

Introduction to Fractal Sums of Pulses

Benoit B. Mandelbrot

Mathematics Department, Yale University, New Haven, CT 06520-8283 USA

Abstract. In this paper, the classical Lévy flights are generalized, their jumps being replaced by more involved “pulses.” This generates a wide family of self-affine random functions. Their versatility makes them useful in modeling. Their structure throws new conceptual light on the difficult issue of global statistical dependence, especially in the case of processes with infinite variance.

Keywords. Fractal sums of pulses. Global statistical dependence. Infinite variance. Lateral attractors. Lévy flights (generalized). Pulses.

A *Lévy flight* or *Lévy stable motion* (SLM) is well-known to be the sum of an infinity of step-functions of widely varying sizes. This paper introduces a generalized construction: the jumps are replaced by suitable affine reductions or dilations of “templates” represented by *kernel functions* K more general than a step. The result can be described as being an *affine convolution*. A template whose kernel is constant except on a (bounded) interval will be called *pulse*, and the resulting sums will be called *fractal sums of pulses* (FSP). A series of papers that started with [1], [2], [3] and [6] will describe the theory of self-affine FSP and (going beyond [4]) some of their concrete applications.

This paper concerns semi-random self-affine FSP constructed as follows. A pulse's height and location follow the same distribution as in a Lévy flight: the probability of the point $\{\lambda, t\}$ being found in an elementary rectangle of the (λ, t) address plane is $\propto \lambda^{-\delta-1} d\lambda dt$. In a Lévy flight the exponent δ is constrained to satisfy $0 < \delta < 2$. A major immediate difference is that in an FSP, the constraints are either $\delta > 0$ or $\delta > 1$, depending on the case. The pulse's width W (the length of the smallest interval in which the pulse varies) satisfies $W = \sigma \Lambda^\delta$, where $\sigma > 0$ is a scale constant, implying that $Pr\{W > w\} \propto w^{-1}$. The resulting FSP is called *semi-random* because Λ and W are functionally related (in fully random FSP, Λ and W are statistically independent). This paper is meant to show the great variety of distinct behaviors that can be found in an FSP, as we vary δ and three properties of the kernels:

- a) Discontinuous versus very smoothly continuous;
- b) Canceling (vanishing outside the interval in which they vary) versus non-canceling.
- c) Atoms versus bursts. This is a useful but elusive distinction. When the pulse is made of a rise and fall followed later by another rise and fall, it can be decomposed into a burst of two indecomposable or atomic pulses.

The resulting templates are exemplified in Table I: cylinders (one discontinuous *rise* followed after the time W by one discontinuous *fall*), and multiple steps, cones (uniform rate rise followed by uniform rate fall), and uniform rate rises. Other templates are discussed in forthcoming papers.

Lévy Flights and Related Phenomena in Physics (Nice, 1994).

Edited by Michael F. Shlesinger, George Zaslavsky, & Uriel Frisch.

(Lecture Notes in Physics: 450.) New York: Springer, 110-123.

In the standard study of attraction of a random function $X(t)$ to a limit such as Brownian motion, an early step is an affine rescaling of the form $T^{-H}X(T\rho)$. Being constructed to be self-affine, all FSP are invariant under this rescaling with the exponent $H = 1/\delta$. That is, using the physicists' language, each FSP defines its own "class of universality" with respect to a suitable affinity. But we shall introduce an alternative rescaling, to be called "lateral," and show that each semi-random FSP has an interesting "lateral attractor." The lateral attractor may be a Lévy flight (SLM), which might have been vaguely expected, because the construction starts with the Lévy measure. But the lateral attractor may also be a fractional Brownian motion (FBM), which is a surprise, and establishes deep new links between two independent theories that are known to have striking formal parallelism. The attractor may also be neither SLM nor FBM.

The increments of the FSP are globally dependent, but those of their SLM limits are independent. Thus, the FSP bring altogether new conceptual light on the probabilistic notion of global dependence and the related notion that a process is attracted by another process. Indeed, depending on the shape of K , global dependence is expressed in either of two ways: a) by a special exponent H familiar in such known contexts as FBM, or b) by a prefactor rather than a special exponent.

Separate papers investigate: A) semi-random FSP using additional classes of kernels and second differences of FSP; B) semi-random FSP with $W = \sigma\Lambda^\theta$, where $\theta \neq \delta$; they are not self-affine. Related papers investigate: C) Fully random FSP in which W is statistically independent of Λ with the measure $\propto w^{-\theta-1}dw$, where $0 < \theta < 1$; they are shown in [3] to be self-affine with $H = (1 - \delta)/\theta$; D) fractal sums of *micropulses* (FSM) which generate (FBM); see [1,2].

1. INTRODUCTION

The "normal" model of natural fluctuations is the Wiener Brownian motion process (WBM). By this standard, however, many natural fluctuations exhibit clear-cut "anomalies" which may be due to large discontinuities ("Noah Effect") and/or non-negligible global statistical dependence ("Joseph Effect"). I have long argued that the geometric features of surprisingly many of these anomalous aspects of nature are fractal. For example, for many large "Noah" discontinuities the tail probability distribution is "hyperbolic." That is, if it is large, a discontinuity U that exceeds the value u has a probability of the hyperbolic form $Pr(U > u) \sim u^{-\delta}$, with δ a positive constant. Second example: for large lags s , many globally correlated "Joseph" fluctuations have a correlation function of the form $C(s) \sim 2H(2H-1)s^{2H-2}$, with $1/2 < H < 1$. [5] shows that one can model various instances of the Noah effect by the classical process of SLM, and various instances of the Joseph effect by the process of FBM.

SLM and FBM, however, are far from exhausting the anomalies found in nature; in particular, neither gives a satisfactory model of the shape of clouds, and many phenomena exhibit *both* the Noah and the Joseph effects and fail to be represented by either SLM or FBM. Hence, fractal modeling of nature demands "bridges," namely random functions (r.f.'s) that combine the infinite variance feature that is characteristic of SLM and the global dependence feature that is characteristic of FBM. One obvious bridge, fractional Lévy motion, is interesting mathematically,

but has found no concrete use.

Furthermore, the mathematical theories of SLM and FBM exhibit striking parallels as well as discrepancies. One major discrepancy is in the allowable value of the exponent H which is defined by the condition that the distribution of $T^{-H}F(T\rho)$ is independent of T . For SLM, $1/2 < H < \infty$, while for FBM, $0 < H < 1$. This mismatch is a challenge. Being unexpected, the parallels are sometimes described as miraculous, but they have deep roots worth exploring.

2. DEFINITIONS

2.1 Stationarity and self-affinity

The function $F(t)$ is said to have *stationary increments* if the translated function

$$F(t_0 + t) - F(t_0)$$

has the same distributions for all values of t_0 . A function $F(t)$ is said to be *self-affine of exponent $H > 0$* when the rescaled function

$$\rho^{-H}[F(t_0 + \rho t) - F(t_0)]$$

has the same distributions for all values of t_0 and $\rho > 0$. Some authors denote self-affinity by the improper term, self-similarity.

2.2 Pulse templates, pulses, affine convolutions, and fractal sums of pulses

The graph of $K(t)$, a one-dimension function of a one-dimensional variable, will be called *generator*, or (*pulse template*), if $K(t)$ is constant outside an interval; we shall set the shortest such interval to be of length 1.

A *pulse* is a translated affine transform $K\left(\frac{t-t_n}{w_n}\right)$ of $K(t)$, where λ_n , t_n , and w_n are called the pulse's height, position, and width.

A *sum of pulses* is a function of the form

$$F(t) = \sum \lambda_n K\left(\frac{t-t_n}{w_n}\right).$$

Figure 1 is an example of a sum of these pulses. When $F(t)$ is a self-affine function, it will be called a *fractal sum of pulses*, FSP.

An *affine convolution* of the sequence $\{\lambda_n, t_n\}$ by the kernel $K(\cdot)$ is obtained when the pulse heights and widths are linked by a relation $w_n = \sigma \lambda_n^\delta$, where $\sigma > 0$ and $\delta > 0$ are prescribed. Thus, the semi-random FSP are affine convolutions of sequences $\{\lambda_n, t_n, w_n\}$.

2.3 The Lévy measure for the probability distribution of pulse height and position

The simplest pulse template is a step function redefined so that $w_n = 1$ for all n . The distribution of $\{\lambda_n, t_n\}$ that insures self-affinity in that case was discovered by Paul Lévy and is classical. The same distribution continues to be required in all FSP. Let the plane of coordinates t and λ be called *address plane*. Take a rectangle

$[\lambda, \lambda + d\lambda] \times [t, t + dt]$ in the address plane, such that $[\lambda, \lambda + d\lambda]$ does not contain $\lambda = 0$. Given an exponent $\delta > 0$ and two scale factors C' and C'' , the Lévy measure is

$$C' \lambda^{-\delta-1} d\lambda dt \quad , \text{if} \quad \lambda > 0, \quad \text{and} \quad C'' |\lambda|^{-\delta-1} d\lambda dt \quad , \text{if} \quad \lambda < 0.$$

Giving C' and C'' is, of course, equivalent to giving the overall scale $C' + C''$ and the skewness factor $C'/(C' + C'')$.

To define an FSP, the probability of finding an address point (λ, t) in the elementary rectangle is set equal to the Lévy measure. The number of address points in a domain \mathcal{D} in the address space is taken to be a Poisson random variable whose expectation is the integral of the Lévy measure over \mathcal{D} . The total number of pulses is countably infinite.

2.4. Semi-random pulse templates

The simplest pulses are the step functions used by Paul Lévy to generate the SLM. The pulses examined in this paper and illustrated in Table 1 are *semi-random*: Λ is random with the Lévy measure, but the height Λ fully determines the width W . To insure that the FSP is self-affine, one must take

$$W = \sigma \Lambda^\delta, \quad \text{where} \quad \sigma > 0.$$

The resulting probability distribution of W is, independently of δ ,

$$Pr\{W > w\} \propto w^{-1}.$$

The units in which Λ and W are measured are arbitrary and unrelated. If those units are identical, either $C' + C''$ or σ can be normalized to 1 by changing the unit. However, up to scale, the distribution of an FSP is determined by $(C' + C'')\sigma$ and the skewness $C'/(C' + C'')$. In the sequel, an important role is played by lateral limit theorems that are expressed most conveniently by fixing $C' + C''$ and allowing $\sigma \rightarrow \infty$. The consequences of $\sigma \rightarrow \infty$ are obvious when the pulses are “cylindrical:” a *rise* followed, after a span of $\Delta t = w$, by a *fall* of equal absolute value and opposite sign. Clearly, the contribution to $F(t + T) - F(t)$ from a pulse such that $w > T$ is not a pulse but an unattached rise or fall. Therefore, as $\sigma \rightarrow \infty$, each cylindrical pulse reduces to a rise or a fall, and the fact that a pulse has a bounded support becomes less and less significant.

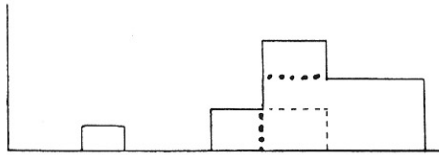


FIGURE 1: Schematic sum of pulses

3. SELF-AFFINITY AND THE EXPONENT $H = 1/\delta$; EXISTENCE OF GLOBAL DEPENDENCE

3.1. The self-affinity property of all FSP

For many combinations of a pulse shape and a value of δ , the semi-random FSP construction yields a well-defined random process, and δ is called *admissible* for these pulses. For other values of δ , the construction diverges and δ is called *excluded*. When the construction converges, it is easy to see that the resulting FSP are self-affine. All semi-random cases yield $H = 1/\delta$, just like in the Lévy case when the pulses are step functions. In one case to be described elsewhere, the construction of $F(t+T) - F(t)$ diverges, but that of the second difference $[F(t+T) - F(t)] - [F(t) - F(t-T)]$ converges and is self-affine.

3.2. First corollary of self-affinity: each FSP defines a special domain of attraction, hence the standard limit problem concerning random processes is degenerate

In the study of random functions, the standard next issue is whether or not there exists an exponent H such that, setting $F(0) = 0$,

$$\text{weak} \lim_{T \rightarrow \infty} T^{-H} F(\rho T)$$

is a non-degenerate function of ρ , called the “attractor” of F . The most familiar attractions are WBM and SLM in the case of independence and FBM in the case of dependence. Now suppose that $F(t)$ is a semi-random FSP. For it, the standard limit problem does not arise, since $T^{-H} F(\rho T)$ independent of T in distribution, each FSP defines its own domain of attraction of exponent $H = 1/\delta$.

Standard domains of attraction to an FSP. Given a self-affine attractor $X(t)$, the next challenge is to describe its domain of attraction, defined as the collection of r.f.’s $G(t)$ for which the rescaling (or renormalizing) function $A(T)$ can be selected so as to insure that

$$\text{weak} \lim_{T \rightarrow \infty} A^{-1}(T) G(\rho T) = X(t).$$

My study of the domain of attraction of a semi-random FSP has limited itself thus far to r.f.’s that are themselves sums of pulses, but involve a density other than Lévy’s or a relation other than $w = \sigma \lambda^\delta$. I have not yet examined templates that depend on height.

Clearly, $\text{weak} \lim_{T \rightarrow \infty} T^{-H} F(\rho T)$ is unchanged if the distribution of pulse heights is changed over a bounded interval, for example if pulse height is restricted to $\lambda > \varepsilon > 0$.

For the next obvious change, SLM suggests replacing the constants C'' and C''' in the Lévy density by functions $C''(\lambda)$ and $C'''(\lambda)$ that vary slowly for $\lambda \rightarrow \infty$. And FBM suggests replacing the relation $w = \sigma \lambda^\delta$ by $w = \sigma(\lambda) \lambda^\delta$, where $\sigma(\lambda)$ is also slowly varying. These changes lead to a nonself-affine generalized FSP. The questions is whether or not, as $T \rightarrow \infty$, there exists a rescaling $A(T)$ that makes $A^{-1}(T) G(T)$ converge weakly to a self-affine FSP. I have been content with verifying (see Section 7.6) that a sufficient condition is that $C''(\lambda) \sigma(\lambda) \rightarrow 1$; in that case, $T^{1/\delta} A(T)$ is the inverse function of $T = \sigma(\lambda) \lambda^\delta$, implying that $A(T)$ is slowly

varying. The condition $C''(\lambda)\sigma(\lambda) \rightarrow 1$ is demanding, resulting in a narrow domain of attraction, a notion discussed in Section 3.4.

3.3. Second corollary of self-affinity: the global dependence property of all FSP

A corollary of Section 3.2 is that if F is a semi-random FSP, then $T^{-H}F(\rho T)$ fails to converge to a standard attractor relative to asymptotically independent increments, namely, either WBM or SLM. This implies that all semi-random FSP must fail to satisfy the usual criteria that express that dependence between increments is local.

In fact, they are uniformly globally dependent. For example, define for each t the following two functions

- the rescaled finite past $T^{-1/\delta}[F(t) - F(t - Tp)]$
- and the rescaled finite future $T^{-1/\delta}[F(t + Tp) - F(t)]$.

Because of pulses that contribute to both past and future, these random functions of p are not statistically independent. Because of self-affinity, their joint distributions are independent of T . This means that strong mixing is contradicted uniformly for all T .

3.4. Thoughts on the role of limit theorems suggested by the degeneracy of the standard limit problem in the case of FSP

To comment on the role of limit theorems in the light of Section 3.3, let us compare the “attractands” with their attractor. One wants the process of going to the limit to destroy the most idiosyncratic features of the attractand, while preserving features that have a degree of “universality.” This is why the most important attractors continue to be: the nonrandom attractor for the laws of large numbers, the Gaussian r.v. for the central limit theorems, and WBM for the functional central limit theorems. These attractors’ domains of attraction are very broad, being largely characterized by the absence of global independence and of significant probabilities for large values. By contrast, when a basin of attraction is narrow, the attractor yields specific information about the attractands. Thus, there is a sharp contrast between broad universality with little information and narrow universality with extensive information.

Down to specifics: for SLM, the dependence can be anything, as long as it is local, but the tails must be long and strictly constrained; for FBM, the tails can be anything, as long as they are short, but the dependence must be global and strictly constrained. Similarly, the sufficient condition $C''(\lambda)\sigma(\lambda) \rightarrow 1$ in Section 3.2 defines for each FSP a (partial) domain of attraction that is tightly constrained, with *the same exponent, for the tails and the global dependence*.

The resulting variety of forms creates a use for additional limit problems that would put order by destroying some of the FSP’s overabundant specifications. That is, the finding in Section 3.3 must spur the search for alternative limit problems, for which the domains of attraction are broader, therefore reveal more “universal” properties of the FSP. Section 4 will advance one possibility.

4. THE CONCEPT OF LATERAL LIMIT PROBLEM AND THE EXPONENT α ; UNISCALING ($\alpha = \delta = 1/H$) AND PLURISCALING ($\alpha = \min[2, \delta] \neq 1/H$) LATERAL ATTRACTORS

4.1. Background of the new “lateral” limit problem

Neither the random walk nor the Poisson process of finite density is self-affine, but both are attracted in the usual way to the Wiener-Brownian motion $B(t)$, which is self-affine with $H = 1/2$. That is, writing $F(0) = 0$, it is true in both cases that $T^{-1/2}F(\rho T)$ depends on T for $T < \infty$, but not in the limit $T \rightarrow \infty$; since the replacement of T by ρT transforms discrete time and F into real variables, rescaling time before taking a limit is necessary in the case of a random walk. But in the Poisson case one can rephrase the standard passage to the limit into a form that avoids rescaling time. One imagines N independent and identically distributed Poisson processes $F_n(t)$, then one forms

$$\tilde{F}_N(t) = \sum_{n=1}^N F_n(t),$$

and one finds that,

$$\text{weak limit}_{N \rightarrow \infty} N^{-1/\alpha} \tilde{F}_N(t) = B(t), \text{ with } 1/H = \alpha = 2.$$

This rephrasing of the passage to the limit is not important in the Poisson case, but it has the virtue that it continues to make sense in the case of FSP. The theory of the addition of independent identically distributed random variables tells us that, for fixed t , one must have $0 < \alpha \leq 2$ and the limit is a stable rv: Gaussian for $\alpha = 2$, and Lévy stable for $0 < \alpha < 2$. Table I lists the values of α for a selection of pulse shapes, and Section 7 gives an example of derivation.

4.2. The lateral limit problem as applied to FSP; the term “lateral”

Let us now observe that forming $\tilde{F}_N(t)$ for a semi-random FSP, and then letting $N \rightarrow \infty$ is equivalent to viewing $F(t)$ as a function of both t and $C' + C''$. Then we keep C'/C'' fixed and let $C' \rightarrow \infty$ and $C'' \rightarrow \infty$. One can think of the axis of $C' + C''$ as orthogonal to the axis of t , hence the term *lateral*.

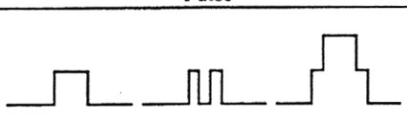

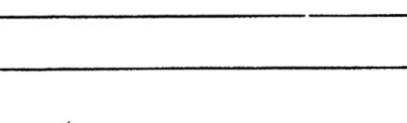
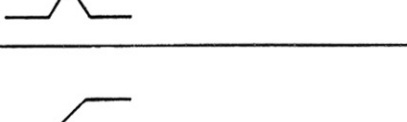
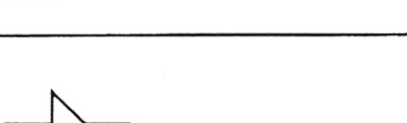
For example, replacing C' by $C'_1 + C'_2$ and C'' by $C''_1 + C''_2$ can be interpreted as follows. The pulses corresponding to C'_1 and C''_1 can be called red and said to add up to $F_1(t)$, and the pulses corresponding to C'_2 and C''_2 can be called blue and said to add up to $F_2(t)$; all pulses together yield $\tilde{F}(2, t) = F_1(t) + F_2(t)$. In order to represent $F(N, t)$ graphically on a page, one must rescale it by the factor $N^{-1/\alpha}$.

4.3. An important corollary of the results in Table I: global dependence can be either “uniscaling” ($H = 1/\alpha$) or “pluriscaling” ($H \neq 1/\alpha$)

Table I uses α to denote the usual exponent of stability. A first glance shows familiar r.f.’s among the lateral limits. The SLM has independent increments and satisfies $0 < \alpha < 2$ and $H = 1/\alpha$. The FBM, except for $H = 1/2$, has globally dependent Gaussian increments, so that $\alpha = 2$, but satisfies $H \neq 1/\alpha = 1/2$.

For many years, my studies of global dependence concentrated on these two examples (broadened by fractional Lévy motion) and on the R/S statistic. This concentration made me think that global dependence can be tested and measured by a single exponent H . Given α , I thought that global dependence could be defined by $H \neq 1/\alpha$ and measured by $H - 1/\alpha$. It is a pleasure to note that I was prudent enough *not* to write of (in)dependence but rather of (R/S)-(in)dependence, but I confess that discrepancies were expected to belong to "mathematical pathology." I was thoroughly mistaken.

TABLE I

Pulse	$0 < \delta < 1$	$1 < \delta < 2$	$2 < \delta$
	$\beta_0 = 0$	$\beta_0 = 0$	FBM: $\alpha = 2; \nu = \delta$
	$\beta_0 = \beta_i$	SLM: $\alpha = \nu = \delta$ $\beta_0 = \beta_i = 0$	Excluded: Divergence for low λ s is non renormalizable
		$\nu = \infty$	$\nu = \delta$
		FBM: $\alpha = 2$	Excluded: Divergence for low λ s is non renormalizable
	Excluded	SLM $\alpha = \delta$ $1/2 < H < 1$	FBM: $\alpha = 2; \nu = \delta$ $H < 1/2$

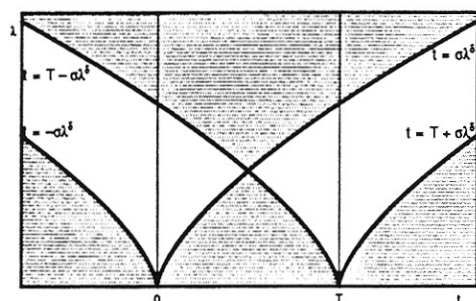


FIGURE 2

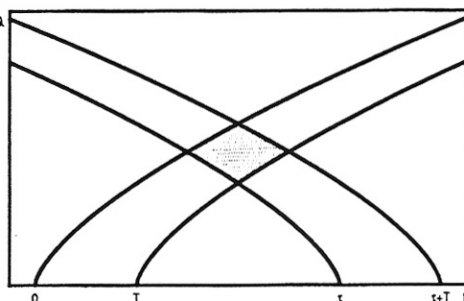


FIGURE 3

5. ADDRESS DIAGRAMS AND THE MECHANISM OF NON-LINEAR GLOBAL DEPENDENCE IN FSP

5.1. Address function and diagram; characteristic function of F

Once again, each semi-random pulse is represented by a point in the address space $\{t, \lambda\}$. Therefore, when a pulse template includes one or more time-intervals, one can define an address function and an address diagram, as seen momentarily. Examples are provided by this paper's illustrations.

When there is one time interval it will be denoted by $[0, T]$. The *address* function of $[0, T]$, denoted by $\varphi(t, \lambda | 0, T)$, will be the contribution of the pulse represented by $\{t, \lambda\}$ to $F(T)$, where we set $F(0) = 0$, and - as may be the case - to other increments of F as well.

Since $\{t, \lambda\}$ has a Poisson distribution, and the contributions of the address points to F are additive, the logarithmic characteristic function of F is simply the integral

$$\psi(\xi) = C(C', C'') \int (e^{i\xi\varphi(t, \lambda|0, T)} - 1) |\lambda|^{-\delta-1} d\lambda$$

carried over the address space. Here,

$$C(C', C'') = C' \text{ for } \lambda > 0 \text{ and } C(C', C'') = C'' \text{ for } \lambda < 0.$$

When pulses are constant outside an interval, it follows that $\varphi = 0$ over a large *excluded* part of the address space. The analytic form of φ may depend on (λ, t) , making it convenient to integrate $\psi(\xi)$ separately over a number of distinct domains. The boundaries of these domains will be said to form an *address diagram*, as exemplified in the case of cylinders by Figure 2.

5.2. Joint address diagrams and the nature of dependence in an FSP; its strongly non-linear character

The interdependence between the ΔF corresponding to two intervals $[0, T]$ and $[t, t+T]$, with $t \geq T$, is investigated by superposing their individual address diagrams. The result can be confusing, therefore it is important to look closely at the simplest case, when the pulse reduces to two steps, either cancelling (up then down or down then up) or noncancelling (up and up or down and down). We assume $0 < \delta < 1$ to postpone the need to face convergence problems.

For both of these pulses $\varphi(t, \lambda | 0, T)$ and $\varphi(t, \lambda | t, t+T)$ are either $= \lambda$ or $= 0$. Therefore, the domain of integration of φ splits into three parts: The first affects only $[0, T]$ and defines a r.v. Δ_{LL} . The second affects both $[0, T]$ and $[t, t+T]$ and defines a r.v. Δ_O . The third affects only $[t, t+T]$ and defines a r.v. Δ_{RR} .

That is,

$$\begin{aligned} \Delta_L &= F(T) - F(0) = \Delta_{LL} + \Delta_O \\ \Delta_R &= F(t+T) - F(t) = \pm \Delta_O + \Delta_{RR}. \end{aligned}$$

Once again, Δ_{LL} , Δ_O and Δ_{RR} are independent and the symbol \pm means $+$ when the pulse is noncancelling and $-$ when the pulse is cancelling.

The origin and the nature of the resulting dependence between Δ_{LL} and Δ_{RR} are clearest when t is several times larger than T (Figure 2), and easiest to follow if $C' = C''$ and their common value C is made to increase from 0.

The theory of Lévy stable variables suggests that, except for small C , the order of magnitude $\lambda(C)$ of the largest contribution to Δ_{LL} and Δ_{RR} , hence to Δ_L and Δ_R , satisfies $C[\lambda(C)]^\delta = 1$, i.e., $\lambda(C) \sim C^{1/\delta}$.

The underlying argument does not apply for small C , i.e., below the cross-over at $C \sim T$. For smaller C , Δ_L and Δ_R are both dominated by $\lambda \sim \pm T^{1/\delta}$. To the contrary, the addend Δ_O comes exclusively from tail values of λ , and is small. The dependence is therefore small, except when $t = T$, when we deal with neighboring intervals.

When C is very large, one finds that $\Delta_{LL} \propto C^{1/\delta}$ and $\Delta_{RR} \propto C^{1/\delta}$, while $\Delta_O \propto C^{1/2}$. Again, the dependence of Δ_L and Δ_R is small.

We are left with the midrange where $C \propto t$. There, both Δ_L and Δ_R are of the order of magnitude of the largest contributing λ . Once again, the dependence between Δ_L and Δ_R can be either of the order of 1 or small, according to whether or not it is the case that the same largest λ contributes to *both* Δ_L and Δ_R , through its contribution to Δ_O . Strong dependence has a probability of the order of 1, say 1/3.

As $C \rightarrow \infty$, the midrange $t \sim C$ also $\rightarrow \infty$, which shows that Δ_L and Δ_R become independent and do so in non-uniform fashion.

6. DISCUSSION OF TABLE I: EFFECTS OF PULSE SHAPE ON THE ADMISSIBLE δ , AND ON THE LATERAL ATTRACTOR

Changing the pulse shape greatly affects an FSP. First, it affects the domain of admissible δ s. Next, it affects the attractor of $F(t)$ and in particular the dependence of α on δ . Several examples are summarized in Table I.

The left column of Table I illustrates a selection of pulse templates. Other templates are examined in follow-up papers. A striking immediate observation is that a discontinuous FSP can have continuous lateral attractors, and continuous FSP can have discontinuous lateral attractors.

The right column states that $\delta > 2$ is excluded when the pulse is non-cancelling but is admissible when the pulse is cancelling. In the latter case, the lateral attractor is FBM. In any event, $H < 1/2$ is incompatible with SLM.

The column second from the right shows that $1 < \delta < 2$ is admissible for all pulse templates and all values of δ , and that it yields varied and interesting results. Combining proven facts with inferences based on compelling heuristics, one is tempted to infer the following.

- For continuous pulses having left and right derivatives bounded away from 0 and ∞ , the lateral attractor is FBM.
- For discontinuous or mixed pulses, the lateral attractor is SLM.

The column shows that the dependence of the limit on the template is far more complicated for $0 < \delta < 1$ than for $1 < \delta < 2$. When the pulse template is discontinuous, a formal extrapolation to $0 < \delta < 1$ of the results relative to SLM with $1 < \delta < 2$ is both meaningful and correct. When the pulse is continuous,

the extrapolation to $0 < \delta < 1$ of the results relative to FBM with $1 < \delta < 2$ is meaningless, in fact, the construction diverges. As shown in a follow-up paper, one finds meaningful results by considering the second difference

$$\Delta\Delta F = [F(t+T) - F(t)] - [F(t) - F(t-T)].$$

The case $1 < \delta < 2$; sensitive dependence of the lateral attractor on the pulse templates; a bridge between FBM and SLM. Table I indicates that the attractor is FBM in the case of a smooth conical pulse, but it is SLM if the pulse has a discontinuity, however small, which occurs, for example, when the pulse is a stepped cone, namely a staircase made of many steps up followed by many steps down. "Intuition" suggests that, for finite C , these two pulse forms could not make much of a difference. This is confirmed by a comparative examination of the address diagrams. The diagram corresponding to the stepped cones approximates that of the smooth cone, except for large λ 's. As the number of steps increase, so does the quality of the approximation, and it also spreads up to increasingly large λ 's.

7. PROOFS OF THE CLAIMS IN TABLE I FOR THE CYLINDRICAL PULSES

The parameter σ is set to 1 in this Section. The address points with $\lambda > 0$ and with $\lambda < 0$ contribute to two independent parts of $F(T)$. The calculations leading to their characteristic functions are the same except for different values C' and C'' . We restrict ourselves to the case $\lambda > 0$ and $C' = 1$.

7.1. The l.c.f. of $F(T)$

From Figure 1 it is clear that the address points that contribute to $F(T)$ fall into two domains described as left and right and denoted by \mathcal{D}_L and \mathcal{D}_R . The strip $(\lambda, \lambda + d\lambda)$, where $\lambda > 0$, makes the following contribution to the l.c.f.

$$\begin{aligned} T[(e^{i\xi\lambda} - 1) + (e^{-i\xi\lambda} - 1)]\lambda^{-\delta-1}d\lambda & \quad \text{if } \lambda > T^{1/\delta}, \\ \lambda^\delta[(e^{i\xi\lambda} - 1) + (e^{-i\xi\lambda} - 1)]\lambda^{-\delta-1}d\lambda & \quad \text{if } \lambda < T^{1/\delta}. \end{aligned}$$

Integrating over λ and transforming to the rescaled variables $x = \lambda T^{-1/\delta}$ and $y = \xi T^{1/\delta}$ yields

$$\int_1^\infty [(e^{iyx} - 1) + (e^{-iyx} - 1)]x^{-\delta-1}dx + \int_0^1 [(e^{iyx} - 1) + (e^{-iyx} - 1)]x^{-1}dx.$$

This expression converges for all $\delta > 0$ and is the l.c.f. of a rescaled r.v. independent of T . Therefore, the rescaled increment $T^{-1/\delta}F(T)$ has a distribution independent of T . This is a property of self-affinity with $H = 1/\delta$. We know the mechanisms of self-affinity: in FBM, it is caused by global dependence without long-tailedness, in SLM it is caused by long-tailedness without global dependence, and in FSP it is caused by both long-tailedness and global dependence, acting together with the

same value of H . The next issue is to separate the long-tailedness and dependence aspects.

7.2. The attractor in the case $\delta < 2$. Lateral attraction to symmetric Lévy stable increments with $\alpha = \delta$

Given N independent r.v.'s $F_n(T)$ with the above distribution, the behavior of $\Delta \tilde{F}_N = \sum_{n=1}^N F_n(T)$ depends sharply on the value of δ . When $\delta < 2$, the l.c.f. of $N^{-1/\delta} \Delta \tilde{F}_N$ can be written in the form

$$T \int_{(T/N)^{1/\delta}}^{\infty} (e^{i\xi u} + e^{-i\xi u} - 2) u^{-\delta-1} du \\ + N \int_0^{T^{1/\delta}} (e^{i\xi \lambda N^{-1/\delta}} + e^{-i\xi \lambda N^{-1/\delta}} - 2) \lambda^{-1} d\lambda.$$

When $N \rightarrow \infty$, the first term converges to the well-known l.c.f. of a symmetric Lévy stable r.v. with the stability parameter $\alpha = \delta$ and the scale parameter proportional to $T^{1/\delta}$. The second term is of order $N^{1-2/\delta}$, therefore converges to zero. Hence, for fixed T , $F(T)$ belongs to the symmetric δ -stable domain of attraction.

7.3. The dependence structure of $F(T)$ in the case $\delta < 2$. Lateral attraction to a Lévy stable r.f.'s with $\alpha = \delta$ and independent increments

We proceed to the multidimensional structure of the FSP. We show that the multidimensional distributions of the FSP are attracted to that of a symmetric SLM. To prove it we need to find the limit in distribution of a linear combination

$$\sum_{i=1}^k \xi_i N^{-1/\delta} \sum_{n=1}^N F_n(T_i),$$

where $F_n(T_i)$, $i = 1, 2, \dots, k$, are nonoverlapping increments of an FSP copy F_n , over (possibly different) time spans T_i . The limit should be the corresponding linear combination of independent δ -stable variables with scale parameters proportional to the respective $T_i^{1/\delta}$. Here, we consider increments over two disjoint time spans T_1 and T_2 , i.e., $k = 2$. The general case is not mathematically more involved but requires an overload of notation. Our assertion follows from the same reasoning as in the one-dimensional case, if we show that the expression

$$N \int_{\mathcal{D}} [(e^{i(\xi_1 - \xi_2)\lambda N^{-1/\delta}} - 1) - (e^{i\xi_1 \lambda N^{-1/\delta}} + e^{-i\xi_2 \lambda N^{-1/\delta}} - 2)] \lambda^{-\delta-1} dtd\lambda$$

converges to zero, where \mathcal{D} is the dotted region depicted in Figure 3. But this expression is again of order $N^{1-2/\delta}$, which establishes the result.

7.4. The attractor in the case $\delta > 2$. Lateral attraction to Gaussian increments

This case is very different, since the variance $EF^2(T)$ is finite, therefore the rescaled sum $N^{1/2}[\tilde{F}_N - E\tilde{F}_N]$ is asymptotically Gaussian, i.e. stable with $\alpha = 2$.

7.5. The dependence structure of $F(T)$ in the case $\delta > 2$. Lateral attraction to a fractional Brownian r.f.'s with $H = 1/\delta < 1/2$

Examination of the characteristic function of a linear combination of nonoverlapping increments of the FSP shows that it has the second derivative at 0, i.e. every linear combination has the second moment, and multidimensional distributions of the FSP are attracted to multidimensional Gaussian distributions. Note that this Gaussian process in the limit must have stationary increments and be self-affine with the constant $H = 1/\delta$ since these properties are preserved under convolution and convergence in distribution. It is well-known that FBM with $H = 1/\delta$ is the unique Gaussian process satisfying these requirements.

7.6. Some semi-random FSP belonging to the domain of standard attraction of a semi-random self-affine FSP

We replace the constants C' and σ with the slowly varying (at ∞) functions $\gamma(\lambda)$ and $\sigma(\lambda)$ such that the function $w = \sigma(\lambda)\lambda^\delta$ is monotonically increasing. Writing the inverse function of $w(\lambda)$ as $\lambda = w^{1/\delta}L(w)$ yields the identity

$$L(w) = \sigma^{-1/\delta}(w^{1/\delta}L(w)),$$

which is an implicit equation for $L(w)$ and will be used momentarily without having to be solved.

When $\lambda > 0$, the strip $(\lambda, \lambda + d\lambda)$ makes the following contribution to the l.c.f.

$$\begin{aligned} T[(e^{i\xi\lambda} - 1) + (e^{-i\xi\lambda} - 1)]\gamma(\lambda)\lambda^{-\delta-1}d\lambda & \quad \text{if } \lambda > T^{1/\delta}L(T), \\ \sigma(\lambda)\lambda^\delta[(e^{i\xi\lambda} - 1) + (e^{-i\xi\lambda} - 1)]\gamma(\lambda)\lambda^{-\delta-1}d\lambda & \quad \text{if } \lambda < T^{1/\delta}L(T). \end{aligned}$$

Integrating over λ and transforming to the rescaled variables $x = \lambda T^{-1/\delta}L^{-1}(T)$ and $y = \xi T^{1/\delta}L(T)$ yields

$$\begin{aligned} & \int_1^\infty [(e^{iyx} - 1) + (e^{-iyx} - 1)]x^{-\delta-1}\{\gamma[xT^{1/\delta}L(T)]L^{-\delta}(T)\}dx \\ & + \int_0^1 [(e^{iyx} - 1) + (e^{-iyx} - 1)]x^{-1}\{\gamma[xT^{1/\delta}L(T)]\sigma[xT^{1/\delta}L(T)]\}dx. \end{aligned}$$

The integral over $(1, \infty)$ converges for all $\delta > 0$. Assuming that the functions γ and σ are such that also the second integral is finite (e.g. γ and σ are both bounded in the neighborhood of zero), the above expression gives the l.c.f. of a rescaled r.v. $T^{-1/\delta}L^{-1}(T)F$.

This l.c.f. may depend on T . If so, the following question arises: under what conditions on $\gamma(\lambda)$ and $\sigma(\lambda)$, and hence on $L(\lambda)$, does this l.c.f. converge to that prevailing in the FSP case $\gamma(\lambda)\sigma(\lambda) = C'\sigma$? (We know that the product $C'\sigma$

determines the type of an FSP.) Because of the identity that links $\sigma(\lambda)$ and $L(w)$ the two factors written between braces are identical (asymptotically, when $T \rightarrow \infty$); therefore the two halves of the l.c.f. yield the same condition on convergence. It is

$$\lim_{\lambda \rightarrow \infty} \gamma(\lambda) \sigma(\lambda) = C' \sigma.$$

In other words, the functions $1/\gamma(\lambda)$ and $\sigma(\lambda)$ must vary slowly with λ and asymptotically proportionally to each other.

The question concerning whether or not these conditions are also necessary has not been addressed yet.

Acknowledgments. In 1977-78, I studied semi-random FSP with a one-dimensional t and a multi-dimensional F : early simulations for the second row of Table I were performed by M.R. Laff, and I conjectured (then J. Hawkes proved, that the closure of the set of values of $F(t)$ remains of Hausdorff-Besicovitch dimension $D = \delta$. In the mid-1980s, I studied an application of semi-random FSP with a multi-dimensional t and one-dimensional F : early simulations performed by S. Lovejoy are reported in [4]. I also made conjectures concerning the random FSP; in due time, they were proved in [1] and [3]. This paper was discussed at length with R. Cioczek-Georges. Diagrams were drawn by H. Kaufman.

REFERENCES

- [1] R. Cioczek-Georges and B.B. Mandelbrot, *A class of micropulses and antipersistent fractional Brownian motion*, to appear.
- [2] R. Cioczek-Georges and B.B. Mandelbrot, *Alternative micropulses and fractional Brownian motion*, to appear.
- [3] R. Cioczek-Georges, B.B. Mandelbrot, G. Samorodnitsky and M. Taqqu, *A class of cylindrical pulse processes*, to appear in *Bernoulli*.
- [4] S. Lovejoy and B.B. Mandelbrot, *Fractal properties of rain and a fractal model*, *Tellus* **37A** (1985), 209-232.
- [5] B.B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman, New York, 1982.
- [6] B.B. Mandelbrot, *Fractal sums of pulses I: Affine convolution, global dependence, and a versatile family of self-affine random functions*, to appear.