

A class of micropulses and antipersistent fractional Brownian motion

R. Cioczek-Georges, B.B. Mandelbrot*

Department of Mathematics, Yale University, New Haven, CT 06520-8283, USA

Received August 1994; revised May 1995

Abstract

We begin with stochastic processes obtained as sums of “up-and-down” pulses with random moments of birth τ and random lifetime w determined by a Poisson random measure. When the pulse amplitude $\varepsilon \rightarrow 0$, while the pulse density δ increases to infinity, one obtains a process of “fractal sum of micropulses.” A CLT style argument shows convergence in the sense of finite dimensional distributions to a Gaussian process with negatively correlated increments. In the most interesting case the limit is fractional Brownian motion (FBM), a self-affine process with the scaling constant $0 < H < \frac{1}{2}$. The construction is extended to the multidimensional FBM field as well as to micropulses of more complicated shape.

Keywords: Fractal sums of pulses; Fractal sums of micropulses; Fractional Brownian motion; Poisson random measure; Self-similarity; Self-affinity; Stationarity of increments

1. Introduction

The paper establishes two facts:

A. Fractional Brownian Motion (FBM) with scaling parameter $0 < H < \frac{1}{2}$ can be obtained as a sum of infinitesimal contributions, called “micropulses,” generated by a Poisson random measure.

B. A micropulse representation is also possible for a wider class of random functions.

The micropulses construction was first mentioned, without detail, in Mandelbrot (1984). Processes obtained as sum of noninfinitesimal pulses are investigated in Mandelbrot (1995a,b) Cioczek-Georges et al. (1995), and Cioczek-Georges and Mandelbrot (1995b). In this paper, a general discussion concerning fact B is presented in Section 2, while the remaining sections are devoted to FBM. Other ways of implementing the idea of micropulses are discussed in Cioczek-Georges and Mandelbrot (1995a).

* Corresponding author. E-mail: mathmath.yale.edu.

In Mandelbrot and VanNess (1968), a process $\{B_H(t), t \geq 0\}$ is called FBM with the scaling parameter $0 < H < 1$ if its multidimensional distributions are Gaussian with zero mean, $B_H(0) = 0$ a.s. and the covariance function is

$$r(s, t) := EB_H(s)B_H(t) = \frac{1}{2}\{|s|^{2H} + |t|^{2H} - |s - t|^{2H}\}EB_H^2(1). \quad (1.1)$$

The process is called *standard* FBM if $EB_H^2(1) = 1$. As with Brownian motion, there exists a version of FBM with a.s. continuous sample paths.

We begin in Section 3 by obtaining FBM as a sum of infinitesimal simple “up-and-down” pulses determined by a Poisson measure as follows. Jumps of size ε appear at uniformly distributed random instants τ , $\tau \in \mathbb{R}$, of density $\delta = \varepsilon^{-2}/2$, and each is followed by a “canceling echo” ($-\varepsilon$) at time $\tau + w$, where $w > 0$ is a generalized random variable with $Pr(w > w_0) \sim w_0^{-\theta}$, $0 < \theta < 1$. We define a process $\{X_\varepsilon(t), t \geq 0\}$ as the sum of heights of pulses alive at time t minus the sum of heights of pulses alive at time 0. Using a CLT-like argument we prove that, when $\varepsilon \rightarrow 0$ the finite dimensional distributions of $\{X_\varepsilon(t), t \geq 0\}$ approach that of FBM with $0 < H < \frac{1}{2}$. We show that FBM also arises when amplitude of an “up-and-down” micropulse is of the form εA , where A is a random variable with finite second moment (Theorem 3.2).

Section 4 extends this construction to multitemporal isotropic FBM with $0 < H < \frac{1}{2}$. Standard FBM with the parameter $t \in \mathbb{R}^d$ is defined in the same way as FBM with one-dimensional t , except that in (1.1) the expressions $|s|$, $|t|$, $|s - t|$ must be understood as Euclidean norms in \mathbb{R}^d , and $B_H(1)$ is the value of the process at any point 1 on the unit sphere S^{d-1} of \mathbb{R}^d .

Finally, in Section 5, we consider two successive formalizations of the notion of micropulses. First, the amplitude is taken to be of the form $\varepsilon A f$, where the compactly supported deterministic function f need not be the indicator of an interval. Second, the amplitude is itself taken to be determined by a random process, which could be FBM of exponent $\eta > H$. This approach to FBM is new, but remains in the spirit of shot-noise processes.

2. Informal background, study of nonfractal sums of micropulses and heuristics concerning the representation of FBM

This section touches a variety of topics, and is written to be understood by readers whose interests do not include the fine mathematical detail.

Pulses and micropulses. In the original context of Mandelbrot (1984), the micropulses construction was motivated by an effort to reinterpret the notion of moving average; eventually the motivation broadened. A Gaussian moving average process is of the form $X(t) = \int dY(\tau)K(t - \tau)$, where $dY(\tau)$ is a Gaussian variable called “innovation at time τ ,” and the kernel $K(t - \tau)$ describes the *linear* decay of the innovation $dY(\tau)$ during the increment $t - \tau$ of *continuously varying* time. However, when dealing with atomic (i.e., discrete) phenomena, linear continuous-time decay is a nonphysical approximation. Thus, radioactive decay is exponential, but individual atoms do NOT decay. They remain unchanged from birth to death, and their lifetime is exponentially distributed. But let atoms be very numerous and very small, denote by $dY(\tau)$ the

number of atoms in an “input” added during the time increment $d\tau$ and follow this “input’s” decay in time. This decay will be usefully approximated by an exponential kernel $K(t - \tau)$. Such is really the reason why moving averages are useful in this type of physical situation. Building upon this situation, Mandelbrot (1984) investigated an alternative form of innovations: during their lifetime, they remain constant, equal to one atom, and their lifetime is a random variable W with a given probability distribution $F(w) = \Pr(W \leq w)$ and the corresponding tail distribution $P(w) = \Pr(W > w) = 1 - F(w)$.

The first task, then, is to determine under what conditions on the covariance one can interpret $\{X(t)\}$ as an integral of discrete innovations of this special form, or of microinnovations that remain equal to an “infinitesimal” $\varepsilon > 0$. In the latter case, each microinnovation would contribute a “self-canceling *rectangular micropulse*” that begins at time τ with a rise of $+\varepsilon$ and ends at time $\tau + w$ with a fall of $-\varepsilon$, a canceling echo. The motivation for speaking of *micropulses* is that the results relative to $\varepsilon > 0$ are always meant to be eventually replaced by a limit. In the asymptotic (“thermodynamical”) limit, $\varepsilon \rightarrow 0$, the number of micropulses per unit time $\rightarrow \infty$, the distribution of W may also change, and the sum of the innovations in the time increment $d\tau$ becomes an asymptotically Gaussian $dY(\tau)$. This leads back to the usual moving average formula $X(t) = \int dY(\tau)K(t - \tau)$. We shall momentarily show that, if $X(t)$ has a micropulse representation, its increments over successive equal time spans are such that their spectral density at 0 is itself 0. Therefore, moving averages that do not satisfy this necessary property cannot be represented as sums of rectangular micropulses.

A vanishing spectral density is a restrictive property, but is broad enough to be interesting. In particular, it holds for FBM with $H < \frac{1}{2}$. However, in order to obtain FBM, it is necessary to take $P(w) = w^{-\theta}$, $\theta > 0$. This expression satisfies $P(0) = \infty$ and must be handled with care. As an introduction to its study, we now discuss the case when $P(0) = 1$.

Non-fractal sums constructed with rectangular micropulses for which $P(0) = 1$. To the positive pulses that have been defined, we could add negative pulses that start with a fall $-\varepsilon$ and end with the rise $+\varepsilon$, but this extension would only change the scale parameter in the distribution of $X(t)$; therefore, it can be omitted with no loss of generality.

To evaluate the variance $EX^2(t)$ if $X(0) = 0$ and t is fixed, assume that the number of positive pulses originating in $(0, t]$ has density δ and introduce the function $\tilde{P}(t) := \int_0^t P(\tau) d\tau$; this function $\tilde{P}(t)$ is always cap convex (or “concave”) like \sqrt{t} . The pulses that make a positive contribution to $X(t) - X(0)$ are those which rise for $\tau > 0$ and do *not* cancel out before time t . Averaging over τ , we see that their expected number takes the form

$$\int_0^t \delta P(t - \tau) d\tau = \delta \tilde{P}(t).$$

The same result holds for the pulses originating before time 0 and canceling out at $\tau + w \in (0, t]$. Thus, the total $X(t)$ is the sum of $2\delta\tilde{P}(t)$ jumps of sign $\pm\varepsilon$. As $\varepsilon \rightarrow 0$

and $\delta \rightarrow \infty$, we want this sum to converge. If $2\delta\epsilon^2 = 1$, it converges to a Gaussian random variable of zero mean and variance

$$EX^2(t) = 2\delta\epsilon^2\tilde{P}(t) = \tilde{P}(t).$$

By the very nature of the pulses, the increments of $\{X(t)\}$ are stationary and the increments over nonoverlapping time spans are negatively correlated.

To investigate $\{X(t)\}$ further, consider the increments of $\{X(t)\}$ over fixed length time intervals of equal duration $s > 0$. Their correlation

$$C(k) = E\{[X(t_0 + (k+1)s) - X(t_0 + ks)][X(t_0 + s) - X(t_0)]\}$$

is independent from t_0 because the increments are stationary. For $k = 0$, $C(0) = \tilde{P}(s)$. For $k \neq 0$, we find

$$C(k) = \frac{1}{2}[\tilde{P}(|k|+1)s - 2\tilde{P}(|k|s) + \tilde{P}(|k|-1)s].$$

Three properties are worth noting. (A) The cap convexity of \tilde{P} confirms that $C(k) < 0$ for $k \neq 0$, as stated above. (B) As $k \rightarrow \infty$, $C(k) \rightarrow 0$ monotonically. (C) $C(k)$ satisfies $\sum_{-\infty}^{\infty} C(k) = 0$, hence the spectral density S of $\{X(t)\}$ satisfies $S(0) = 0$.

Heuristic study of fractal sums of rectangular micropulses that yield FBM. The last-written property, $\sum_{-\infty}^{\infty} C(k) = 0$, is compatible with asymptotic or exact self-affinity. In the case of greatest interest, we want $\{X(t)\}$ to be FBM, which satisfies $EX^2(t) = t^{2H}$. The fact that $\{X(t)\}$ must be negatively correlated (antipersistent) will immediately restrict the discussion to FBM with $0 < H < \frac{1}{2}$. (However, the case $\frac{1}{2} < H < 1$ can be handled by generalizing the shape of pulses – cf. Cioczek-Georges and Mandelbrot (1995a).)

In the present informal argument, let us assume that the relation between $EX^2(t)$ and $P(w)$ continues to hold even if $P(0) \neq 1$. This assumption yields $P(w) = 2Hw^{2H-1}$, which is indeed a possible asymptotic behavior for the tail probability when $0 < H < \frac{1}{2}$. The corresponding density is $-P'(w) \sim w^{2H-2}$ for $1 \ll w$. Replacing the jump height ϵ by $t^{-H}\tilde{\epsilon}$ and keeping $2\delta\tilde{\epsilon}^2 = 1$, we find that, if a Gaussian limit does indeed exist, it is independent of t . Uniqueness of the self-affine Gaussian process with stationary increments causes this limit to be FBM. However, this heuristic argument does not suffice, and the proof advanced in Section 3 will use a slightly different approach.

An alternative: reinforcing micropulses, in which the final fall is replaced by a second rise of equal value. To make this generalization possible, it becomes necessary to allow negative pulses, yielding a symmetric distribution of $X(t)$. The correlation for $k \neq 0$ continues to take the same expression as for the canceling pulses, except that now it is positive. It follows that $0 < \sum_{-\infty}^{\infty} C(k) < \infty$, which expresses the fact that dependence is local (short range). Therefore, the reinforcing micropulses are incompatible with asymptotic or exact self-affinity. In particular, positively correlated FBM's, which correspond to $\frac{1}{2} < H < 1$, cannot be obtained as sums of reinforcing micropulses.

An argument more complicated than for canceling echoes gives the following expression for the second moment of the limit $X(t)$ for reinforcing pulses. The three

terms correspond, respectively, to pulses originating before time 0 with the second rise in $(0, t]$, pulses originating in $(0, t]$ with the second rise also in $(0, t]$, and pulses originating in $(0, t]$ with the second rise after t .

$$\begin{aligned} EX^2(t) &= \int_{-\infty}^0 Pr[-\tau < W < t - \tau] d\tau + 4 \int_0^t Pr[W \leq t - \tau] d\tau \\ &\quad + \int_0^t Pr[W > t - \tau] d\tau \\ &= \tilde{P}(t) + 4 \int_0^t (1 - P(t - \tau)) d\tau + \tilde{P}(t) = 4t - 2\tilde{P}(t). \end{aligned}$$

This expression confirms that it is impossible to find a distribution for W such that $EX^2(t) \sim t^{2H}$ for $H > \frac{1}{2}$. Moreover, let us take the density of W to be proportional to w^{2H-2} , more precisely to be $-P'(w) = cw^{2H-2}/M^{2H-1}$ for $0 < w < M$ and $P'(w) = 0$ otherwise. It follows that $\lim_{M \rightarrow \infty} P(w) = 1$ for $w > 0$ and $\lim_{M \rightarrow \infty} \tilde{P}(t) = t$. The finite dimensional distributions of the process $\{X(t)\}$ depend on M ; when $M \rightarrow \infty$, they approach those of the Brownian motion; in other words, the original H is replaced by $H = \frac{1}{2}$. This defeats, once again, any hope of using discontinuous pulses to obtain FBM with $\frac{1}{2} < H < 1$.

More general micropulses whose amplitude is no longer constant during their lifetime, but varies according to some, possibly random, function. Some such micropulses are discussed in Section 5. For example, the pulses can itself follow an FBM, with time argument restricted to $(0, 1)$, and with parameter η . To obtain FBM with the scale parameter H via micropulses, we shall find that it suffices that $\eta > H$. One obvious complication comes from the fact that nonzero contributions to $X(t) - X(0)$ are no longer solely due to pulses that start before time 0 and end in $(0, t]$ or start in $(0, t]$ and end after t . Pulses which cover the interval $(0, t]$ also contribute. In the limit, each such micropulse contributes to a Gaussian random variable of variance $(t/w)^{2\eta}$. Hence, their total variance equals

$$\int_t^\infty \left(\frac{t}{w}\right)^{2\eta} (w - t) \text{const } w^{2H-2} dw = \text{const } t^{2H} \int_1^\infty (w - 1) w^{-2\eta+2H-2} dw.$$

A necessary condition for this expression to be finite is $\eta > H$.

A companion paper, Cioczek-Georges and Mandelbrot (1995a), considers a different micropulse construction also leading to FBM. While discontinuous (e.g. right-triangular) micropulses again can only produce negatively correlated FBM's, we show that continuous micropulses allow for positively correlated FBM's as well. The continuous pulses investigated in that companion paper are cones (i.e. isosceles triangles) and their generalizations, and the quantity that is made to decrease to zero in the limit is the base angle (not the height). The exponent in the density $-P'(w)$ of the pulse width is related to H by a different formula.

3. One-dimensional FBM of one-dimensional time

Lévy showed long ago (cf. e.g. Itô (1969)) how to construct stable motions with independent increments by adding an infinite number of jumps. In this construction, the number of jumps occurring in an interval of time is governed by a Poisson random measure with mean (intensity) uniform in time and hyperbolic in jump height. Unfortunately, the number of small jumps increases when the index of stability α approaches 2 (the Gaussian case) and, for $\alpha = 2$, no compensating constant can ensure convergence of the series of jump heights. Therefore, one needs a different technique when building FBM or other Gaussian processes. As in the Central Limit Theorem (CLT), one has to add many infinitesimally small summands, in this case jumps with heights decreasing to zero. Using a Poisson random measure, however, to determine the number of jumps is still very much desired. Its intensity measure need not be finite and can be used to specify random moments of pulse births (τ) and pulse lifetimes, or widths (w), for simple “up-and-down” pulses of size ε described above.

When $\varepsilon > 0$ we can introduce the familiar pulse address space $E = \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$. Each pulse is represented by a point (τ, w) in E , where τ and w correspond to time of birth and width (duration) of a pulse, respectively.

Consider a Poisson random measure N_ε defined on Borel sets of E , with the intensity $n_\varepsilon \equiv EN_\varepsilon$ given by

$$dn_\varepsilon(\tau, w) = \frac{1}{2}\varepsilon^{-2}w^{-\theta-1}d\tau dw, \quad (3.1)$$

where $0 < \theta < 1$ and $\varepsilon > 0$ are fixed.

If $\{(\tau_j, w_j), j = 1, 2, \dots\}$ is an enumeration of points of the random measure N_ε , then the process $\{X_\varepsilon(t), t \geq 0\}$ described in the introduction can be formally defined as follows:

$$\begin{aligned} X_\varepsilon(0) &= 0, \\ X_\varepsilon(t) &= \sum_j \varepsilon I[0 < \tau_j < t, t - \tau_j < w_j] - \sum_j \varepsilon I[\tau_j < 0, -\tau_j < w_j < t - \tau_j]. \end{aligned} \quad (3.2)$$

But ε is constant, so we may write

$$X_\varepsilon(t) = \varepsilon[N_\varepsilon(S_{0,t}^+) - N_\varepsilon(S_{0,t}^-)],$$

where

$$\begin{aligned} S_{0,t}^+ &:= \{(\tau, w): 0 < \tau < t, t - \tau < w\}, \\ S_{0,t}^- &:= \{(\tau, w): -\infty < \tau < 0, -\tau < w < t - \tau\}. \end{aligned}$$

Since

$$n_\varepsilon(S_{0,t}^+) = n_\varepsilon(S_{0,t}^-) = \int_0^t \int_{t-\tau}^\infty \frac{1}{2}\varepsilon^{-2}w^{-\theta-1}d\tau dw = \frac{1}{2}\varepsilon^{-2}\theta^{-1}(1-\theta)^{-1}t^{1-\theta} < \infty, \quad (3.3)$$

both $N_\varepsilon(S_{0,t}^+)$ and $N_\varepsilon(S_{0,t}^-)$ are finite a.s. and $X_\varepsilon(t)$ is well-defined for every $t \geq 0$.

Let us calculate the finite dimensional distributions of $\{X_\varepsilon(t), t \geq 0\}$, or equivalently find the characteristic function of a linear combination $\sum_{k=1}^n \xi_k X_\varepsilon(t_k)$, $t_k \geq 0$, $\xi_k \in \mathbb{R}$, $k = 1, 2, \dots, n$, $n \in \mathbb{N}$. First, however, recall that an integral $\int_E g dN$ w.r.t. a Poisson measure N satisfies (cf. e.g. Takács, 1954; Westcott, 1976; or Resnick, 1987)

$$E \exp \left(i \int_E g dN \right) = \exp \left\{ \int_E (e^{ig} - 1) dn \right\} \quad (3.4)$$

and also notice that

$$\sum_k \xi_k N(A_k) = \int_E \sum_k \xi_k I[A_k] dN.$$

The last two facts and (3.3) imply

$$\begin{aligned} E \exp \left\{ i \sum_{k=1}^n \xi_k X_\varepsilon(t_k) \right\} &= E \exp \left\{ i \int_E \varepsilon \sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-]) dN_\varepsilon(\tau, w) \right\} \\ &= \exp \left\{ \int_E [\exp(i\varepsilon \sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-])) - 1] dn_\varepsilon(\tau, w) \right\} \\ &= \exp \left\{ \int_E [\exp(i\varepsilon \sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-])) \right. \\ &\quad \left. - 1 - i\varepsilon \sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-]) \right] \frac{1}{2} \varepsilon^{-2} w^{-\theta-1} d\tau dw \right\}. \end{aligned}$$

Using $|\exp(ix) - 1 - ix| \leq x^2/2$ we can show that the above integrand is bounded by an integrable function uniformly in ε . Hence, applying the Lebesgue Dominated Convergence Theorem we obtain the following limit

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} E \exp \left\{ i \sum_{k=1}^n \xi_k X_\varepsilon(t_k) \right\} &= \exp \left\{ -\frac{1}{2} \int_E \left[\sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-]) \right]^2 \frac{1}{2} w^{-\theta-1} d\tau dw \right\} \\ &= \exp \left\{ -\frac{1}{2} \sum_{k,j} \xi_k \xi_j \int_E (I[S_{0,t_k}^+] I[S_{0,t_j}^+] \right. \\ &\quad \left. + I[S_{0,t_k}^-] I[S_{0,t_j}^-]) \frac{1}{2} w^{-\theta-1} d\tau dw \right\}. \quad (3.5) \end{aligned}$$

The last equality is true since S_{0,t_k}^+ and S_{0,t_j}^- are disjoint for any choice of k and j . Now notice that, for $t_k > t_j$,

$$\begin{aligned} &\int_E (I[S_{0,t_k}^+] I[S_{0,t_j}^+] + I[S_{0,t_k}^-] I[S_{0,t_j}^-]) \frac{1}{2} w^{-\theta-1} d\tau dw \\ &= \frac{1}{2} \theta^{-1} (1 - \theta)^{-1} (t_k^{(1-\theta)} - (t_k - t_j)^{(1-\theta)} + t_j^{(1-\theta)}), \end{aligned}$$

and for $t_k = t_j$,

$$\int_E (I[S_{0,t_k}^+] I[S_{0,t_j}^+] + I[S_{0,t_k}^-] I[S_{0,t_j}^-]) \frac{1}{2} w^{-\theta-1} d\tau dw = \theta^{-1} (1 - \theta)^{-1} t_k^{1-\theta}.$$

The above limit is the characteristic function of a vector from a Gaussian process with the covariance function as in (1.1). Consequently, we have proved

Theorem 3.1. *The finite dimensional distributions of $\{X_\varepsilon(t), t \geq 0\}$ converge, as $\varepsilon \rightarrow 0$, to those of FBM with the scale parameter $H = (1 - \theta)/2$ and variance $EB_H^2(1) = \theta^{-1}(1 - \theta)^{-1}$.*

We can generalize the process given in (3.2) by replacing ε with εX , where X is a random variable with finite second moment. More precisely, define $X'_\varepsilon(0) = 0$ and

$$X'_\varepsilon(t) = \sum_j \varepsilon X_j I[0 < \tau_j < t, t - \tau_j < w_j] - \sum_j \varepsilon X_j I[\tau_j < 0, -\tau_j < w_j < t - \tau_j]$$

for $t > 0$, where $\{X_j, j = 1, 2, \dots\}$ is a sequence of independent identically distributed (i.i.d) random variables, independent from the measure N_ε , such that $EX^2 < \infty$. The process is well defined since both series converge a.s. Now, the process $\{X'_\varepsilon(t), t \geq 0\}$ is the superposition of pulses of various heights, including negative ones. Moreover, the heights of different pulses are independent from one another and from the moments of pulse birth and the pulse widths.

Theorem 3.2. *The finite dimensional distributions of $\{X'_\varepsilon(t), t \geq 0\}$ converge, as $\varepsilon \rightarrow 0$, to those of FBM with the scale parameter $H = (1 - \theta)/2$ and variance $EB_H^2(1) = \theta^{-1}(1 - \theta)^{-1}EX^2$.*

Proof. We are going to use reasoning similar to that which led to Theorem 3.1. Again, it is useful to represent $X'_\varepsilon(t)$ as an integral w.r.t. N_ε , i.e.,

$$X'_\varepsilon(t) = \varepsilon \int_{S_{0,t}^+} X(\tau, w) dN_\varepsilon(\tau, w) - \varepsilon \int_{S_{0,t}^-} X(\tau, w) dN_\varepsilon(\tau, w),$$

where $\{X(\tau, w)\}$ are independent from N_ε and i.i.d., with the distribution F equal to that of X . Instead of (3.4) we use the following more general formula allowing for random integrands

$$E \exp \left(i \int_E G(s) dN(s) \right) = \exp \left\{ \int_E (E \exp \{iG(s)\} - 1) dn(s) \right\}. \quad (3.6)$$

Note that

$$\begin{aligned} & E \exp \left\{ i \sum_{k=1}^n \xi_k X'_\varepsilon(t_k) \right\} \\ &= E \exp \left\{ i \int_E \sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-]) \varepsilon X(\tau, w) dN_\varepsilon(\tau, w) \right\} \\ &= \exp \left\{ \int_E (E \exp \left\{ i \sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-]) \varepsilon X(\tau, w) \right\} - 1) dn_\varepsilon(\tau, w) \right\} \end{aligned}$$

$$= \exp \left\{ \int_E \int_{\mathbb{R}} [\exp(i \sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-]) \varepsilon x) - 1 \right. \\ \left. - i \sum_{k=1}^n \xi_k (I[S_{0,t_k}^+] - I[S_{0,t_k}^-]) \varepsilon x F(dx) \frac{1}{2} \varepsilon^{-2} w^{-\theta-1} d\tau dw \right\}.$$

Since $EX^2 < \infty$, we obtain the limit as in (3.5) with the exponent multiplied by EX^2 . \square

Remark 1. We could also think about the process $\{X'_\varepsilon(t), t \geq 0\}$ as an integral $\varepsilon \int_{\mathbb{R} \times S_{0,t}^+} x dN'_\varepsilon(x, \tau, w) - \varepsilon \int_{\mathbb{R} \times S_{0,t}^-} x dN'_\varepsilon(x, \tau, w)$ w.r.t. a Poisson measure N'_ε defined on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$ with the intensity n'_ε given by $dn'_\varepsilon(x, \tau, w) = \frac{1}{2} \varepsilon^{-2} w^{-\theta-1} F(dx) d\tau dw$. Then the last equality in the proof above is a consequence of (3.4) rather than (3.6).

Remark 2. Taking $\varepsilon = 1/\sqrt{n}$ we get

$$\lim_{n \rightarrow \infty} \sum_j \frac{X_j}{\sqrt{n}} (I[0 < \tau_j < t, t - \tau_j < w_j] - I[\tau_j < 0, -\tau_j < w_j < t - \tau_j]) \stackrel{d}{=} B_H(t),$$

where $\stackrel{d}{=}$ denotes equality of finite-dimensional distributions. This is a more CLT-like statement, analogous to the known result:

$$\lim_{n \rightarrow \infty} \sum_j \frac{X_j - EX_j}{\sqrt{n}} I[0 < \tau_j < t] = \lim_{n \rightarrow \infty} \sum_k \frac{X_k - EX_k}{\sqrt{n}} \stackrel{d}{=} B_{\frac{1}{2}}(t),$$

where $N_{1/\sqrt{n}}$ is a Poisson measure on \mathbb{R} with $dEN_{1/\sqrt{n}} = nd\tau$, or just a Poisson process $\{N(t), t \geq 0\}$ with the constant rate n , and $\{B_{\frac{1}{2}}(t), t \geq 0\}$ is Brownian motion.

4. One-dimensional FBM of multidimensional time

The multidimensional FBM defined as a Gaussian vector process with independent one-dimensional FBM coordinates can be obtained using the results of the previous Section. Below we give a generalization of the construction presented in Section 3 to a multidimensional fractional Brownian field, i.e. a one-dimensional Gaussian process parameterized by multidimensional time. It will be more convenient, however, to start with a slightly different, but equivalent, pulse address space. Let us consider the following change of variables ϕ :

$$z = \tau + w/2, \quad r = w/2.$$

Then $\{X'_\varepsilon(t), t \geq 0\}$ has the same finite-dimensional distributions as the process $\{Y_\varepsilon(t), t \geq 0\}$ defined by

$$Y_\varepsilon(t) = \varepsilon \int_{S_{0,t}^+} X(z, r) dM_\varepsilon(z, r) - \varepsilon \int_{S_{0,t}^+} X(z, r) dM_\varepsilon(z, r),$$

where $M_\varepsilon \equiv N_\varepsilon \circ \phi^{-1}$ is a Poisson random measure on $\mathbb{R} \times \mathbb{R}_+$ with the intensity m satisfying

$$dm_\varepsilon(z, r) = 2^{-\theta-1} \varepsilon^{-2} r^{-\theta-1} dr dz,$$

$\{X(z, r)\}$ are i.i.d. random variables, independent from M_ε , with the distribution F as before, and finally,

$$S'_{0,t}{}^+ = \{(z, r): r > t - z, 0 < z - r < t\} = \{(z, r): |z - t| < r, |z| > r\},$$

$$S'_{0,t}{}^- = \{(z, r): -z < r < t - z, z - r < 0\} = \{(z, r): |z - t| > r, |z| < r\}.$$

(To check that this type of change of variables holds, see Proposition 3.7 in Resnick (1987).) Point (z, r) of the new address space has the obvious interpretation as the center and the radius of a pulse.

Now it is easy to extend the definition of the process $\{Y_\varepsilon(t), t \geq 0\}$ to the multidimensional parameter case. A pulse is a cylinder in \mathbb{R}^{d+1} space. It is completely described by the radius r , the center (z) of its circular base, and its height $\varepsilon X(z, r)$. To get $Y_\varepsilon(t)$ one adds the pulses which contain t but not 0 in their base, and subtract those with 0 but not t in their base. Formally,

$$Y_\varepsilon(0) = 0,$$

$$Y_\varepsilon(t) = \varepsilon \int_{\mathbb{R}^d \times \mathbb{R}_+} X(z, r) I[\|z - t\| < r, \|z\| > r] dM_\varepsilon^d(z, r) \\ - \varepsilon \int_{\mathbb{R}^d \times \mathbb{R}_+} X(z, r) I[\|z - t\| > r, \|z\| < r] dM_\varepsilon^d(z, r),$$

where M_ε^d is a Poisson measure on $\mathbb{R}^d \times \mathbb{R}_+$ with intensity

$$dm_\varepsilon^d(z, r) = \frac{1}{2} \varepsilon^{-2} r^{-\theta-d} dz dr,$$

and $\{X(z, r)\}$ are i.i.d., independent from M_ε^d , with finite second moment EX^2 . (We dropped the constant $2^{-\theta}$ as this normalization is meaningless in higher dimensions.)

Define the constant C by

$$C = \int_{\mathbb{R}^d} (\|z - (1, 0, \dots, 0)\|^{1-\theta-d} - \|z\|^{1-\theta-d}) I[\|z - (1, 0, \dots, 0)\| < \|z\|] dz.$$

Theorem 4.1. *The finite dimensional distributions of $\{Y_\varepsilon(t), t \in \mathbb{R}^d\}$ converge, as $\varepsilon \rightarrow 0$, to those of multidimensional FBM with the scale parameter $H = (1 - \theta)/2$ and variance $EB_H^2(1) = C(d - 1 + \theta)^{-1} EX^2$.*

Proof. Following the proofs of Theorems 3.1 and 3.2 we get that (cf. (3.5))

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} E \exp \left\{ i \sum_{k=1}^n \xi_k Y_\varepsilon(t_k) \right\} \\ &= \exp \left\{ -\frac{EX^2}{2} \sum_{k,j} \xi_k \xi_j \int_{\mathbb{R}^d \times \mathbb{R}_+} (I[\|z - t_k\| < r, \|z - t_j\| < r, \|z\| > r] \right. \\ & \quad \left. + I[\|z - t_k\| > r, \|z - t_j\| > r, \|z\| < r]) \frac{1}{2} r^{-\theta-d} dz dr \right\}. \end{aligned} \quad (4.1)$$

Note that the integral in (4.1) can be further written as

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}_+} \frac{1}{2} (I[\|z - t_k\| < r, \|z\| > r] + I[\|z - t_j\| < r, \|z\| > r] \\ & \quad - I[\|z - t_k\| < r, \|z - t_j\| > r, \|z\| > r] \\ & \quad - I[\|z - t_k\| > r, \|z - t_j\| < r, \|z\| > r] \\ & \quad + I[\|z - t_k\| > r, \|z\| < r] + I[\|z - t_j\| > r, \|z\| < r] \\ & \quad - I[\|z - t_k\| > r, \|z - t_j\| < r, \|z\| < r] \\ & \quad - I[\|z - t_k\| < r, \|z - t_j\| > r, \|z\| < r]) \frac{1}{2} r^{-\theta-d} dz dr \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}_+} (I[\|z - t_k\| < r, \|z\| > r] \\ & \quad + I[\|z - t_k\| > r, \|z\| < r]) \frac{1}{2} r^{-\theta-d} dz dr \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}_+} (I[\|z - t_j\| < r, \|z\| > r] \\ & \quad + I[\|z - t_j\| > r, \|z\| < r]) \frac{1}{2} r^{-\theta-d} dz dr \\ & \quad - \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}_+} (I[\|z - t_k\| < r, \|z - t_j\| > r] \\ & \quad + I[\|z - t_k\| > r, \|z - t_j\| < r]) \frac{1}{2} r^{-\theta-d} dz dr. \end{aligned}$$

Appropriate changes of variables (in z) show that each of the three terms above is equal to an integral

$$\int_{\mathbb{R}^d \times \mathbb{R}_+} I[\|z - t\| < r, \|z\| > r] \frac{1}{2} r^{-\theta-d} dz dr,$$

with $\mathbf{t} = \mathbf{t}_k$, \mathbf{t}_j or $\mathbf{t}_k - \mathbf{t}_j$, and this integral, in turn, equals

$$\frac{1}{2} \frac{\|\mathbf{t}\|^{1-\theta}}{d-1+\theta} \int_{\mathbb{R}^d} \left(\left\| \mathbf{z} - \frac{\mathbf{t}}{\|\mathbf{t}\|} \right\|^{1-\theta-d} - \|\mathbf{z}\|^{1-\theta-d} \right) I \left[\left\| \mathbf{z} - \frac{\mathbf{t}}{\|\mathbf{t}\|} \right\| < \|z\| \right] dz.$$

However, the value of the integral over \mathbb{R}^d does not depend on t and equals the constant C defined before the statement of the theorem. Hence, we have shown that the integral in (4.1) is indeed the covariance function of the asserted multidimensional FBM. \square

5. FBM of one-dimensional time as a sum of micropulses of a more general shape

Let $E = \mathbb{R} \times \mathbb{R}_+$ continue to be the pulse address space defined in Section 3 with $(\tau, w) \in E$ having the same interpretation as moment of pulse birth and pulse width, respectively. From now on, however, the pulse is no longer a rectangle described by the indicator of an interval. Instead, the shape is determined by a bounded function f with the support in $[0, 1]$. The amplitude at time t of a pulse triggered at time τ and with width w equals

$$\varepsilon X f\left(\frac{t - \tau}{w}\right),$$

where $\varepsilon > 0$ and X is a random variable such that $EX^2 < \infty$.

As before, let the occurrence of pulses be determined by a Poisson random measure N_ε , with the intensity given by (3.1) for $\varepsilon > 0$ and $0 < \theta < 1$, and $\{X(\tau, w)\}$ be independent from N_ε and i.i.d., with the distribution F equal to that of X . Put $\tilde{X}_\varepsilon(0) = 0$ and, for $t > 0$,

$$\tilde{X}_\varepsilon(t) = \varepsilon \int_E X(\tau, w) \left[f\left(\frac{t - \tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] dN_\varepsilon(\tau, w). \quad (5.1)$$

The variable $\tilde{X}_\varepsilon(t)$, for $t > 0$, again represents the difference in the sum of pulses existing at time t and at time 0. The question is when it is well-defined, i.e., when the integral in (5.1) converges a.s. Note that the region of integration in (5.1) may be much larger than $S_{0,t}^+ \cup S_{0,t}^-$. This is due to the fact that pulses originating at $\tau < 0$ and vanishing after t (i.e. $\tau + w > t$) may have nonzero contributions to $\tilde{X}_\varepsilon(t)$ in contrast to simple rectangular pulses for which this contribution was zero. In fact, if $\tilde{X}_\varepsilon(t)$ exists then it can be written

$$\begin{aligned} \tilde{X}_\varepsilon(t) = & \varepsilon \int_{S_{0,t}^+} X(\tau, w) f\left(\frac{t - \tau}{w}\right) dN_\varepsilon(\tau, w) \\ & - \varepsilon \int_{S_{0,t}^-} X(\tau, w) f\left(\frac{-\tau}{w}\right) dN_\varepsilon(\tau, w) \\ & + \varepsilon \int_{S_{0,t}} X(\tau, w) \left[f\left(\frac{t - \tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] dN_\varepsilon(\tau, w), \end{aligned} \quad (5.2)$$

where

$$S_{0,t} = \{(\tau, w) : -\infty < \tau < 0, t - \tau < w\}.$$

One can see that the first two integrals in (5.2) converge a.s. since we assume that f is bounded and (3.3) holds. It is the integral over $S_{0,t}$ which requires special care. We need to ensure that this integral is finite a.s. First, however, we give general conditions for the existence of $\tilde{X}_\varepsilon(t)$.

Proposition 5.1. (a) If f is a function with the support in $[0, 1]$ satisfying

$$\int_E \left| f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right| w^{-\theta-1} d\tau dw < \infty, \quad (5.3)$$

then $\tilde{X}_\varepsilon(t)$ in (5.2) is well-defined.

(b) If f is a function with the support in $[0, 1]$ satisfying

$$\int_E \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right]^2 w^{-\theta-1} d\tau dw < \infty, \quad (5.4)$$

then the series

$$\sum_{n=0}^{\infty} \varepsilon \int_{E_n} X(\tau, w) \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] dN_\varepsilon(\tau, w), \quad (5.5)$$

where $E_0 = \mathbb{R} \times (0, 1]$ and $E_n = \mathbb{R} \times (2^{n-1}, 2^n]$, $n = 1, 2, \dots$, converges a.s., i.e. $\tilde{X}_\varepsilon(t)$ is defined in the sense of conditional a.s. convergence.

Proof. Part (a) follows from the general theory of Poisson integrals (cf. Resnick, 1987, p. 127) and the fact that $EX < \infty$. To prove (b) first notice that in (5.5) the random variables

$$V_n = \varepsilon \int_{E_n} X(\tau, w) \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] dN_\varepsilon(\tau, w),$$

$n = 0, 1, \dots$, are well-defined since

$$\begin{aligned} & \int_{E_n} \left| f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right| w^{-\theta-1} d\tau dw \\ & \leq \left(\int_{E_n} I(S_{0,t}^+ \cup S_{0,t}^- \cup S_{0,t}) w^{-\theta-1} d\tau dw \right. \\ & \quad \left. \int_{E_n} \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right]^2 w^{-\theta-1} d\tau dw \right)^{\frac{1}{2}} < \infty \end{aligned}$$

and $\theta < 1$. Moreover,

$$\int_{E_n} \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] w^{-\theta-1} d\tau dw = 0. \quad (5.6)$$

It is enough to show only convergence in distribution since the terms V_n in (5.5) are independent. Using (3.6), the logarithm of the characteristic function of V_n at $\xi \in \mathbb{R}$ equals, by (5.6)

$$\begin{aligned} & \int_{E_n} \int_{\mathbb{R}} \left(\exp \left\{ i\varepsilon \xi x \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] \right\} - 1 \right. \\ & \quad \left. - i\varepsilon \xi x \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] \right) F(dx) \frac{1}{2} \varepsilon^{-2} w^{-\theta-1} d\tau dw, \end{aligned} \quad (5.7)$$

for each $n = 0, 1, \dots$, and their sum over n can be bounded by

$$\frac{1}{4} \xi^2 EX^2 \int_E \left[f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right]^2 w^{-\theta-1} d\tau dw.$$

Hence, indeed, the series in (5.5) converges in distribution (the logarithm of its characteristic function is equal to the expression in (5.7) with E_n replaced by E). \square

The next Proposition provides examples of functions that fulfill assumptions of Proposition 5.1, i.e. it specifies some classes of functions for which $\tilde{X}_\varepsilon(t)$ is well-defined.

Proposition 5.2. *Let f be a bounded function with the support in $[0, 1]$ and satisfying either of the two conditions:*

(i) *f is Hölder continuous in $[0, 1]$ with an exponent $\alpha > 0$, i.e.*

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for some positive M and any $x, y \in [0, 1]$;

(ii) *f is a step function, i.e. there exist $a_i \in \mathbb{R}$, $i = 1, 2, \dots, k \in \mathbb{N}$, such that*

$$f(x) = \sum_{i=1}^k a_i I[s_{i-1} < x \leq s_i],$$

where $0 = s_0 < s_1 < \dots < s_k = 1$.

Then (i) with $\alpha > 1 - \theta$ or (ii) imply (5.3) and (i) with $\alpha > (1 - \theta)/2$ implies (5.4).

Proof. The relations (5.3) and (5.4) hold trivially under condition (i) with respective α . To show (5.3) under (ii) let us focus only on the integral over $S_{0,t}$ and integrate w.r.t. τ ($t - w < \tau < 0$) first and then w.r.t. w ($t < w < \infty$). Notice that the divergence of the integral may be caused only by high values of w . On the other hand, when w is large enough, such that $t < \min_i (s_i - s_{i-1})w$, the integrand (for fixed w) is nonzero in at most k intervals of the length t . Hence, for $A = (\min_i (s_i - s_{i-1}))^{-1}t$,

$$\begin{aligned} & \int_A^\infty \int_{t-w}^0 \left| f\left(\frac{t-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right| w^{-\theta-1} d\tau dw \\ & \leq \int_A^\infty kt \max_i |a_i - a_{i-1}| w^{-\theta-1} dw < \infty, \end{aligned}$$

which establishes (5.3). \square

In the statement of the main result we will need the following constant $C(f)$:

$$\begin{aligned} C(f) &:= \int_{S_{0,1}^+} f^2\left(\frac{1-\tau}{w}\right) w^{-\theta-1} d\tau dw \\ &+ \int_{S_{0,1}^-} f^2\left(\frac{-\tau}{w}\right) w^{-\theta-1} d\tau dw \\ &+ \int_{S_{0,1}} \left[f\left(\frac{1-\tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right]^2 w^{-\theta-1} d\tau dw. \end{aligned}$$

Clearly, $C(f) < \infty$ not only under the assumption of Proposition 5.1(b), but also for any bounded f satisfying (5.3). In particular, the finiteness of the third integral follows from the fact that bounded integrable functions are also square-integrable.

Theorem 5.1. *The finite dimensional distributions of $\{\tilde{X}_\varepsilon(t), t \geq 0\}$, for f bounded and satisfying (5.3) or f satisfying (5.4), converge, as $\varepsilon \rightarrow 0$, to those of FBM with the scale parameter $H = (1 - \theta)/2$ and variance $EB_H^2(1) = C(f)EX^2/2$.*

Proof. Note that (5.6) holds also with E_n replaced by E in the case when (5.3) is satisfied. To prove it, use the Fubini Theorem, integrate w.r.t. τ first and use a change of variables. Hence, as in the proof of Theorems 3.1 and 3.2 we can show that when $\varepsilon \rightarrow 0$, the characteristic function of $\sum_{k=1}^n \xi_k \tilde{X}_\varepsilon(t_k)$ approaches that of $\sum_{k=1}^n \xi_k X(t_k)$ where $(X(t_1), X(t_2), \dots, X(t_k))$ is a Gaussian vector with covariance

$$\begin{aligned} \text{Cov}(X(t_k), X(t_j)) &= \frac{1}{2} EX^2 \int_E \left[f\left(\frac{t_k - \tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] \left[f\left(\frac{t_j - \tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right] \\ &\quad w^{-\theta-1} d\tau dw \\ &= -\frac{1}{4} EX^2 \left\{ \int_E \left[f\left(\frac{t_k - \tau}{w}\right) - f\left(\frac{t_j - \tau}{w}\right) \right]^2 w^{-\theta-1} d\tau dw \right. \\ &\quad - \int_E \left[f\left(\frac{t_k - \tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right]^2 w^{-\theta-1} d\tau dw \\ &\quad \left. - \int_E \left[f\left(\frac{t_j - \tau}{w}\right) - f\left(\frac{-\tau}{w}\right) \right]^2 w^{-\theta-1} d\tau dw \right\}. \end{aligned}$$

Translation by t_j in the first integral and scaling in both τ and w in all three integrals show that

$$\text{Cov}(X(t_k), X(t_j)) = -\frac{1}{4} EX^2 [|t_k - t_j|^{1-\theta} - t_k^{1-\theta} - t_j^{1-\theta}] C(f).$$

This proves that the limiting process is FBM. \square

Example. Consider as a pulse shape f the graph of a typical sample path of FBM with the scale parameter $\eta > (1 - \theta)/2$. More precisely, take a rescaled part of the graph with starting and ending points at 0 (e.g. a part between time 0 and time of a return to 0). Such a “typical” (in the a.s. sense) graph is Hölder continuous with any $\alpha < \eta$, hence, also with some $\alpha > (1 - \theta)/2$. Note, however, that this templet always produces the final FBM with smaller scale parameter, i.e. with less regular sample paths. Thus, the scale parameter of the final FBM is affected only by the exponent of the pulse width, not the smoothness of the pulse shape.

We can continue the example and the above discussion about variously shaped random micropulses by considering the *pulse shape* (not only its height) to be random. A pulse triggered at a starting point τ and lasting for time w can itself be a random

process. In our case, we assume it is a rescaled and shifted FBM with the self-affinity parameter η , i.e. the pulse amplitude at time t , $\tau < t < \tau + w$, is

$$\varepsilon B_\eta\left(\frac{t-\tau}{w}\right).$$

Then the process $\{\tilde{X}'_e(t), t \geq 0\}$ is defined as a superposition of the changes between times 0 and t of such shifted and rescaled independent copies of FBM:

$$\tilde{X}'_e(t) = \varepsilon \sum_j \left(X_j\left(\frac{t-\tau_j}{w_j}\right) - X_j\left(\frac{-\tau_j}{w_j}\right) \right), \quad (5.8)$$

where

$$X_j(s) = B_{\eta_j}(s)I[0 < s < 1]$$

and $\{(\tau_j, w_j), j = 1, 2, \dots\}$ is an enumeration of points of the random measure N_ε . The above series converges in L^2 and a.s. if $\eta > (1 - \theta)/2$, since

$$\begin{aligned} & E\left(\varepsilon \sum_j \left(X_j\left(\frac{t-\tau_j}{w_j}\right) - X_j\left(\frac{-\tau_j}{w_j}\right) \right)\right)^2 \\ &= E\left[E\left(\varepsilon \sum_j \left(X_j\left(\frac{t-\tau_j}{w_j}\right) - X_j\left(\frac{-\tau_j}{w_j}\right) \right)\right)^2 \middle| \{\tau_j, w_j\}\right] \\ &= \varepsilon^2 E\left[\sum_j E\left(X_j\left(\frac{t-\tau_j}{w_j}\right) - X_j\left(\frac{-\tau_j}{w_j}\right)\right)^2 \middle| \{\tau_j, w_j\}\right] \\ &= \frac{1}{2} \int_{-\infty}^0 \int_{-\tau}^{t-\tau} (-\tau)^{2\eta} w^{-2\eta-\theta-1} dw d\tau + \frac{1}{2} \int_{-\infty}^0 \int_{t-\tau}^{\infty} t^{2\eta} w^{-2\eta-\theta-1} dw d\tau \\ &\quad + \frac{1}{2} \int_0^t \int_{t-\tau}^{\infty} (t-\tau)^{2\eta} w^{-2\eta-\theta-1} dw d\tau \\ &= C_1 t^{1-\theta} < \infty, \end{aligned} \quad (5.9)$$

where

$$C_1 = \frac{1}{2} \left(\int_{-\infty}^0 \int_{-\tau}^{t-\tau} (-\tau)^{2\eta} w^{-2\eta-\theta-1} dw d\tau + \frac{1}{2\eta + \theta} \frac{1}{2\eta + \theta - 1} + \frac{1}{2\eta + \theta} \frac{1}{1 - \theta} \right).$$

We used the independence of $\{X_j(\cdot), j = 1, 2, \dots\}$ and the following formulae implied by the FBM variance:

$$\text{Var} \left(X_j\left(\frac{t-\tau}{w}\right) - X_j\left(\frac{-\tau}{w}\right) \right) = \begin{cases} \left(\frac{-\tau}{w}\right)^{2\eta} \text{ for } 0 < \frac{-\tau}{w} < 1 \text{ and } 1 < \frac{t-\tau}{w}, \\ \left(\frac{t}{w}\right)^{2\eta} \text{ for } 0 < \frac{-\tau}{w} < 1 \text{ and } 0 < \frac{t-\tau}{w} < 1, \\ \left(\frac{t-\tau}{w}\right)^{2\eta} \text{ for } \frac{-\tau}{w} < 0 \text{ and } 0 < \frac{t-\tau}{w} < 1. \end{cases}$$

Again, the process $\{\tilde{X}'_\varepsilon(t), t \geq 0\}$ can be formally written as an integral with respect to Poisson measure N_ε ,

$$\tilde{X}'_\varepsilon(t) = \varepsilon \int_E \left(X\left(\frac{t-\tau}{w}\right) - X\left(\frac{-\tau}{w}\right) \right) dN_\varepsilon(\tau, w),$$

and the convergence of this integral is understood as the convergence of the series (5.8). Note that $\tilde{X}'_\varepsilon(t)$ has finite variance if and only if $\eta > (1 - \theta)/2$, $0 < \theta < 1$, and then its variance is given by (5.9). The characteristic function of a linear combination $\sum_{k=1}^n \xi_k \tilde{X}'_\varepsilon(t_k)$ equals (cf. (3.6))

$$\begin{aligned} & \exp \left\{ \int_E \left(E \exp \left\{ i \varepsilon \sum_{k=1}^n \xi_k \left(X\left(\frac{t_k - \tau}{w}\right) - X\left(\frac{-\tau}{w}\right) \right) \right\} - 1 \right) \frac{1}{2} \varepsilon^{-2} w^{-\theta-1} d\tau dw \right\} \\ &= \exp \left\{ \int_E \left(\exp \left\{ -\frac{\varepsilon^2}{2} \sum_{j,k} \xi_j \xi_k \text{Cov} \left(X\left(\frac{t_j - \tau}{w}\right) - X\left(\frac{-\tau}{w}\right), \right. \right. \right. \right. \\ & \quad \left. \left. \left. X\left(\frac{t_k - \tau}{w}\right) - X\left(\frac{-\tau}{w}\right) \right) \right\} - 1 \right) \times \frac{1}{2} \varepsilon^{-2} w^{-\theta-1} d\tau dw \right\}, \end{aligned} \quad (5.10)$$

and some elementary (though tedious) calculations show that the limiting (when $\varepsilon \rightarrow 0$) Gaussian process has the FBM covariance structure with $H = (1 - \theta)/2$ and variance equal to $C_1 t^{1-\theta}$. (On the other hand, the same conclusion can be drawn without calculating the covariance: since we notice that the limiting process is Gaussian with stationary increments and variance $C_1 t^{1-\theta}$, it must be FBM.) Hence:

Theorem 5.2. *The finite dimensional distributions of $\{\tilde{X}'_\varepsilon(t), t \geq 0\}$ defined in (5.8), where the random pulse shape is determined by an FBM with the scale parameter $\eta > (1 - \theta)/2$, converge, as $\varepsilon \rightarrow 0$, to those of FBM with the scale parameter $H = (1 - \theta)/2$ and variance given by (5.9).*

It is clear (from (5.10)) that B_η in the definition of $\tilde{X}'_\varepsilon(t)$ can be replaced by a more general process, e.g. any zero-mean process with stationary increments and covariance function identical to that of FBM with a scale parameter $\eta > (1 - \theta)/2$. The same conclusion can be drawn as in the Example above. Increasing η increases smoothness of the sample paths of the templet $\{X(s), s \in \mathbb{R}\}$ (for example, $\{B_\eta(s)/[0 < s < 1]\}$), but it does not influence the self-affinity constant of the final FBM. However, finiteness of the second moment of $\tilde{X}'_\varepsilon(t)$ in the above construction requires $\eta > (1 - \theta)/2$. In particular, this implies that we may always use the ordinary Brownian motion for which $\eta = \frac{1}{2}$.

References

- R. Cioczek-Georges and B.B. Mandelbrot, Alternative micropulses and fractional Brownian motion, Preprint (1995a).
- R. Cioczek-Georges, and B.B. Mandelbrot, Stable fractal sums of pulses: the general case, Preprint (1995b).
- R. Cioczek-Georges, B.B. Mandelbrot, G. Samorodnitsky, and M. Taqqu, Stable fractal sums of pulses: the cylindrical case, To appear in Bernoulli (1995).
- K. Itô, Stochastic Processes, Aarhus University Lecture Notes Series, Vol. 16, (1969).
- B.B. Mandelbrot, Fractal sum of pulses, and new random variables and functions, Unpublished (1984).

- B.B. Mandelbrot, Introduction to fractal sums of pulses, in: G. Zaslowsky, M.F. Schlesinger and U. Frisch, eds., *Lévy Flights and Related Phenomena in Physics* (Nice, 1994), Lecture Notes in Physics, (Springer, New York, 1995a).
- B.B. Mandelbrot, Fractal sums of pulses: self-affine global dependence and lateral limit theorems, Preprint (1995b).
- B.B. Mandelbrot and J.W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.* 10 (1968) 422–437.
- S.I. Resnick, *Extreme Values, Regular Variation and Point Processes* (Springer, New York, 1987).
- L. Takács, On secondary processes generated by a Poisson process and their applications in physics, *Acta Math. Acad. Sci. Hungar.* 5 (1954) 203–235.
- M. Westcott, On the existence of a generalized shot-noise process, in: *Studies in Probability and Statistics* (North Holland, Amsterdam, 1976) pp. 73–88.