions become in this case

$$|C_{j,k}| \leqslant C2^{-\alpha j}(1+|2^jx_0-k|)^{\beta} \tag{11}$$

and we will denote this condition $S^{\alpha,-\beta}(x_0)$ for any value of α and β. We remark that for $\alpha \geqslant 1$ this condition is not related to the two-microlocal condition $C^{\alpha,-\beta}(x_0)$ (because, for instance, if $F \in C^{\alpha,-\beta}(x_0)$, then $F \in C^{\alpha-\beta}(\mathbb{R})$, whereas it is obvious to construct nondifferentiable functions satisfying the corresponding criterium $S^{\alpha,-\beta}(x_0)$). However, we will often call such conditions "two-microlocal conditions" for short.

In this part we want to establish regularity criteria for a function whose expansion on the Schauder (or Haar) basis is known. We consider scalar-valued functions; the extension to vector-valued functions (which we need for the Polya function) is straightforward. Let us elaborate on the main problem that we will meet by using two very simple examples:

• The function $\Lambda(x)$ is not differentiable at $\frac{1}{2}$, whereas $\Lambda(x) + \frac{1}{2}\Lambda(2x) + \frac{1}{2}\Lambda(2x-1)$ is C^{∞} at $\frac{1}{2}$, nonetheless, these two functions have the same coefficients on the Schauder basis for $j \geqslant 2$. This example shows that no simple sufficient condition of regularity can be found for Hölder exponents larger than 1.

• The function x^2 has Schauder coefficients $C_{j,k} = 2^{-2j}$ but it is C^{∞}; this decay is rather slow because the biorthogonal system of the Schauder basis has only its first two moments vanishing. Thus no simple necessary condition of regularity is available for Hölder exponents larger than 2.

We remark that the functions of the first example are C^{∞} except at some dyadic points, so that the usual sufficient wavelet regularity condition might work at points where all the functions of the Schauder basis are smooth, i.e., at points which are not dyadic. This is certainly too optimistic, but it turns out to be true for points which are "far enough from the dyadics." Let us first give a precise definition of this notion.

DEFINITION 1.   Let $x \in [0, 1]$. The rate of approximation of x by dyadics is by definition

$$r(x) = \limsup \frac{\log \operatorname{dist}(2^l x, \mathbb{Z})}{\log 2^{-l}}, \tag{12}$$

One always has $r(x) \geqslant 1$. If $r(x) > 1 + \varepsilon$, it means that one can find arbitrarily large J's such that the binary expansion of x contains only 0's or only 1's between ranks J and $J + \varepsilon J$.

The results of this part are ordered in roughly increasing difficulty; Proposition 2 is best possible if we disregard the exact values of the coefficients corresponding to Schauder functions having singularities "very close" to x_0; whereas Proposition 4 takes that behavior into account at the "worst point" $x_0 = \frac{1}{2}$. Theorem 2 addresses the general case without any restriction on x_0. Once and for all, we will state and prove our results in the case of the Schauder basis. Proposition 3 shows how they adapt to spline wavelets.

PROPOSITION 2.   Let $F(x) = \sum C_{j,k} \Lambda(2^j x - k)$. If F is $C^\alpha(x_0)$ for $\alpha < 3$, there exists a constant $A \in \mathbb{R}$ such that

$$|C_{j,k} - A 2^{-2j}| \leqslant C 2^{-\alpha j}(1 + |2^j x_0 - k|)^\alpha. \tag{13}$$

Conversely, if there exists a constant $A \in \mathbb{R}$ such that

$$|C_{j,k} - A 2^{-2j}| \leqslant C 2^{-\alpha j}(1 + |2^j x_0 - k|)^\beta \tag{14}$$

for $\alpha \geqslant 1$ and $\beta < \alpha$, then

$$\alpha(x_0) \geqslant 1 + \frac{\alpha - 1}{r(x_0)}.$$

For almost all x, we have $r(x) = 1$, so that for almost all x there is no loss between this criterium and the one given in Proposition 1. For the sake of simplicity, we suppose that $\alpha < 3$; the reader will easily check that criterion (13) can be easily extended to larger Hölder exponents by substracting the Schauder coefficients of $(x - k_0 2^{-j})^3, \ldots$.

The first criterion is not very convenient to use, but (13) implies that

$$|C_{j,k} - C_{j,k+1}| \leqslant C 2^{-\alpha j}(1 + |2^j x_0 - k|)^\alpha, \tag{15}$$

and we will rather use (15), or higher order differences, in Part 3 as a (non-optimal) necessary criterion of regularity.

Proof of Proposition 2. Let $k_0 = [2^j x_0]$; if F is $C^\alpha(x_0)$,

$$C_{j,k} - C_{j,k_0} = 2F\left(\frac{k+1/2}{2^j}\right) - \left(F\left(\frac{k}{2^j}\right) + F\left(\frac{k+1}{2^j}\right)\right) - 2F\left(\frac{k_0+1/2}{2^j}\right)$$

$$+ \left(F\left(\frac{k_0}{2^j}\right) + F\left(\frac{k_0+1}{2^j}\right)\right)$$

which is bounded in modulus by $C(|k-k_0|/2^j)^\alpha$ because it is an order 3 difference and $\alpha < 3$. Thus

$$C_{j,k} = C_{j,k_0} + O(2^{-\alpha j}(1 + |2^j x_0 - k|)^\alpha). \tag{16}$$

Let now $k_0' = [2^{j-1} x_0]$; $C_{j,k_0} - \frac{1}{4}C_{j-1,k_0'}$ is an order 3 difference of values of F near x_0, so that

$$C_{j,k_0} - \frac{1}{4}C_{j-1,k_0'} = O(2^{-\alpha j}). \tag{17}$$

But (17) implies that there exists a constant A such that

$$C_{j,k_0} = A2^{-2j} + O(2^{-\alpha j})$$

(consider the sequence $d_j = 2^{2j}C_{j,k_0}$), which together with (16) implies (13).

In the converse result, we can suppose that $A = 0$ because 2^{-2j} are the Schauder coefficients of the function x^2 which is C^∞. Actually, by substracting the Schauder basis coefficients of $B(x - k_0 2^{-j})^3$, $C(x - k_0 2^{-j})^4, \ldots$ where $k_0 = [2^j x_0])$ the reader will immediately obtain general criteria similar to (14) for a arbitrarily large, which will be optimal not only when $\alpha < 3$, as Proposition 2, but for larger values of α.

We thus suppose $A = 0$. consider the quantity

$$F(x) - F(x_0) - (x - x_0) \sum C_{j,k} 2^j \Lambda'(2^j x_0 - k). \tag{18}$$

The series at the right-hand side is convergent because $\Lambda'(2^j x_0 - k) = 0$ if $|2^j x_0 - k| \geq 1$, so that (14), implies that the general term of this series is bounded by $2^j 2^{-\alpha j}$; (18) can be written

$$\sum C_{j,k}(\Lambda(2^j x - k) - \Lambda(2^j x_0 - k) - (x - x_0) 2^j \Lambda'(2^j x_0 - k)). \tag{19}$$

We define J by

$$2^{-J-1} < h \leq 2^{-J};$$

we define J' as the first integer such that x and x_0 are not in the same dyadic interval of size $2^{-J'}$. We consider three cases:

• If $j \leqslant J'$, then $\Lambda(2^j y - k)$ is an affine function on the interval bounded by x and x_0 so that

$$\Lambda(2^j x - k) - \Lambda(2^j x_0 - k) - (x - x_0) \, 2^j \Lambda'(2^j x_0 - k) = 0.$$

• If $j \geqslant J$, bounding each term independently, we bound (19) by

$$C \sum_{j \geqslant J} 2^{-\alpha j}(2^j \, |h|)^\beta + C \sum_{j \geqslant J} 2^{-\alpha j} + C \sum_{j \geqslant J} 2^{-\alpha j}(2^j \, |h|) \leqslant C 2^{-\alpha J} \leqslant C \, |h|^\alpha.$$

• If $J' < j < J$, we can only bound $|\Lambda(2^j x - k) - \Lambda(2^j x_0 - k) - (x - x_0) \, 2^j \Lambda'(2^j x_0 - k)|$ by $C 2^j \, |x_0|$ and the corresponding terms in (19) are bounded by $C 2^{(\alpha - 1) J'} \, |x - x_0|$. This is the worst estimate, which cannot be improved, because of the irregularity of the Schauder basis. However, it is bad only if $J - J'$ is large. By definition of $r(x)$, for any $\varepsilon > 0$, if J is large enough, $J' \geqslant (J/r(x)) - \varepsilon$; hence we have Proposition 2.

Only one feature of the Schauder basis plays a particular role in this proof: the locations of its singularities. The same result holds for expansions on the Haar basis or piecewise linear wavelets of Strömberg and Lemarié and it adapts (with obvious modifications) to higher order spline wavelets Let us just make this translation once and state a regularity criterium for spline wavelet bases expansions; the proof exactly follows the previous one and we leave it as an exercise.

We call *spline wavelet basis of order N* an orthonormal wavelet basis (or a set of two biorthogonal bases) such that ψ is Lipschitz of order N (the derivatives of order N are in L^∞) and is piecewise polynomial of degree N between the half integers (see [6 or 9] for such examples). The Haar basis is a spline wavelet basis of order 0.

PROPOSITION 3. *Let $C_{j,k}$ be the coefficients of a function F on a spline wavelet basis of order N and suppose that we are in the "nontrivial" case, where $\alpha \geqslant N$, so that Proposition 1 does not apply. Let $K = [\alpha]$ and let*

$$M_{j,k}^n = \int \psi_{j,k}(x)(x - x_0)^n \, dx.$$

If there exist constants $A_2, ..., A_N$ such that

$$\left| C_{j,k} - \sum_{n=N}^{k} A_n M_{j,k}^n \right| \leqslant C 2^{-(1/2 + \alpha) j}(1 + |2^j x_0 - k|)^\beta$$

for a $\beta < \alpha$, *then*

$$\alpha(x_0) \geqslant N + \frac{\alpha - N}{r(x_0)}.$$

We thus have necessary criteria of regularity and sufficient criteria of regularity which are quite sharp except when $r(x)$ is large; this was to be expected since such a point x is close to the singularities of the functions in the Schauder basis. We now focus our analysis at such points. In order to understand what sharp regularity conditions can be, consider the "worse points," i.e., the dyadics themselves. Since the analysis at all dyadics is the same, we only consider the point $\frac{1}{2}$. Regularity will be obtained under two conditions of different nature: a natural "two microlocal" condition and an "algebraic" condition which expresses the fact that the graph of F at $\frac{1}{2}$ should not have an angle. The necessity of such a condition is clear if one considers again the first counterexample mentioned at the beginning of this part.

PROPOSITION 4. *Suppose that the coefficients* $C_{j,k}$ *satisfy the following condition: there exists* $A \in \mathbb{R}$ *such that*

$$|C_{j,k} - A2^{-2j}| \leqslant C2^{-\alpha j}(1 + |2^j \cdot 1/2 - k|)^{\beta} \tag{20}$$

for $\beta < \alpha$ *and* $\alpha > 1$. *If*

$$C_{0,0} = (C_{1,0} + C_{1,1}) \cdots + 2^{j-1}(C_{j,2^{j-1}-1} + C_{j,2^j-1}) + \cdots, \tag{21}$$

F is $C^{\alpha}(\frac{1}{2})$ *but if* (21) *does not hold, F is not differentiable at* $\frac{1}{2}$.

Proof. The term $A2^{-2j}$ in (20) can be dismissed in the proof because it amounts to adding a C^{∞} function and because (21) remains invariant under this change. Because of (20) the series at the right-hand side of (21) is convergent.

We separate the sum $\sum C_{j,k} \Lambda_{j,k}$ into two parts. The first one contains the indexes that do not appear in (21); the proof of Proposition 2 takes care of this sum because $\frac{1}{2}$ is "badly approximated" by dyadics different from itself.

As regards the indexes that appear in (21), we rewrite the corresponding sum,

$$\sum_{j \geqslant 1} \tfrac{1}{2}(C_{j,2^{j-1}-1} + C_{j,2^j-1})(\Lambda_{j,2^{j-1}-1} + \Lambda_{j,2^j-1} + 2^{j-1}\Lambda_{0,0})$$
$$+ \tfrac{1}{2}(C_{j,2^{j-1}-1} - C_{j,2^j-1})(\Lambda_{j,2^{j-1}-1} - \Lambda_{j,2^j-1});$$

the functions $\Lambda_{j,\,2^{j-1}-1} - \Lambda_{j,\,2^{j-1}}$ have a wavelet type localization and are linear between $\frac{1}{2} - 2^{-j-1}$ and $\frac{1}{2} + 2^{-j-1}$, thus the proof of Proposition 2 adapts immediately to the term

$$\sum \tfrac{1}{2}(C_{j,\,2^{j-1}-1} - C_{j,\,2^{j-1}})(\Lambda_{j,\,2^{j-1}-1} - \Lambda_{j,\,2^{j-1}})$$

which is thus C^{α} at $\frac{1}{2}$.

As regards the first term,

$$g(x) = \sum_{j \geqslant 1} \tfrac{1}{2}(C_{j,\,2^{j-1}-1} + C_{j,\,2^{j-1}})(\Lambda_{j,\,2^{j-1}-1} + \Lambda_{j,\,2^{j-1}} + 2^{j-1}\Lambda_{0,\,0}),$$

the functions $\omega_j = \Lambda_{j,\,2^{j-1}-1} + \Lambda_{j,\,2^{j-1}} + 2^{j-1}\Lambda_{0,\,0}$ are constant in the interval $[\frac{1}{2} - 2^{-j-1}, \frac{1}{2} + 2^{-j-1}]$. Let $h > 0$ and let J be the integer defined by

$$\tfrac{1}{2} 2^{-J} < h \leqslant 2^{-J}.$$

We have

$$g(\tfrac{1}{2} + h) - g(\tfrac{1}{2}) = \sum_{j=1}^{J} \tfrac{1}{2}(C_{j,\,2^{j-1}-1} + C_{j,\,2^{j-1}})(\omega_j(\tfrac{1}{2} + h) - \omega_j(\tfrac{1}{2})).$$

Using the mean value theorem and (20), we obtain

$$|g(\tfrac{1}{2} + h) - g(\tfrac{1}{2})| \leqslant C \sum_{j=1}^{j} C 2^{-\alpha j}(2^j h) \leqslant C 2^{-\alpha J} \leqslant C h^{\alpha};$$

hence we have the first part of the proposition. The converse part is straightforward; if (21) does not hold, there exists $a \neq 0$ such that $F(x) + a\Lambda(x)$ satisfies (21); thus $F(x) + a\Lambda(x)$ is differentiable at $\frac{1}{2}$; since $a\Lambda(x)$ is not differentiable at $\frac{1}{2}$, neither is $F(x)$.

Let now x be not dyadic, but such that $r(x)$ may be larger than 1. For each j we define k_j by

$$\left| \frac{k_j}{2^j} - x \right| = \inf_{k \in \mathbb{Z}} \left| \frac{k}{2^j} - x \right|$$

and $S \subset \mathbb{N}$ by

$$j \in S \quad \text{if} \quad \frac{k_j}{2^j} \neq \frac{k_{j-1}}{2^{j-1}}, \qquad \frac{k_j}{2^j} = \frac{k_{j+1}}{2^{j+1}}.$$

The sparcity of S is clearly related to $r(x)$; for instance, if $r(x) > 1 + \varepsilon$, we can find arbitrarily large J's such that S does not contain $J, J+1, ..., J+[\varepsilon J]$.

Suppose that J belongs to S and let J' be the first index such that $|k_J/2^J - x| \geqslant 2^{-J'}/4$. Let

$$E_J = C_{J-1,\,(k_J-1)/2} - \sum_{j=J}^{J'} 2^{j-J}(C_{j,\,2^j2^{-J}k_j-1} + C_{j,\,2^j2^{-J}k_j}) \qquad (22)$$

(if $x = \frac{1}{2}$, the only element of S is 1 and E_1 is exactly the difference between the left and the right-hand side of (21)).

THEOREM 2.   *Suppose that the Schauder coefficients of F satisfy*

$$|C_{j,\,k} - A2^{-2j}| \leqslant C2^{-\alpha j}(1 + |2^j x - k|)^\beta$$

for $\beta < \alpha$. Let

$$r_j(x) = \frac{\log |k_j/2^j - x|}{\log 2^{-j}}, \qquad \alpha_J = \frac{\log E_J}{\log 2^{-J}}.$$

If $\alpha > \liminf(1 + (\alpha_J - 1)/r_J(x))$, the Hölder exponent of F at x is

$$\alpha(x) = \liminf \left(1 + \frac{\alpha_J - 1}{r_J(x)} \right); \qquad (23)$$

else

$$\alpha(x) \geqslant \alpha. \qquad (24)$$

This theorem improves Proposition 2, especially when $r(x)$ is large, and it is actually close to a necessary and sufficient condition of regularity for $\alpha < 3$ (if $\alpha \geqslant 3$, we have to make the modifications mentioned above).

Proof.   We split the sum $\sum C_{j,\,k} \Lambda_{j,\,k}$ into three pieces:

The first one corresponds to indexes that do not appear in the sum at the right hand side of equality (22). In this case, the singularities of the corresponding $\Lambda_{j,\,k}$ are always at a distance at least $2^{-j}/2$ of x, and the situation is the same as in Proposition 2 for a point such that $r(x) = 1$; the same proof shows that this sum is $C^\alpha(x)$.

The second sum corresponds to the sum on the remaining indexes except for $C_{J-1,\,(k_J-1)/2}$ that we replace by $C_{J-1,\,(k_J-1)/2} - E_J$. The purpose of this substitution is to have no singularity at $k_J 2^{-J}$: for a given J, the corresponding sum

$$(C_{J-1,\,(k_J-1)/2} - E_J)\,\Lambda_{J-1,\,(k_J-1)/2} + \sum_J^{J'} \sum_k C_{j,\,k}\,\Lambda_{j,\,k}$$

(where k takes the two values $2^{j-J}k_j - 1, 2^{j-J}k_j$) is C^∞ at k_j2^{-J} and the same argument as in Proposition 4 shows that the sum (over all values of J) of these blocks is $C^\alpha(x)$ (because the two-microlocal estimate implies that $|E_J| \leqslant C2^{-\alpha J}$).

The remaining term is $g(x) = \sum E_J\Lambda(2^{J-1}x - (k_J-1)/2)$. Here the proof of Proposition 2 immediately adapts and shows that the Hölder exponent of this term satisfies

$$\alpha(x) \geqslant \liminf_{J \in S}\left(1 + \frac{\alpha_J - 1}{r_J(x)}\right);$$

hence (24) and the proposition will be proved if we get the converse inequality in the case where $\alpha < \liminf(1 + (\alpha_J - 1)/r_J(x))$ (since in that case the two previous terms bring a contribution which is smoother than $\alpha(x)$). For that we construct, when $J \in S$, a finite difference of sufficiently large order, centered near k_J, which vanishes for all functions $\Lambda(2^l x - k_l)$, when $l \in S$, except for $\Lambda(2^{J-1}x - (k_J-1)/2)$.

Let $j = J' - 1$ (recall that J' is the first index such that $|k_J/2^J - x| \geqslant 2^{-J'}/4$). We split the interval $[k_j2^{-j}, (k_j+1)2^{-j}]$ into four subintervals of equal length $A = [k_j2^{-j}, (k_j+1/4)2^{-j}], \ldots, D = [(k_j+\frac{3}{4})2^{-j}, (k_j+1)2^{-j}]$. The finite difference Δ_j we construct is the sum of three finite differences:

$$\Delta_1 f = 2f((k_j+1/2)2^{-j}) - f(k_j2^{-j}) - f((k_j+1)2^{-j}).$$

Δ_2 is a finite difference of order 2 centered on A and Δ_3 is a finite difference of order 2 centered on D. One can clearly choose Δ_2 and Δ_3 such that $\Delta_j = \Delta_1 + \Delta_2 + \Delta_3$ has an arbitrarily large order.

If $l < J-1$, $\Lambda(2^l x - k_l)$ is linear on $[k_j2^{-j}, (k_j+1)2^{-j}]$ so that $\Delta_j(\Lambda(2^l x - k_l)) = 0$.

If $l = J-1$, $\Delta_2(\Lambda(2^{J-1}x - (k_J-1)/2)) = \Delta_3(\Lambda(2^{J-1}x - (k_J-1)/2)) = 0$ so that

$$\Delta_j\Lambda(2^J x - k_J)) = \Delta_1\Lambda(2^J x - k_J)) = 2 \cdot 2^{-j+J}.$$

If $l \geqslant J$, the support of $\Lambda(2^l x - k_l)$ is included in B or D so that

$$\Delta_1(\Lambda(2^l x - k_l)) = \Delta_2(\Lambda(2^l x - k_l)) = \Delta_3(\Lambda(2^l x - k_l)) = 0.$$

We remark that $j = Jr_J(x)$ so that

$$\Delta_j(g) = E_J2^{-j+J} = 2^{-\alpha_J J - 1r_J(x)+J},$$

but, if a function F is $C^\alpha(x)$, $|\Delta_j(f)| \leqslant C2^{-j\alpha}$, thus

$$\alpha(x) \leqslant \liminf \frac{-\alpha_J J - Jr_J(x) + J}{Jr_J}$$

which ends the proof of Theorem 2.

3. Pointwise Regularity of the Polya Function F_θ

We will now use the criteria of regularity given in the previous section in order to determine the pointwise Hölder regularity of the Polya function F_θ. We first remark that (8) implies that the order of magnitude of $C_{j,k}$ is, if $k2^{-j} = 0 \cdot t_1 \cdots t_j$,

$$|C_{j,k}| \sim \prod_{i=1}^{j} (\sin \theta)^{(1-t_i)} (\cos \theta)^{t_i}. \tag{25}$$

THEOREM 3. *Let*

$$D_j = \sup_{|k2^{-j} - x_0| \leqslant 2 \cdot 2^{-j}} |C_{j,k}|$$

and

$$\beta(x_0) = \liminf \frac{\log D_j}{\log 2^{-j}};$$

for any $x_0 \in [0, 1]$, the Hölder exponent of F_θ at x_0 is $\beta(x_0)$.

Proof. If $\beta(x_0) < 2$, Proposition 2 implies that

$$\alpha(x_0) \leqslant \beta(x_0). \tag{26}$$

If $2 \leqslant \beta(x_0) < 3$, the difference of two consecutive values of $C_{j,k}$ is

$$C_{j-1, [k/2]}(\sin \theta \overline{S_0}(-\cos 2\theta, \sin 2\theta) + \cos \theta \overline{S_1}(-\cos 2\theta, \sin 2\theta))$$

which is of the order of magnitude of $C_{j,k}$, hence (26) in that case. Taking differences of higher order does not make the order of magnitude of the coefficients smaller, hence (26) even when $\beta(x_0)$ is arbitrarily large. We now prove the converse inequality, which will be a consequence of Theorem 2.

Let us first prove a "two-microlocal" estimate. Consider now a given coefficient $C_{j,k}$. Let J be defined by

$$2^{-J} \leqslant |k2^{-j} - x_0| < 2 \cdot 2^{-J}, \tag{27}$$

In the product (25), the first J terms are the same as for one of the $C_{J,l}$ which appear in the supremum defining D_J; thus $|C_{j,k}| \leqslant D_J \cos \theta^{j-J}$ (because $\cos \theta \geqslant \sin \theta$), which can be written

$$|C_{j,k}| \leqslant D_J 2^{-\alpha_{\min}(j-J)} \tag{28}$$

where $\alpha_{\min} = \log \cos \theta / \log (\frac{1}{2})$. But (27) and (28) imply that

$$|C_{j,k}| \leqslant C 2^{-(\beta(x_0)-\varepsilon)j} |2^j x_0 - k|^{\beta(x_0)-\varepsilon-\alpha_{\min}}. \tag{29}$$

This implies Theorem 3 when $\beta(x_0) < 1$. Suppose $\beta(x_0) \geqslant 1$; we write $x_0 = 0 \cdot t_1 \cdots t_n$. Then $J \in S$ if $t_{J+1} = t_J$ and $t_{J-1} \neq t_J$. Let J' as in (22). Then

$$D_{J'} \sim \prod_{i=1}^{J} (\sin \theta)^{1-t_i} (\cos \theta)^{t_i} (\cos \theta)^{J'-J}$$

and

$$|E_J| \leqslant \prod_{i=1}^{J} (\sin \theta)^{1-t_i} (\cos \theta)^{t_i} \left(1 + \sum_{j=J}^{J'} 2^{j-J}((\sin \theta)^{j-J} + \cos \theta)^{j-J}) \right)$$

$$\leqslant 4 \prod_{i=1}^{J} (\sin \theta)^{1-t_i} (\cos \theta)^{t_i} 2^{J'-J} (\cos \theta)^{J'-J}$$

(because $\cos \theta > \sqrt{2}/2$). Thus,

$$E_J \leqslant C 2^{J'-J} D_{J'};$$

but $r_J(x) = J'/J$, so that

$$1 + \frac{\alpha_J - 1}{r_J(x)} \geqslant \frac{\log D_{J'}}{\log 2^{-J'}}$$

and thus (24) implies that $\alpha(x) \geqslant \lim \inf(\log D_{J'} / \log 2^{-J'})$; hence we have Theorem 3.

If x is normal in base 2, i.e., if it satisfies $N(j)/j \to \frac{1}{2}$, the existence of this limit implies $r(x) = 1$, and the Hölder exponent at x is $(\log \cos \theta - \log \sin \theta)/2 \log 2$. Since almost every point is normal in base 2, we have proved the following proposition which improves Lax's result.

PROPOSITION 5. *For almost every x, the pointwise Hölder regularity of the Polya function at x satisfies*

$$\alpha(x) = \frac{-\log (\cos \theta \sin \theta)}{2 \log 2}$$

If x satisfies $N(j)/j \to p \ 0 \leqslant p \leqslant 1$, again $r(x) = 1$ and Theorem 3 implies that

$$\alpha(x) = \frac{p \log \cos \theta + (1+p) \log \sin \theta}{\log (1/2)}. \tag{30}$$

We will note the function of p at the right-hand side $e(p)$. Points for which $N(j)/j \to p$ have Hausdorff dimension

$$d_H(p) = \frac{p \log p + (1-p) \log (1-p)}{\log (1/2)}$$

(see [1]); hence, if we define $f(\alpha)$ between $\alpha_{\min}$ and $\alpha_{\max}$ by

$$f(\alpha) = d_H(e^{-1}(\alpha)) \tag{31}$$

the Hölder spectrum of the Polya function is larger than $f(\alpha)$. Actually it is easy to check (cf. [3]) that the dimension of the points such that $p = \lim \sup N(j)/j$ is exactly $d_H(p)$. Since these points have Hölder exponent $\alpha(x)$ given by (30), the following result holds.

COROLLARY 1.   *Let* $\alpha_{\min} = \log \cos \theta / \log(1/2)$ *and* $\alpha_{\max} = \log \sin \theta / \log(1/2)$. *All points have Hölder exponents between these two values, and in the interval* $[\alpha_{\min}, \alpha_{\max}]$ *the Hölder spectrum of* F_θ, *is equal to the function* $f(\alpha)$ *defined by* (31).

Remark.   A straightforward computation shows that

$$d(\alpha) = \frac{\begin{bmatrix} [\log (2^\alpha \sin \theta)\log (\log (2^\alpha \sin \theta)) \\ - \log (2^\alpha \cos \theta) \log (\log (2^\alpha \cos \theta)) - \log (\log (\sin \theta \cos \theta))] \end{bmatrix}}{\log (\sqrt{\sin \theta \cos \theta})}$$
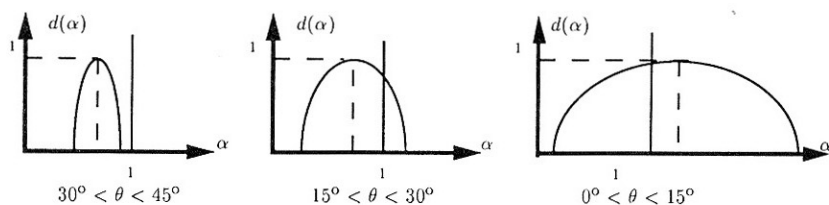

FIG. 2.   Hölder spectrum of the Polya function: In order to compare easily with Theorem 1, the possibilities distinguished by Lax are plotted separately.

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