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Multifractal Formalism for Infinite Multinomial Measures

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There are strong reasons to believe that the multifractal spectrum of DLA shows anomalies which have been termed *left-sided*. In order to show that this is compatible with strictly multiplicative structures Mandelbrot and co-workers introduced a one-parameter family of multifractal measures, invariant under infinitely many linear maps, on the real line. Under the assumption that the usual multifractal formalism holds, they showed that the multifractal spectrum of these measures is indeed left-sided, i.e., increasing over the whole α range $]\alpha_{\min}, \infty[$. Here, it is shown that the multifractal formalism for self-similar measures does indeed hold also in the infinite case, in particular that the singularity exponents $\tau(q)$ satisfy the usual equation $\sum p_i^q \lambda_i^\tau = 1$ and that the spectrum $f(\alpha)$ is the Legendre transform of $\tau(q)$.

1. INTRODUCTION AND SUMMARY

In 1990, [M, MEH] introduced self-similar multifractal measures μ which are constructed with an infinitely multiplicative cascade. Data suggest that the measures considered in [MEH] show the same anomalies as observed with DLA, namely, the partition sum $\chi(q, \varepsilon)$ fails to scale like $\varepsilon^{-\tau(q)}$ for $q < 0$. This behaviour is linked with the existence of infinite Hölder exponents and with a multifractal spectrum which is increasing over the whole range $]\alpha_{\min}, \infty[$, hereafter called a *left-sided* spectrum. This observation had led some authors to describe DLA as "nonmultifractal." On the other hand, the infinite multinomial measures (1) show that left-sided spectra are compatible with strictly multiplicative, hence renormalizable structures. This paper provides a rigorous mathematical frame for the statements made in [MEH].[1]

[1] *Note added in proof*. Infinite systems of conformal contractions are studied in [MU] comparing packing and Hausdorff measures of invariant sets.

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The organization is as follows. Following Hutchinson [Hut] we will start by defining the codespace I, a set of sequences $i_1 i_2 \dots$ suitable to model the invariant set of a given infinite family of contractions $\{w_i\}_{i \in \mathbb{N}}$ of $\mathbb{R}^d$. There is a probability distribution on I such that an address picked randomly satisfies $i_n = k$ with probability p_k independent of n. Via the addressing this translates into a probability measure μ on $\mathbb{R}^d$ which is the only one satisfying the invariance

$$\mu = \sum_{i=1}^{\infty} p_i \, \mu\big(w_i^{-1}(\cdot)\big). \tag{1}$$

Its support is the (only) compact set satisfying

$$K = \overline{\bigcup_{i=1}^{\infty} w_i(K)}. \tag{2}$$

The necessity of taking the closure $\overline{(\cdot)}$ on the right-hand side emphasizes the difference from the "classical" case. Having established μ we proceed to estimate the Hausdorff dimension $d_{\mathrm{HD}}(K_\alpha)$ of the sets K_α of points with Hölder exponent α (see (8) below). It is easy to show that $d_{\mathrm{HD}}(K_\alpha)$ is bounded from above by the Legendre transform $f(\alpha)$ of the function $\tau(q)$ which is defined as follows: Let $\lambda_i := \mathrm{Lip}(w_i)$. For given $q \in \mathbb{R}$ the implicit equation

$$\sum_{i=1}^{\infty} p_i^q \lambda_i^\tau = 1 \tag{3}$$

has either one or no solution. We define $\tau(q)$ to be the solution of (3) if there is one. Otherwise we set $\tau(q) = \infty$.

Finally, we give two conditions which together imply that $d_{\mathrm{HD}}(K_\alpha) = f(\alpha)$. First, the copies $w_i(K)$ of the support should not overlap too much. Hence, we require a separation condition similar to the well-known open set condition. Second, we will use the spectra f_M of finite approximations μ_M of μ as lower bounds of $d_{\mathrm{HD}}(K_\alpha)$. The intuitive idea behind this is the fact that at a certain scale the details provided by the maps w_i with too large i cannot be observed. Indeed, the convergence of f_M to f is in agreement with numerical measurements, where one finds a sequence of preasymptotic spectra which increase to f (Fig. 1). To perform the limit we require some control on the rates with which $\{p_i\}_{i \in \mathbb{N}}$ and $\{\lambda_i\}_{i \in \mathbb{N}}$ tend to zero. We consider this second condition to be purely technical.

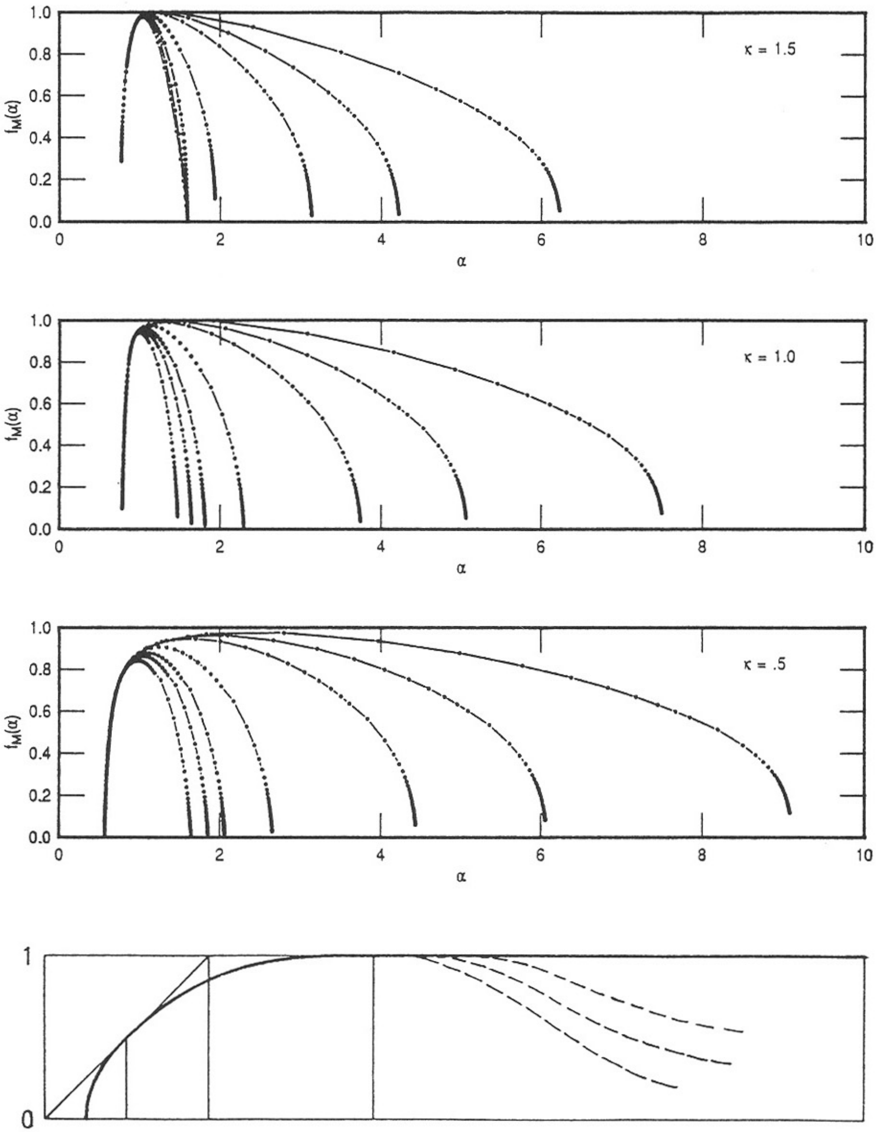

FIG. 1. The approximation of the spectrum $f(\alpha)$ as provided through f_M for the left-sided multifractal measure of Example 1 for $\kappa = 0.5$, 1, and 1.5. Note that the different rates of convergence, depending on the parameter κ, are in agreement with the asymptotic behaviour of $f(\alpha)$ for $\alpha \to \alpha_0$ (Fig. 2). The approximation resembles the convergence of the "pre-asymptotic" spectra $\rho_k(\alpha) + 1$ ($\rho_k(\alpha)$ is the probability of finding a coarse Hölder exponent α at resolution 2^{-k}) as described in [M] (see figure at the bottom).

2. Codespace

As pointed out with (2), having infinitely many copies of a set or a measure requires one to consider the closure of sets. In other words, accumulation points have to be taken into account which can contribute considerably to the geometry of the invariant measure. Their influence, however, is captured in the geometry of the fixpoints a_n of the contractions w_n. In order to have "primitive" addresses for these fixpoints, as well as for their accumulation points, we identify $\mathbb{N}$ with $\{a_i\}_{i \in \mathbb{N}}$ through $i \mapsto a_i$. Being a subset of $\mathbb{R}^d$ the set $\mathbb{N}^c$ of "points at infinity" may contain more than one element. More precisely,

$$\overline{\mathbb{N}} := \overline{\{a_i\}_{i \in \mathbb{N}}} \subset \mathbb{R}^d, \qquad \mathbb{N}^c := \overline{\mathbb{N}} \setminus \{a_i\}_{i \in \mathbb{N}}. \tag{4}$$

Although $\mathbb{N}$ is embedded in space we will still think of it as being the set of natural numbers. As will be seen, it is essential for the construction of μ to assume that $\{a_i\}_{i \in \mathbb{N}}$ is bounded, i.e., that $\overline{\mathbb{N}}$ is compact (see Lemma 1). Equivalently, we can require that there is a bounded open set O, called *cell*, such that $w_i(O) \subset O$ for all $i \in \mathbb{N}$. We recall some separation conditions which are widely used and which apply to finite or countably infinite families $\{w_i\}_{i \in J}$. They are said to hold if there is a cell which satisfies the indicated condition: the *open set condition* (OSC) if

$$w_i(O) \cap w_j(O) = \varnothing \qquad \forall i \neq j \in J,$$

the *disconnected open set condition* (DOSC) if the sets $w_i(\overline{O})$ $(i \in J)$ are mutually disjoint and the *strong open set condition* (SOSC) if, in addition to the OSC, the cell O intersects K.

The desired *codespace* is

$$I := \overline{\mathbb{N}}^{\mathbb{N}}.$$

It is convenient to define for $a \in \mathbb{N}^c$: $w_a(x) \equiv a$, i.e., a trivial contraction with ratio $\lambda_a := \mathrm{Lip}(w_a) = 0$, and to set $p_a := 0$. For $\mathbf{i} \in I$ set $\mathbf{i}|n := i_1 \cdots i_n$, $\lambda_{\mathbf{i}|n} := \lambda_{i_1} \cdot \cdots \cdot \lambda_{i_n}$, $p_{\mathbf{i}|n} := p_{i_1} \cdot \cdots \cdot p_{i_n}$, $w_{\mathbf{i}|n} := w_{i_1} \circ \cdots \circ w_{i_n}$, and, once a cell O has been chosen, $V_{\mathbf{i}|n} := w_{\mathbf{i}|n}(\overline{O})$.

Lemma 1. *Assume that* $\overline{\mathbb{N}}$ *is compact and that* $\mathrm{Lip}(w_i) = \lambda_i \to 0$ $(i \to \infty)$. *This is certainly satisfied if the OSC holds. Then,* $\bigcap_{n \in \mathbb{N}} V_{\mathbf{i}|n}$ *is a singleton, say* $\{x_{\mathbf{i}}\}$, *for every given* $\mathbf{i} \in I$. *Moreover, the* coordinate map π: $I \to \mathbb{R}^d$ $\mathbf{i} \mapsto x_{\mathbf{i}}$ *is continuous.*

Proof. Since $V_{\mathbf{i}|n} = w_{\mathbf{i}|n-1} \circ w_{i_n}(\overline{O}) \subset w_{\mathbf{i}|n-1}(\overline{O}) = V_{\mathbf{i}|n-1}$, the compact sets $V_{\mathbf{i}|n}$ form a decreasing sequence. So, their intersection cannot be empty. But it has diameter zero, since $|V_{\mathbf{i}|n}| = \lambda_{\mathbf{i}|n} \cdot |\overline{O}| \to 0$ $(n \to \infty)$.

This proves the first part. The continuity of π is a direct consequence of our choice of the metric on $\overline{\mathbb{N}}$, which we denote by $d(\cdot,\cdot)$. Take $x = \pi(\mathbf{i}) \in K$ and $\varepsilon > 0$. We have to find a neighbourhood U of $\mathbf{i}$ which is mapped into $B_\varepsilon(x)$. First, there is a n_0 such that $V_{\mathbf{i}|n_0} \subset B_\varepsilon(x)$. In case $i_k \in \mathbb{N}$ for all $k \leq n_0$ it is enough to choose $U = \{\mathbf{j} \in I\colon\ j_k = i_k\ (1 \leq k \leq n_0)\}$. Otherwise let n_1 be such that $i_k \in \mathbb{N}$ for all $k < n_1$ and that $\bar{a} := i_{n_1} \in \mathbb{N}^c$. Since $\lambda_i \to 0$ $(i \to \infty)$, there is $m_0 \in \mathbb{N}$ s.t. $|V_k| \leq \varepsilon/3$ for all $k \geq m_0$. Now, we can choose $\delta < \varepsilon/3$ such that $m \in \mathbb{N}$ and $d(m, \bar{a}) \leq \delta$ imply that $m \geq m_0$. Set $A := \{a \in \overline{\mathbb{N}}\colon d(a, \bar{a}) \leq \delta\}$. We claim that it is enough to choose $U = \{\mathbf{j} \in I\colon\ j_k = i_k\ (1 \leq k < n_1),\ j_{n_1} \in A\}$. Note first that the fixpoint of w_a lies in $w_a(K) \subset V_a = w_a(\overline{O})$ for all $a \in \overline{\mathbb{N}}$. So, since $|V_a| \leq \varepsilon/3$ and $d(a, \bar{a}) \leq \varepsilon/3$ for $a \in A$, the union $\bigcup_{a \in A} V_a$ is a subset of $B_\varepsilon(\bar{a})$. Finally, $\bar{a} = w_{\bar{a}}(\bar{a}) = \pi(\mathbf{k})$ for any $\mathbf{k}$ starting with $\bar{a}$. Consider $\mathbf{k} = i_{n_1} i_{n_1+1} \cdots$. Then, $w_{\mathbf{i}|n_1-1}(\bar{a}) = w_{\mathbf{i}|n_1}(\bar{a}) = \pi(\mathbf{i})$ and we have $\pi(U) = w_{\mathbf{i}|n_1-1}(\bigcup_{a \in A} w_a(K)) \subset w_{\mathbf{i}|n_1-1}(B_\varepsilon(\bar{a})) = B_{\varepsilon'}(\pi(\mathbf{i}))$ with $\varepsilon' = \lambda_{\mathbf{i}|n_1-1}\varepsilon \leq \varepsilon$. ∎

3. The Invariant Measure

Let $\{p_i\}_{i \in \mathbb{N}}$ be a probability vector, i.e., $0 < p_i$ and $\sum_{i \in \mathbb{N}} p_i = 1$. Due to Kolmogorov's consistency theorem [Pth, p. 144] there is a probability measure $\hat{\mu}$ on I corresponding to the measures $\sum_{n \in \mathbb{N}} p_n \delta_{a_n}$ (δ_x is the Dirac measure concentrated in x) on the factors $\overline{\mathbb{N}}$ of I. More precisely,

$$\hat{\mu}\Big[\big\{\mathbf{i} \in I\colon i_{k_1} = j_1, \ldots, i_{k_m} = j_m\big\}\Big] = p_{\mathbf{j}|m} \tag{5}$$

for all $\mathbf{j} \in I$, $k_1 < \cdots < k_m$ and $m \in \mathbb{N}$. Due to its product structure (5), $\hat{\mu}$ is invariant under the shift operators $\sigma_a\colon \mathbf{i} \mapsto a i_1 i_2 \ldots$ of I:

$$\hat{\mu} = \sum_{a \in \overline{\mathbb{N}}} p_a\, \hat{\mu}\big(\sigma_a^{-1}(\cdot)\big) = \sum_{i=1}^{\infty} p_i\, \hat{\mu}\big(\sigma_i^{-1}(\cdot)\big). \tag{6}$$

Provided that π is continuous,

$$\mu := \hat{\mu}\big(\pi^{-1}(\cdot)\big) \tag{7}$$

is a Borel measure, more precisely a probability measure, and (6) translates immediately into (1). Moreover, $K = \pi(I)$ is the support of μ and the shift invariance $I = \bigcup_{a \in \overline{\mathbb{N}}} \{a\} \times I$ yields

$$K = \bigcup_{a \in \overline{\mathbb{N}}} w_a(K) = \overline{\bigcup_{i=1}^{\infty} w_i(K)},$$

hence (2).

Proof. The first equation is immediate. Since π is continuous K is compact and hence contains the right hand set. For $a \in \mathbb{N}^c$ we have $w_a(K) = \{a\} \subset \overline{\mathbb{N}} = \overline{\{a_i\}}_{i \in \mathbb{N}}$. Together with $a_i = \pi(iii\ldots) = w_i(\pi(iii\ldots)) \in w_i(K)$ this implies that the middle set is contained in the right hand set. ■

LEMMA 2. *Assume that the SOSC holds. Then* $\mu(V_{\mathbf{i}|n}) = p_{\mathbf{i}|n}$ *for all* $\mathbf{i} \in I$ *and all* $n \in \mathbb{N}$.

Proof. First note that it is enough to prove $\mu(O) = 1$. Indeed,

$$\mu\big(w_j(O)\big) = \sum_{k=1}^{\infty} p_k\, \mu\Big(w_k^{-1}\big(w_j(O)\big)\Big) = p_j\, \mu(O) = p_j,$$

since $\operatorname{supp}(\mu) \subset \overline{O}$ and $w_k^{-1}(w_j(O)) \cap \overline{O} = \varnothing$ $(k \neq j)$ due to the OSC. Moreover, given $i \in \overline{\mathbb{N}}$, the sets V_i and $w_j(O)$ $(j \in \overline{\mathbb{N}},\ j \neq i)$ are mutually disjoint again by the OSC. So,

$$1 = \mu(\mathbb{R}^d) \geq \mu(V_i) + \sum_{j \neq i} p_j = \mu(V_i) + (1 - p_i).$$

With $\mu(V_i) \geq \mu(\{\mathbf{j}\colon j_1 = i\}) = p_i$ we obtain equality. The same argument applies to words $i_1 \cdots i_n$ of length n instead of i. To show that $\mu(O) = 1$ pick a point $x = \pi(\mathbf{i})$ in O. In case that $\mathbf{i}$ contains letters from $\mathbb{N}^c$ let $i_{n+1} = a \in \mathbb{N}^c$ denote the first one. Then, $x = w_{\mathbf{i}|n}(a)$ and by definition of $\mathbb{N}^c$ there is $m \in \mathbb{N}$ such that the open set O also contains $w_{\mathbf{i}|n}(a_m) = \pi(i_1 \cdots i_n mmm \cdots)$. So, we always find $\mathbf{i} \in \mathbb{N}^{\mathbb{N}}$ with $\pi(\mathbf{i}) \in O$. Since O is open, there is $n \in \mathbb{N}$ such that $V_{\mathbf{i}|n} \subset O$. By switching to the iterated family $w_{i_1 \cdots i_n}$ $(i_1 \cdots i_n \in \overline{\mathbb{N}}^n)$ we can assume that $n = 1$. For simplicity we assume $i_1 = 1$. In particular, $V_1 \subset O$ and $\mu(O) \geq \mu(V_1) \geq p_1 > 0$. Since O is a cell we have $V_{j1} = w_j(V_1) \subset w_j(O) \subset O$. From $w_j(O) \cap V_1 = \varnothing$ $(j \neq 1)$ and $\mu(V_{j1}) \geq p_{j1} = p_j p_1$, we conclude that

$$\mu(O) \geq p_1 + \sum_{j \neq 1} p_j p_1 = p_1 + p_1(1 - p_1).$$

Similarly $V_{kj1} \subset w_{kj}(O) \subset w_k(O) \subset O$ and $w_{kj}(O) \cap V_{k1} = \varnothing$ $(k \neq 1)$. So, V_{kj1}, V_{j1}, and V_1 are all disjoint, provided $k, j \neq 1$. Hence,

$$\begin{aligned}\mu(O) &\geq p_1 + p_1(1 - p_1) + \sum_{k, j \neq 1} p_k p_j p_1 \\ &= p_1 + p_1(1 - p_1) + p_1(1 - p_1)^2.\end{aligned}$$

By induction, $\mu(O) \geq p_1 \Sigma_{n \in \mathbb{N}} (1 - p_1)^n = 1.$ ■

For convenience we focus on the following class of measures.

DEFINITION 3. We will call μ an *infinite self-similar measure* if it is defined by (7), $\{w_i\}_{i \in \mathbb{N}}$ being a family of similarities with $|w_i(x) - w_i(y)| / |x - y| \equiv \lambda_i \in (0, 1)$ and such that the SOSC holds.

A straightforward generalization of the fixpoint argument given by Hutchinson [Hut, p. 733] shows that (1) and (2) define μ and K uniquely.

4. UPPER BOUND

Our results concern the sets

$$K_\alpha := \left\{ x = \pi(\mathbf{i}) \colon \frac{\log \mu(V_{\mathbf{i}|n})}{\log |V_{\mathbf{i}|n}|} \text{ is defined for all } n \text{ and tends to } \alpha \right\}, \quad (8)$$

where $|E|$ denotes the diameter of the set E. The limit $\alpha = \alpha_{\mathbf{i}}$ involved is called the *Hölder exponent* of μ at $x = \pi(\mathbf{i})$ and depends on the address $\mathbf{i}$. As with the examples in the last section, points with multiple addresses can often be disregarded. The definition implies that we disregard the set E of all addresses containing any letter from $\mathbb{N}^c$ and focus on the set of points with "finite addresses" $\pi(F)$, where

$$F := \mathbb{N}^{\mathbb{N}} = I \setminus E.$$

Note, that E is a $\hat{\mu}$-nullset since $\mathbb{N}^c$ is a compact subset of $\overline{\mathbb{N}}$ of zero measure in each factor. The corresponding points require a special treatment. In the example considered here, however, this turns out to be trivial. Moreover, although E is always uncountable, $\pi(E)$ is certainly countable if $\mathbb{N}^c$ is. Disregarding such a set does not affect Hausdorff dimensions [F2]. Finally, it is impossible to observe infinitely many maps in simulations but rather some finite approximations like μ_M defined below and their behaviour with increasing M. For these measures μ_M the usual definition of Hölder exponent coincides with ours [CM, R: Theorem 18 iii].

An upper bound of $d_{\mathrm{HD}}(K_\alpha)$ is easy to obtain. Let

$$B_\alpha := \pi\left(\left\{\mathbf{i} \in I \setminus E \colon \limsup_{n \to \infty} \left(\log \mu(V_{\mathbf{i}|n}) / \log |V_{\mathbf{i}|n}|\right) \le \alpha \right\}\right)$$

$$C_\alpha := \pi\left(\left\{\mathbf{i} \in I \setminus E \colon \liminf_{n \to \infty} \left(\log \mu(V_{\mathbf{i}|n}) / \log |V_{\mathbf{i}|n}|\right) \ge \alpha \right\}\right).$$

PROPOSITION 4. *For any infinite self-similar measure we have*

$$d_{\mathrm{HD}}(B_\alpha) \le \inf\{\tau(q) + q\alpha : q \ge 0\},$$
$$d_{\mathrm{HD}}(C_\alpha) \le \inf\{\tau(q) + q\alpha : q \le 0\}.$$

Moreover, $d_{\mathrm{HD}}(K) \le \tau(0)$.

Remark. The same result holds if we barely impose $|w_i(x) - w_i(y)| \le \lambda_i |x - y|$.

Proof. For simplicity assume that the cell O has diameter $|O| = 1$; hence $|V_{\mathbf{i}|n}| = \lambda_{\mathbf{i}|n}$. The SOSC implies that $\bar{\lambda} := \max\{\lambda_i : i \in \mathbb{N}\} < 1$ and $\mu(V_{\mathbf{i}|n}) = p_{\mathbf{i}|n}$. Take $q \in \mathbb{R}$. If $\tau(q) = \infty$ there is nothing to show. Otherwise take $\varepsilon > 0$. Let

$$L_N := \pi\Big(\Big\{\mathbf{i} \in I \setminus E : p^q_{\mathbf{i}|n} \ge \lambda^{q(\alpha+\varepsilon)}_{\mathbf{i}|n} \text{ for all } n \ge N\Big\}\Big).$$

First we estimate $d_{\mathrm{HD}}(L_N)$. For $n \ge N$ let $A_n = \{\mathbf{j} = j_1, \ldots, j_n \in \mathbb{N}^n : p^q_{\mathbf{j}} \ge \lambda^{q(\alpha+\varepsilon)}_{\mathbf{j}}\}$. We have

$$\begin{aligned} 1 = \Big(\sum_{i \in \mathbb{N}} p_i^q \lambda_i^{\tau(q)}\Big)^n &\ge \sum_{\mathbf{j} \in A_n} p^q_{\mathbf{j}} \lambda^{\tau(q)}_{\mathbf{j}} \\ &\ge \sum_{\mathbf{j} \in A_n} \lambda^{q\alpha + q\varepsilon + \tau(q)}_{\mathbf{j}} = \sum_{\mathbf{j} \in A_n} |V_{\mathbf{j}}|^{q\alpha + q\varepsilon + \tau(q)}. \end{aligned}$$

Since $(V_{\mathbf{j}})_{\mathbf{j} \in A_n}$ provides a covering of L_N with $|V_{\mathbf{j}}| \le \bar{\lambda}^n$, this implies that $d_{\mathrm{HD}}(L_N) \le q\alpha + q\varepsilon + \tau(q)$ for all N. The union $\bigcup_{N \in \mathbb{N}} L_N$ contains B_α if $q \ge 0$ (resp. C_α if $q \le 0$). Recalling that $d_{\mathrm{HD}}(\bigcup_{N \in \mathbb{N}} L_N) = \sup_{N \in \mathbb{N}} d_{\mathrm{HD}}(L_N)$ [F2] and then using that $\varepsilon > 0$ was arbitrary the first two estimates follow. Finally, for $q = 0$ the collection $(V_{\mathbf{j}})_{\mathbf{j} \in A_n}$ covers the whole support K of μ. This completes the proof. ■

COROLLARY 5. *For any infinite self-similar measure we have*

$$d_{\mathrm{HD}}(K_\alpha) \le \inf\{\tau(q) + q\alpha : q \in \mathbb{R}\}.$$

Proof. $K_\alpha \subset B_\alpha \cap C_\alpha$. ■

5. SEPARATION CONDITION

To estimate $d_{\mathrm{HD}}(K_\alpha)$ from below we consider "finite approximations" of μ, i.e., self-similar measures μ_M invariant under the first M maps $w_1, \ldots, w_M$. In a certain sense, the spectra f_M of μ_M approximate f from below which is again very helpful from the point of view of application. Fix a natural number M and let μ_M be the unique self-similar measure invariant under the maps $(w_1, \ldots, w_M)$ and the probabilities

$(p_1^{(M)}, \ldots, p_M^{(M)})$, where $p_k^{(M)} = p_k/N_M$ and $N_M := \Sigma_{i=1}^M p_i$. Hence,

$$\mu_M = \sum_{i=1}^{M} p_i^{(M)} \mu_M\big(w_i^{-1}(\cdot)\big).$$

Assuming that $w_1, \ldots, w_M$ satisfies the DOSC, Cawley and Mauldin [CM] showed that the multifractal formalism holds for μ_M. This means that the spectrum $f_M(\alpha)$ equals the Legendre transform of the singularity exponents $\tau_M(q)$ which are given by

$$\sum_{i=1}^{M} \big(p_i^{(M)}\big)^q \lambda_i^{\tau_M(q)} = 1. \tag{9}$$

In particular, denoting the set where μ_M attains the Hölder exponent α by $K_\alpha^{(M)}$, we have

$$d_{\mathrm{HD}}\big(K_\alpha^{(M)}\big) = f_M(\alpha) = \tau_M(q) - q\tau_M'(q)$$

for $\alpha = -\tau_M'(q)$. (A short proof and the spectrum obtained by box counting are given in [R].) The separation condition we require is therefore the following one.

Definition 6 (Finite Separation Condition). A family $\{w_i\}_{i\in\mathbb{N}}$ and its self-similar measure μ is called *finitely separated* if for all $M \in \mathbb{N}$ there is a bounded open set O_M such that the sets $w_i(\overline{O_M})$ $(i = 1, \ldots, M)$ are mutually disjoint subsets of $\overline{O_M}$; i.e., $(w_1, \ldots, w_M)$ satisfies the DOSC.

Of course, a family $\{w_i\}_{i\in\mathbb{N}}$ satisfying the DOSC is finitely separated, but not vice versa. Even connected supports K are possible, provided the copies $w_i(K)$ accumulate "nicely" at the boundary of a cell.

Proposition 7. *Let $\{w_i\}_{i\in\mathbb{N}}$ be a family of similarities on the line, i.e., $w_i(x) = \theta_i\lambda_i x + u_i$ with $\theta_i \in \{-1, 1\}$. Assume that the OSC holds with some open interval $O = \,]u, v[$. If $\mathbb{N}^c$ contains both boundary points of O then $\{w_i\}_{i\in\mathbb{N}}$ is finitely separated. The same is true if all θ_i are equal and at least one boundary point of O lies in $\mathbb{N}^c$.*

Proof. The basic idea is to shrink O to become a smaller open set O_M. First, consider the case $u \in \mathbb{N}^c$. We show that u cannot lie in any $V_i = w_i([u, v])$ $(i \in \mathbb{N})$. Assume the contrary. This implies that $V_i = [u, u + \lambda_i(v - u)]$. By definition of $\mathbb{N}^c$ there is $k \in \mathbb{N}$ such that $|u - a_k| \le \lambda_i(v - u)/2$. Since $a_k \in V_k \subset [u, v]$ the fixpoint a_k must lie in the interior of V_i which contradicts the OSC. Now take any $M \in \mathbb{N}$. Since no V_i contains u there is $\varepsilon = \varepsilon(M)$ such that $V_i \subset O_M := \,]u + \varepsilon, v[\, \subset O$ for $i = 1, \ldots, M$. Since $w_i(O_M) \subset V_i$, O_M satisfies the OSC for $w_1, \ldots, w_M$. Finally, the OSC for O and the equal signs of the θ_i imply that the sets

$w_i(\overline{O_M})$ $(i = 1, \dots, M)$ are mutually disjoint. This proves the claim. In the case $v \in \mathbb{N}^c$ a similar argument applies to some $O_M :=]u, v - \varepsilon[$. If u and v are both contained in $\mathbb{N}^c$ then we can choose $O_M :=]u + \varepsilon, v - \varepsilon[$ and the signs of the θ_i are not important. ■

6. Lower Bound

Now we proceed to estimate $d_{\mathrm{HD}}(K_\alpha)$. We recall some details of the proof of (9) which are useful here. Fix q and set $\bar{p}_k := (p_k^{(M)})^q (\lambda_k)^{\tau_M(q)}$. The main idea is to focus on the words $\mathbf{i}$ which produce the desired Hölder exponent $\alpha = \alpha_M(q)$, where according to (9)

$$\alpha_M(q) := \frac{\sum_{i=1}^M \log\left(p_i^{(M)}\right)\left(p_i^{(M)}\right)^q \lambda_i^{\tau_M(q)}}{\sum_{i=1}^M \log(\lambda_i)\left(p_i^{(M)}\right)^q \lambda_i^{\tau_M(q)}} = \frac{\sum_{i=1}^M \log\left(p_i^{(M)}\right)\bar{p}_i}{\sum_{i=1}^M \log(\lambda_i)\bar{p}_i} = -\tau_M'(q).$$

As in (5), let $\hat{\mu}_{q,M}$ be the product measure on the product space $I_M := \{1, \dots, M\}^{\mathbb{N}}$ corresponding to the factor measures $\bar{p}_1\delta_1 + \cdots + \bar{p}_M\delta_M$. The tools to observe the Hölder exponent are the random variables $X_n := \log p_{i_n}^{(M)}$ and $Y_n := \log \lambda_{i_n}$ which are i.i.d., due to the product structure (5) of $\hat{\mu}_{q,M}$. Consider the set $\hat{K}_q^{(M)}$ of all words $\mathbf{i} \in \{1, \dots, M\}^{\mathbb{N}}$ such that simultaneously

$$\frac{1}{n}\log p_{\mathbf{i}|n}^{(M)} = \frac{1}{n}\sum_{i=1}^n X_k \to E[X_n] = \sum_{i=1}^M \log\left(p_i^{(M)}\right)\bar{p}_k \tag{10}$$

and $(1/n)\log \lambda_{\mathbf{i}|n} \to E[Y_n] = \sum_{i=1}^M \log(\lambda_i)\bar{p}_k$. Obviously, $K_\alpha^{(M)} = \pi(\hat{K}_q^{(M)})$. By the Law of Large Numbers (10) is a $\hat{\mu}_{q,M}$-almost sure event. Hence, $\hat{\mu}_{q,M}(\hat{K}_q^{(M)}) = 1$ and Frostman's lemma [F2, 4.9] yields $d_{\mathrm{HD}}(K_\alpha^{(M)})$. For μ we conclude that

$$\begin{aligned}\frac{\log \mu(V_{\mathbf{i}|n})}{\log|V_{\mathbf{i}|n}|} = \frac{\log p_{\mathbf{i}|n}}{\log \lambda_{\mathbf{i}|n}} &\to \beta_M(q) \\ &:= \alpha_M(q) + \frac{\log(N_M)}{\sum_{i=1}^M \log(\lambda_i)\left(p_i^{(M)}\right)^q \lambda_i^{\tau_M(q)}} \qquad (n \to \infty)\end{aligned} \tag{11}$$

for every $\mathbf{i} \in \hat{K}_q^{(M)}$. Hence $K_\alpha^{(M)} \subset K_{\beta_M(q)}$ and

$$d_{\mathrm{HD}}\left(K_{\beta_M(q)}\right) \geq \tau_M(q) - q\tau_M'(q). \tag{12}$$

It remains to consider the limit $M \to \infty$. Let

$$\mathbb{D} := \{q \in \mathbb{R} \colon \tau \text{ is finite in a neighbourhood of } q\}.$$

LEMMA 8. *For all q with finite $\tau(q)$ we have $\tau_M(q) \to \tau(q)$ as $M \to \infty$.*

Proof. Assume that there is $\gamma > 0$ and a sequence of integers $(M_n)_{n \in \mathbb{N}}$ such that $\tau_{M_n}(q) \geq \tau(q) + \gamma$ for all n. Then, using monotonicity, (3), $N_M \to 1$, and (9) we obtain

$$1 > \lim_{n \to \infty} \sum_{i=1}^{M_n} p_i^q \lambda_i^{\tau(q)+\gamma} \geq \lim_{n \to \infty} \left(N_{M_n}\right)^q \sum_{i=1}^{M_n} \left(p_i^{(M_n)}\right)^q \lambda_i^{\tau_{M_n}(q)} = 1,$$

a contradiction. A similar argument shows that $\liminf_{M \to \infty} \tau_M(q) \geq \tau(q)$. ∎

LEMMA 9. *Let μ be an infinite self-similar measure. Assume that there are numbers r, R such that $-\infty < \log r \leq (1/n)\log p_n \leq \log R < 0$ $\forall n$. Then, we distinguish two cases:*

(a) (*left-sided*) *If* $\lim_{n \to \infty} 1/n \log \lambda_n = 0$, *then* $\mathbb{D} = (0, \infty)$.

(b) (*right-sided*) *If* $\lim_{n \to \infty} 1/n \log \lambda_n = -\infty$, *then* $\mathbb{D} = (-\infty, 1)$.

In both cases,

$$\lim_{M \to \infty} \beta_M(q) = \lim_{M \to \infty} \alpha_M(q) = \alpha(q) := \frac{\sum_{i \in \mathbb{N}} \log(p_i) p_i^q \lambda_i^{\tau(q)}}{\sum_{i \in \mathbb{N}} \log(\lambda_i) p_i^q \lambda_i^{\tau(q)}}, \quad (13)$$

locally uniform in $\mathbb{D}$. In particular, $\tau'(q) = -\alpha(q)$ for $q \in \mathbb{D}$. Moreover, all statements remain true if the sequences $\{p_i\}_{i \in \mathbb{N}}$ and $\{\lambda_i\}_{i \in \mathbb{N}}$ as well as the assumptions of (a) and (b) are exchanged. In particular, we are then in the left-sided case $\mathbb{D} = (0, \infty)$ if $1/n \log p_n \to -\infty$.

Remark. The case (a) is called *left-sided*, since $\tau(q)$ is not defined for $q < 0$ and its Legendre transform $f(\alpha)$ has only an increasing (= left) part. If $(1/n)\log p_n$ and $(1/n)\log \lambda_n$ are both bounded away from $-\infty$ and 0 then $\mathbb{D} = \mathbb{R}$. In this case, a sufficient condition for (13) is that $(1/n)\log p_n$ and $(1/n)\log \lambda_n$ both converge with limits different from 0 and $-\infty$. Then, f has the usual $\cap$-shape. Finally, it is easy to construct examples which have $\mathbb{D} = (0, 1)$.

Proof. First, note that $\beta_M(q)$ and $\alpha_M(q)$ converge simultaneously with the same (uniform) limit as $M \to \infty$. Indeed, $N_M \to 1$ and a mean value argument gives

$$\left| \sum_{i=1}^{M} \log(\lambda_i) \left(p_i^{(M)}\right)^q \lambda_i^{\tau_M(q)} \right| \geq -\log(\max\{\lambda_i : i \in \mathbb{N}\}) > 0. \quad (14)$$

Let us compute $\mathbb{D}$. Due to monotonicity and convexity of the exponential $\mathbb{D}$ must be an interval and τ is decreasing and convex, hence continuous in $\mathbb{D}$. We show that $(0,1) \subset \mathbb{D}$ without further assumption. Let $q_1 \in (0,1)$. Due to the OSC $\sum_{i \in \mathbb{N}} p_i^{q_1} \lambda_i^d < \sum_{i \in \mathbb{N}} \lambda_i^d \le 1 < \sum_{i \in \mathbb{N}} p_i^{q_1}$. Also, $h(x) := \sum_{i \in \mathbb{N}} p_i^{q_1} \lambda_i^x \le \sum_{i \in \mathbb{N}} p_i^{q_1}$ converges uniformly for $x \in [0,d]$. Hence, h is continuous. Since it is strictly decreasing there must be a (unique) solution $\tau(q_1)$ of (3) which must lie in $]0,d[$.

Now, assume that $(1/n)\log \lambda_n \to 0$. Let $q_2 > 1$. There is $x_0 < 0$ such that $p_1^{q_2} \lambda_1^{x_0} > 1$. Choose $s \in (0,1)$ such that $\gamma := R^{q_1} s^{x_0} < 1$. By assumption there is $i_0(s)$ such that $\lambda_i > s^i$ for $i \ge i_0$. So, $\sum_{i \in \mathbb{N}} p_i^{q_2} \lambda_i^{x_0}$ converges to a number > 1. Since $\sum_{i \in \mathbb{N}} p_i^{q_2} < 1$, there must be a solution $\tau(q_2)$ of (3) which must lie in $[x_0, 0]$. Due to monotonicity, (3) has a solution $\tau(q)$ which must lie in $[x_0, d]$ for all $q \in [q_1, q_2]$. Similar estimates show that (3) has no solution for $q < 0$. Hence, $\mathbb{D} = (0, \infty)$. For further use note that for every $q \in [q_1, q_2]$ and every $i \ge i_0(s)$,

$$p_i^q \lambda_i^{\tau(q)} \le p_i^{q_1} \lambda_i^{x_0} \le \left(R^{q_1} s^{x_0}\right)^i = \gamma^i.$$

In order to show that $\beta_M(q) \to \alpha(q)$ it is enough to prove that

$$\sum_{i=1}^{M} \log(x_i)\left(p_i^{(M)}\right)^q \lambda_i^{\tau_M(q)} - \sum_{i \in \mathbb{N}} \log(x_i) p_i^q \lambda_i^{\tau(q)} \to 0 \qquad (M \to \infty) \quad (15)$$

for $\{x_i\}_{i \in \mathbb{N}} = \{p_i\}_{i \in \mathbb{N}}$ and for $\{x_i\}_{i \in \mathbb{N}} = \{\lambda_i\}_{i \in \mathbb{N}}$. We split the difference (15) into two parts $A + B$. A mean value argument yields

$$A := \sum_{i=1}^{M} \log(1/x_i)\bar{p}_i - \sum_{i=1}^{M} \log(1/x_i) p_i^q \lambda_i^{\tau(q)} = \log(1/\xi_M) \sum_{i=1}^{M} \bar{p}_i - p_i^q \lambda_i^{\tau(q)}$$

with $\min\{x_i\colon 1 \le i \le M\} \le \xi_M \le \max\{x_i\colon 1 \le i \le M\}$. Since $\sum_{i=1}^{M} \bar{p}_i = 1$, we have

$$A = \log(1/\xi_M) \sum_{i=M+1}^{\infty} p_i^q \lambda_i^{\tau(q)}.$$

For both choices of $\{x_i\}_{i \in \mathbb{N}}$ we must have $\xi_M \ge r^M$ for large M. Hence,

$$0 \le A \le M \log(1/r) \cdot \sum_{i=M+1}^{\infty} \gamma^i \to 0 \qquad (M \to \infty).$$

The rest of the difference (15) can be treated with similar arguments:

$$B := \sum_{i=M+1}^{\infty} \log(1/x_i) p_i^q \lambda_i^{\tau(q)} \le \sum_{i=M+1}^{\infty} i \log(1/r) \gamma^i \to 0 \qquad (M \to \infty).$$

This proves that (15) holds uniformly for $q \in [q_1, q_2]$ which implies (13). In fact, using (14) and (11) one obtains that $\tau'_M(q)$ converges uniformly to $-\alpha(q)$ in every compact subset of $\mathbb{D}$.

Next, assume $(1/n)\log \lambda_n \to -\infty$. Again, straightforward estimates yield $\mathbb{D}$. The locally uniform convergence of (15) can be proven in a similar manner as above, but with one adoption. In this case the "strong" sequence is $\{\lambda_i\}_{i \in \mathbb{N}}$, the rate of convergence of which is not known. But we may assume that it is monotonously decreasing, since reordering changes neither the geometry nor the value of positive sums. Fix $q_3 < 0$ and choose $S \in (0,1)$ such that $\gamma := r^{q_3} S^{\tau(q_1)/2} < 1$. Recall that $\tau(q_1) > 0$. By assumption, $\lambda_i \leq S^i$ for $i \geq i_1(S)$. For both choices of $\{x_i\}_{i \in \mathbb{N}}$, the estimate

$$\log(1/\xi_M) \leq \log(1/\lambda_M) \leq \log(1/\lambda_i) \leq \lambda_i^{-\tau(q_1)/2}$$

holds for $M \geq M_0$, M_0 depending on q_1 and on $\{p_i\}_{i \in \mathbb{N}}$, and for $i \geq M$. For $q \in [q_3, q_1]$ we obtain

$$\begin{aligned}\log(1/\xi_M) \sum_{i=M+1}^{\infty} p_i^q \lambda_i^{\tau(q)} &\leq \sum_{i=M+1}^{\infty} p_i^{q_3} \lambda_i^{\tau(q_1) - \tau(q_1)/2} \\ &\leq \sum_{i=M+1}^{\infty} \left(r^{q_3} S^{\tau(q_1)/2}\right)^i = \sum_{i=M+1}^{\infty} \gamma^i,\end{aligned}$$

provided $M \geq \max(M_0, i_1)$. It is now easy to complete the proof of (b).

Finally, exchanging the sequences $\{p_i\}_{i \in \mathbb{N}}$ and $\{\lambda_i\}_{i \in \mathbb{N}}$ means to exchange τ and q, hence to switch the left- and right-sided case. Indeed, the additional assumption on the asymptotic behaviour of the sequences were only used to prove (15) which is symmetric in p and λ, so to say. The only break of symmetry is $\Sigma_{i \in \mathbb{N}} p_i = 1$ while $\Sigma_{i \in \mathbb{N}} \lambda_i^d \leq 1$. So, precisely speaking, τ becomes $q \cdot d$ and $\{\lambda_i\}_{i \in \mathbb{N}}$ has to be "normalized" by multiplication with a suitable constant. Then, the same argumentation applies to give the obvious results. ∎

7. Main Result

THEOREM 10 (Multifractal Formalism). *Let μ be a finitely separated infinite self-similar measure. Assume that there are numbers r, R such that $-\infty < \log r \leq (1/n)\log p_n \leq \log R < 0$ $\forall n$. Furthermore, assume that $\lim_{n \to \infty}(1/n)\log \lambda_n$ exists and is either 0 or $-\infty$. Then, for $\alpha = \alpha(q) = -\tau'(q)$ $(q \in \mathbb{D})$,*

$$d_{\mathrm{HD}}(K_\alpha) = f(\alpha) := \inf\{\tau(q) + q\alpha : q \in \mathbb{R}\}.$$

The same holds if there are numbers s, S *such that* $-\infty < \log s \leq (1/n)\log \lambda_n \leq \log S < 0\ \forall n$ *and* $\lim_{n\to\infty}(1/n)\log p_n$ *is either* 0 *or* $-\infty$.

Remark. Together with Proposition 4 we get the Hausdorff dimensions of B_α and C_α at $\alpha = \alpha(q)$ for $q \geq 0$ and $q \leq 0$, respectively.[2]

To give a more precise statement let us agree on the notation $d_{\mathrm{HD}}(\varnothing) = -\infty$.

COROLLARY 11. *Let* μ *be a finitely separated infinite self-similar measure. Assume there are numbers* r, R *such that* $-\infty < \log r \leq (1/n)\log p_n \leq \log R < 0\ \forall n$. *Then,*

(a) **left-sided.** *If* $\lim_{n\to\infty}(1/n)\log \lambda_n = 0$ *then*

$$d_{\mathrm{HD}}(K_\alpha) = \begin{cases} \tau(0) = d_{\mathrm{HD}}(K) & \text{if } \alpha \geq \alpha_0 \\ \tau(q) - q\tau'(q) & \text{if } \alpha_0 > \alpha = \alpha(q) > \alpha(\infty) \\ D_\infty & \text{if } \alpha = \alpha(\infty) \\ -\infty & \text{if } \alpha < \alpha(\infty), \end{cases}$$

where $\alpha_0 := \alpha(0+) = -\tau'(0+)$, $\alpha(\infty) := \lim_{q\to\infty}\alpha(q) = \inf\{\alpha(q) \colon q \in \mathbb{R}\}$ *and* D_∞ *is defined through* $\Sigma'\lambda_i^{D_\infty} = 1$, *the sum* Σ' *running over all i with* $\log p_i/\log \lambda_i = \alpha(\infty)$. *Note that* α_0 *may be infinite.*

(b) **right-sided.** *If* $\lim_{n\to\infty}(1/n)\log \lambda_n = -\infty$ *then*

$$d_{\mathrm{HD}}(K_\alpha) = \begin{cases} \alpha & \text{if } 0 \leq \alpha \leq \alpha_1 \\ \tau(q) - q\tau'(q) & \text{if } \alpha_1 < \alpha = \alpha(q) < \alpha(-\infty) \\ D_{-\infty} & \text{if } \alpha = \alpha(-\infty) \\ -\infty & \text{if } \alpha > \alpha(-\infty), \end{cases}$$

where $\alpha_1 := \alpha(1-) = -\tau'(1-)$, $\alpha(-\infty) := \lim_{q\to-\infty}\alpha(q) = \sup\{\alpha(q)\colon q \in \mathbb{R}\}$ *and* $D_{-\infty}$ *is defined through* $\Sigma''\lambda_i^{D_{-\infty}} = 1$, *the sum* Σ'' *running over all i with* $\log p_i/\log \lambda_i = \alpha(-\infty)$. *Note that* α_1 *may be zero.*

Remark. For any left-sided infinite self-similar measure μ on the line with $\Sigma\lambda_i = 1$ there is a right-sided "reciprocal" measure μ^* obtained by exchanging probabilities p_i and ratios λ_i. Reciprocal measures can be defined in more general situations which will be discussed in a forthcoming paper [MPR].

[2]*Note added in proof.* According to the recent result of Arbeiter and Patzschke [AP] the finitely speration condition can be replaced with the weaker OSC.

Proof. (a) Since $A_i := \log p_i/\log \lambda_i \to \infty$ as $i \to \infty$ it must reach its minimum, say A_k. Obviously, A_k is the smallest Hölder exponent under consideration here (8). It is known [CM, R] that $\alpha_M(\infty) = \min\{\log p_i/\log \lambda_i\colon 1 \le i \le M\} = \inf\{\alpha_M(q)\colon q \in \mathbb{R}\}$ is the minimal Hölder exponent of μ_M. Consequently, $\alpha_M(\infty) = A_k$ for $M \ge k$. This shows that $\alpha(\infty) = A_k$ exists trivially and also that $K_\alpha = \emptyset$ for $\alpha < \alpha(\infty)$.

It is clear that only words $\mathbf{i}$ with $\log p_{i_n}/\log \lambda_{i_n} = A_k$ for all $n \ge n_0(\mathbf{i})$ have a Hölder exponent $\alpha(\infty)$. Moreover, there is an integer $l \ge k$ such that $A_i > \alpha(\infty)$ for all $i > l$. Hence, $K_{\alpha(\infty)} = \bigcup' w_{j_1 \dots j_n}(K^{(l)}_{\alpha_l(\infty)})$ with the union $\bigcup'$ running over all finite sequences $j_1 \dots j_n$ of integers. Recall that $\alpha_l(\infty) = \alpha(\infty)$. In particular, $\bigcup'$ is countable. It is known [R] that $d_{\mathrm{HD}}(K^{(l)}_{\alpha_l(\infty)}) = f_l(\alpha_l(\infty)+\ ) = \gamma_l$, where $\Sigma' \lambda_i^{\gamma_l} = 1$ with Σ' running over all $i \le l$ with $A_i = \alpha_l(\infty)$. Since any set E has the same dimension as $w_i(E)$ and since $d_{\mathrm{HD}}(\bigcup_{\mathbb{N}} E_n) = \sup_{\mathbb{N}} d_{\mathrm{HD}}(E_n)$ for any countable collection of sets E_n [F2], we obtain $d_{\mathrm{HD}}(K_{\alpha(\infty)}) = \gamma_l$. Note also, that we can choose l as large as we wish without changing the result. This shows that the definition of D_∞ makes sense and that $D_\infty = \gamma_l$. In particular, $d_{\mathrm{HD}}(K_{\alpha(\infty)}) = f(\alpha(\infty)+\ ) = \gamma_l$ for every large enough l.

To compute the limit of the left-hand side of (12) we have to use the continuity of f which is guaranteed by Lemma 9. Let $\beta = \alpha(q)$ with $q \in \mathbb{D}$. Recall that $\mathbb{D}$ is an interval and that $\alpha(\cdot)$ is continuous and monotonous. Hence, there are q_1 and q_2 such that $\alpha(q_1) < \beta < \alpha(q_2)$. From Lemma 9 we conclude that there is M_0 such that $\alpha_M(q_1) < \beta < \alpha_M(q_2)$ for all $M \ge M_0$. Due to concavity of f_M and due to (12) $d_{\mathrm{HD}}(K_\beta) \ge \min\{f_M(\alpha_M(q_1)), f_M(\alpha_M(q_2))\}$. Letting $M \to \infty$, Lemmata 8 and 9 give $d_{\mathrm{HD}}(K_\beta) \ge \min\{f(\alpha(q_1)), f(\alpha(q_2))\}$. Letting $(q_2 - q_1) \to 0$ we obtain $d_{\mathrm{HD}}(K_\beta) \ge f(\beta)$ by continuity of f. Proposition 4 gives equality.

The well-known fact $d_{\mathrm{HD}}(\mathrm{supp}(\mu_M)) = \tau_M(0)$ and Corollary 5 imply that $d_{\mathrm{HD}}(K) = \tau(0)$.

If $\alpha_0 = \infty$ there is nothing more to show. Otherwise, i.e., if $\alpha_0 < \infty$, note that $\alpha_M(-\infty) = \lim_{q \to -\infty} \alpha_M(q) = \sup\{\alpha_M(q)\colon q \in \mathbb{R}\} = \max\{\log p_i/\log \lambda_i\colon 1 \le i \le M\} \to \infty$ $(M \to \infty)$. From the concavity of f_M and (12) one obtains $d_{\mathrm{HD}}(K_\beta) \ge l_{M,q}(\alpha)$, where α and β are related through (11), where $l_{M,q}$ is the line through the points $(\alpha_M(q), f_M(\alpha_M(q)))$ and $(\alpha_M(-\infty), 0)$ and where $\alpha_M(q) \le \alpha \le \alpha_M(-\infty)$. Fix $q > 0$ and let $M \to \infty$. The estimate just obtained, Lemma 9 and $\alpha_M(-\infty) \to \infty$ give $d_{\mathrm{HD}}(K_\beta) \ge f(\alpha(q))$ for $\beta > \alpha(q)$. Finally, as $q \downarrow 0$, $\alpha(q) \uparrow \alpha_0$ and $f(\alpha(q)) = \tau(q) - q\tau'(q) \to \tau(0) - 0 \cdot \alpha_0 = d_{\mathrm{HD}}(K)$. Hence, $d_{\mathrm{HD}}(K_\beta) \ge d_{\mathrm{HD}}(K)$ for $\beta \ge \alpha_0$.

(b) The conditions on the asymptotic behaviour of the sequences $\{p_i\}_{i \in \mathbb{N}}$ and $\{\lambda_i\}_{i \in \mathbb{N}}$ are only used to determine $\mathbb{D}$ and to prove $f_M \to f$. According to Lemma 9, exchanging the two sequences means basically to exchange q and τ, hence to exchange left- and right-sided cases. ■

8. EXAMPLES

Three examples of spectra are given which cannot be observed with finite self-similar measures. The first is the left-sided spectrum given in [MEH]. The second one is left-sided as well, showing that infinite Hölder exponents can occur for multiplicative cascades with "binomial" structure. The third example gives a kind of "inverse" left sided spectrum that we called right-sided. It shows that Hölder exponent zero is possible without the presence of atoms. This possibility has already been mentioned in [M].

EXAMPLE 1 (The left-sided spectrum of [MEH]). The infinite multinomial measure considered by Mandelbrot *et al.* [MEH] fits perfectly into our framework. Proposition 7 and Theorem 10 apply with the cell $O :=(0,1)$ showing that the usual multifractal formalism holds. Let

$$w_i(x) := \frac{1}{(i+1)^{\kappa}} + \left(\frac{1}{i^{\kappa}} - \frac{1}{(i+1)^{\kappa}}\right)x, \qquad p_i := \frac{1}{2^i},$$

where $\kappa > 0$ is a parameter. The support of μ is $K = [0,1]$. Some lengthy but straightforward calculations show [MEH, Appendix] that the spectrum is left-sided, in particular $f(\alpha)\uparrow 1$ for $\alpha \to \infty$. It is worth noting that the rate of approach of $f(\alpha)$ towards 1 depends on κ as indicated in Fig. 2. Set

$$\alpha_0 := -\tau'(0+) = \begin{cases} \infty, & \text{if } \kappa \leq 1, \\ \sum_{i\in\mathbb{N}} \log(p_i)\,\lambda_i / \sum_{i\in\mathbb{N}} \log(\lambda_i)\,\lambda_i, & \text{otherwise}. \end{cases}$$

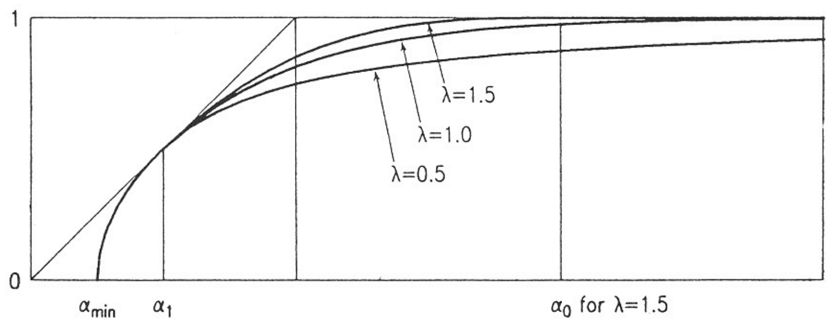

FIG. 2. The spectrum $f(\alpha)$ of the infinite multinomial measure from Example 1 tends towards 1 with a rate that depends on the parameter.

As $\alpha \uparrow \alpha_0$ we have

$$f(\alpha) \simeq \begin{cases} 1 - c(\alpha_0 - \alpha)^{\kappa'} & \text{if } \kappa > 1, \\ 1 - c\exp(-c'\alpha) & \text{if } \kappa = 1, \\ 1 - c\alpha^{\kappa''} & \text{if } \kappa < 1, \end{cases}$$

where $\kappa' = \max\{2, \kappa/(\kappa - 1)\}$ and $\kappa'' = \kappa/(\kappa - 1)$.

Let us add some remarks on the peculiar Hölder exponent $\alpha = \infty$. First, $\mathbb{N}^c = \{0\}$. The Hölder exponent is obviously infinite there and at all the iterated images, e.g., at the boundary points $1/i^\kappa$ of the intervals appearing in the construction. These points are the one with multiple addresses and with "infinite addresses" from E. Furthermore, the measure is supported by $K = [0, 1]$. Disregarding the countable number of points with multiple addresses we have

$$\frac{\log \mu(V_{\mathbf{i}|n})}{\log|V_{\mathbf{i}|n}|} = \frac{\log p_{\mathbf{i}|n}}{\log \lambda_{\mathbf{i}|n}} \to \alpha(0) = -\tau'(0+)$$

for (Lebesgue) almost every $x = \pi(\mathbf{i})$, as will be shown. This Hölder exponent $\alpha(0)$ is infinite for $\kappa \leq 1$.

With the "asymptotic" Hölder exponent $\alpha(0) = \infty$ also unbounded "pre-asymptotic" Hölder exponents are present. Indeed, the arguments given in the proof of Theorem 10 and in Fig. 1 show that this behaviour can be observed also in numerical simulations, since the spectrum is determined by "finite approximations." On the other hand, these "finite approximations" are constituted by points with bounded addresses, i.e., with $i_n \leq M$ for all n, which lie on fractal sets of dimensions D_M strictly less than 1, in particular $\sum_{i=1}^{M} \lambda_i^{D_M} = 1$. Hence, these points form a Lebesgue nullset. Finally, note that

$$\frac{\log \mu(V_{\mathbf{i}|n})}{\log|V_{\mathbf{i}|n}|} = \frac{\log p_{\mathbf{i}|n}}{\log \lambda_{\mathbf{i}|n}} \to \alpha_1 := \tau'(1) = \frac{\sum_{i \in \mathbb{N}} \log(p_i) p_i}{\sum_{i \in \mathbb{N}} \log(\lambda_i) p_i}$$

for μ almost every $x = \pi(\mathbf{i})$.

Proof. The argument is essentially the same as (10) with "$M = \infty$." Pick any $q \geq 0$ and let $\hat{\mu}_q$ be the product measure on I corresponding to the probabilities $\bar{p}_i := p_i^q \lambda_i^{\tau(q)}$. Then, $\mu_q := \hat{\mu}_q(\pi^{-1}(\cdot))$ satisfies (1) with $\bar{p}_i$

instead of p_i. Consider the i.i.d. random variables $X_n := \log p_{i_n}$ and $Y_n := \log \lambda_{i_n}$. Then,

$$\frac{\log p_{\mathbf{i}|n}}{\log \lambda_{\mathbf{i}|n}} = \frac{(1/n)\log(p_{\mathbf{i}|n})}{(1/n)\log(\lambda_{\mathbf{i}|n})} \to \frac{E[X_n]}{E[Y_n]}$$

$$= \frac{\sum_{i\in\mathbb{N}} \log(p_i)\, p_i^q \lambda_i^{\tau(q)}}{\sum_{i\in\mathbb{N}} \log(\lambda_i)\, p_i^q \lambda_i^{\tau(q)}} \qquad (n\to\infty),$$

$\hat{\mu}_q$ almost surely. Note that the denominator $E[Y_n]$ is always finite and that the LLN holds also if the numerator $E[X_n]$ is infinite. Since only countably many points have multiple addresses this translates into the desired μ_q-almost sure statement. Finally, $\mu_1 = \mu$ is obvious. For $q = 0$ we have $\bar{p}_i = \lambda_i$ and the Lebesgue measure restricted to $[0, 1]$ satisfies (1) with $p_i = \lambda_i$. By uniqueness it must be μ_0. ■

EXAMPLE 2 (A left-sided spectrum with binomial structure). An example of a self-similar measure with geometrically decreasing ratios λ_i and left-sided spectrum is given by

$$w_n(x) = \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}x, \qquad p_n := \rho\cdot\frac{1}{2^{2^{n/\kappa}}},$$

where $\kappa > 0$ is a parameter and ρ is the obvious normalization constant. Similar calculations as in [MEH] give exactly the same behaviour of $f(\alpha)$ as in Example 1. This example has been included for its transparent geometry. In particular, it shows the binomial structure as found with the binomial measure invariant under $w_1(x) = x/2$ and $w_2(x) = x/2 + 1/2$, however, with a different mass distribution.

EXAMPLE 3 (A right-sided spectrum). Exchange the ratios λ_i and the probabilities p_i of Example 1. Again, similar calculations as in [MEH] give the different rates of convergence of $f(\alpha) \to \alpha$ as $(\alpha \downarrow \alpha_1)$, where $\alpha_1 := -\tau'(1+)$. In particular, if $\kappa > 1$ then α_1 is positive; otherwise it is zero. As $\alpha \downarrow \alpha_1$ the difference $f(\alpha) - \alpha$ behaves like a power of $\alpha - \alpha_1$ for $\kappa > 1$ (and vanishes for $\alpha < \alpha_1$), like an exponential of α for $\kappa = 1$, and like a power of α for $\kappa < 1$.

But it is even simpler to argue in the following way: Let us denote the singularity exponents and the spectrum of Example 1 by τ^* and f^*, respectively. Then τ is simply the inverse function of τ^*. In other words, $\tau = q^*$ and $q = \tau^*$. Applying the Legendre transform shows that the spectrum $f(\alpha)$ is related to $f^*(\alpha)$ by

$$f(\alpha) = \alpha f^*(1/\alpha).$$

This relation is generally valid for reciprocal multifractal measures, which is proven in a forthcoming paper [MPR]. Here, it shows that the different rates of $f(\alpha) \to \alpha$ are directly related to the one of $f^* \to 1$. Compare Fig. 1. In conclusion, with this measure one finds the Hölder exponent zero without presence of atoms.

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