Fractal sums of pulses and a practical challenge to the distinction between local and global dependence

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Abstract. The partly random fractal sums of pulses (PFSP) are a family of random functions that depend on a kernel function K and at least two positive parameters C and δ. Given K, the construction of $F(t; C, \delta)$ consists in adding N affine versions of a pulse as follows. The pulse height Λ and its width W are random variables related by w/λ^{δ} = a constant. The width is distributed according to the Poisson measure $Cw^{-1}dwdt$ in the "address plane" of coordinates w and t. For finite C, the increments of $F(t; C, \delta)$ fail to be strongly mixing therefore they exhibit global dependence. Indeed some PFSP resemble the icon of global dependence, which is fractional Brownian motion (FBM) with $H \neq 1/2$. When the presence of strong mixing must be tested empirically, many tests rely on the comparison of two exponents of diffusion: that of a r.f. $X(t)$ and of a "shuffled" r.f. $\tilde{X}(t)$ whose increments for $\Delta t = 1$ (say) are independent and follow the same distribution as $X(t)$. In FBM, the diffusion exponent is H for the process itself and $\frac{1}{2}$ for its shuffled variant. Therefore, $H \neq 1/2$ is a token of global dependence. For the Lévy stable motion (LSM) to the contrary, the diffusion exponent is the same as for the independent process with the same marginal distribution. The PFSP are not so clear-cut. The dependence is always global. But consider those tests of globality versus locality that, like R/S testing, are founded on the exponent of diffusion. Those tests will classify the dependence in many PFSP as local. Therefore, the PFSP are challenging borderline cases, while the conceptual fact is important, more important is the concrete fact that their rich properties and the absence of arbitrary grids make them excellent candidates for modeling phenomena that combine global dependence with long distribution tails. Furthermore, related structures that are discussed elsewhere, namely, the multifractal measures obtained as products of pulses, are grid-free and provide a great improvement over the now-classical multifractal measures generated by multiplicative cascades in a grid.

Keywords. Fractal sums of pulses. Global statistical dependence. Infinite variance. Lateral attractors. Lévy flights (generalized). Pulses.

1 INTRODUCTION

1.1 "Normal" and "anomalous" fluctuations, the Noah and Joseph effects and the distinction between the local and the global form of statistical dependence

The "normal" model of natural fluctuations is the Wiener Brownian motion process (WBM). By this standard, however, many natural fluctuations exhibit clear-cut "anomalies". Some are due to large discontinuities ("Noah Effect"), others to non-negligible global statistical dependence ("Joseph Effect"). I have long argued - especially in [7] and [9] - that the geometric features of surprisingly many of these anomalous aspects of nature are fractal. For example, for many large "Noah" discontinuities, the tail probability distribution follows a power-law. That is, for large u, a discontinuity U that exceeds the value u has a probability of the form $Pr(U > u) \sim u^{-\alpha}$, with α a positive constant. Second example: for large lags s, many globally correlated "Joseph" fluctuations have a correlation function of the power-law form $C(s) \sim 2H(2H-1)s^{2H-2}$, with $1/2 < H < 1$. [7] shows that one can model various instances of the Noah effect by the classical process of SLM, and various instances of the Joseph effect by the process of FBM.

In the classical cases of WBM, FBM and SLM, the distinction between local and global dependence is straightforward. The increment of WBM and LSM are independent, hence locally (short-run) dependent. But if $H \neq \frac{1}{2}$, the increments of FBM are globally (long-run) dependent. This contrast makes the task of laying down a boundary between local and global that can be implemented by a statistical test.

SLM and FBM, however, are far from exhausting the anomalies found in nature. In particular, many phenomena exhibit *both* the Noah and the Joseph effects and fail to be represented by either SLM or FBM. Hence, fractal modeling of nature demands "bridges," namely, r.f.'s that combine the infinite variance feature that is characteristic of SLM and the global dependence feature that is characteristic of FBM. Such is the case of a new family of random functions described in this paper and called "partly random sums of pulses" or PFSP. A very different and more classical bridge, is *fractional Lévy motion.* It is interesting mathematically, but not versatile and has found no concrete use.

Two general issues come up.

Firstly, when the tail exponent α satisfies $\alpha < 2$, the exponents of PFSP satisfy $\alpha = 1/H$. This identity strongly links the tail and the correlation. As a result, while the PFSP are globally dependent, their dependence seems from certain viewpoints to be only local.

Furthermore, the mathematical theories of SLM and FBM are known to exhibit striking formal parallels as well as often-noted discrepancies. One major discrepancy is in the allowable range of the exponent H which is defined by the self-affinity condition that the distribution of $T^{-H}F(T\rho)$ is independent of T. For SLM, $1/2 < H < \infty$, while for FBM, $0 < H < 1$. This mismatch poses a conceptual challenge. Being unexpected, the formal parallels between SLM and

FBM are sometimes described as miraculous. But in fact they have deep roots worth exploring. Some such roots are brought forward by the PFSP.

1.2 Inspiration for the partly random fractal sums of pulses (PFSP)

The PFSP focus on a property of *Lévy flight* or *Lévy stable motion*, modify it, achieve a major generalization. SLM is well-known to be the sum of an infinity of step-functions whose sizes follow a power-law distribution. This paper preserves this classical distribution of sizes but replaces the step-functions by suitable affine reductions or dilations of more general "templates", obtained as graphs of a *kernel function* K. The result can be described as being an *affine convolution.* A template whose kernel function only varies on a (bounded) interval will be called *pulse.*

1.3 Sketch of the construction of the PFSP

A PFSP built with only positive pulses, involves a single parameter C. It will be denoted as $F(t; C, \delta)$ and constructed as follows. A pulse's height and location are ruled by the same distribution as in a Lévy flight: the probability of the point $\{\lambda, t\}$ being found in an elementary rectangle of the (λ, t) address plane is $C\lambda^{-\delta-1}d\lambda dt$. This probability is carried over from the original paper on cutouts or tremas on the line, [6].

Recall that in a Lévy flight, the exponent δ is constrained to satisfy $0 < \delta < 2$. Many applications view this limitation as unduly restrictive. A major immediate novelty is that in an PFSP, the constraints are far weaker: either $\delta > 0$ or $\delta > 1$, depending on the kernel function K. The pulse's width W (the length of the smallest interval in which the pulse varies) satisfies $W = \sigma|\Lambda^{\delta}|$, where $\sigma > 0$ is a scale constant, implying that $Pr\{W > w\} \propto w^{-1}$. The reason why the resulting $F(t, C, \delta)$ is called *semi-random* is because Λ and W are functionally related. In fully random PFSP, as discussed in [4], Λ and W are statistically independent.

When the pulses can be either positive or negative, there are two parameters C' and C'' and the PFSP will be denoted as either $F(t : C', C'', \delta)$ or as $F(t; C, \delta)$ where C represents the combination of two posoitive real numbers C' and C''.

1.4 The roles of δ as tail exponent and $H = 1/\delta$ as self-affinity exponent

The key property of the PFSP is that for many kernels the same exponent δ plays a fundamental role for both longtailedness and global dependence.

A first role of δ is that, over any time increment Δt, the increment of F follows a power-low probability distribution of exponent δ.

A second role of δ is that, by design, all PFSP are self-affine, namely, invariant under an affine rescaling of the form $T^{-H}X(T\rho)$ with the exponent $H = 1/\delta$. (In physicists' language, each PFSP defines its own "class of universality" with respect to a suitable affinity).

As a result, there is no counterpart to the standard study of attraction of a random function $X(t)$ to a limit such as Wiener or fractional Brownian motion or Lévy motion.

1.5 When $\delta < 2$, the link between the tail and self-affinity exponents complicates the testing for global dependence

Assume $\delta < 2$, write $X = F(t+1, c, \delta) - F(t, c, \delta)$, and "shuffle" the X to obtain a sequence of independent r.v. $\tilde{X}$ with the same distribution as X. That sum is attracted by the LSM. Therefore, both the original X and the shuffled $\tilde{X}$ have the same self-affinity exponent $H = 1/\delta$. Tests like R/S which are based on the comparison of those exponents will declare the X to be (R/S) independent. But we know that they are globally dependent.

1.6 Lateral rescaling and lateral attractors

It is important to introduce an alternative "lateral" rescaling, and show that each semi-random PFSP has an interesting "lateral attractor." This attractor is sometimes a Lévy flight (SLM), a conclusion that might have been vaguely expected, because the construction starts with the Lévy measure. But there exists another possibility, the lateral attractor may be a fractional Brownian motion (FBM), which is a surprise. This is readily implemented when $\delta > 2$. The possibilities of merging FBM and SLM into a broader common universe is enlightening. It establishes the existence of deep links - already mentioned and described as desirable - between two independent theories that are known to be very different yet strikingly parallel. Furthermore, the attractor may also be neither SLM nor FBM.

For finite C, the increments of the $F(t; C, \delta)$ are globally dependent, but when the lateral limit is SLM, they become independent. Thus, the PFSP bring altogether new conceptual light on the probabilistic notion of global dependence and the related notion that a process is attracted by another process. Indeed, depending on the K, global dependence is expressed in either of two ways: a) by a special exponent H familiar in such known contexts as FBM, or b) by a prefactor rather than a special exponent.

1.7 Summary

This paper is far from exhaustion but shows the great variety of distinct behaviors that can be found in an PFSP, as we vary δ and three properties of the kernels:

a) Discontinuous versus very smoothly continuous;

b) Canceling (i.e., vanishing outside the interval in which they vary) versus non-canceling.

c) Atoms versus bursts. This is a useful but elusive distinction. When the pulse is made of a rise and fall followed later by another rise and fall, it can be decomposed into a burst of two indecomposable or atomic pulses.

The resulting templates are exemplified in Table I: cylinders (one discontinuous *rise* followed after the time W by one discontinuous *fall*), multiple steps, cones (uniform rate rise followed by uniform rate fall), and uniform rate rises. Other templates are discussed in forthcoming papers.

1.8 Related papers

They investigate: A) additional classes of PFSP with kernels for which convergence demands second differences;

B) semi-random PFSP with $W = \sigma \Lambda^{\theta}$, where $\theta \neq \delta$; they are not self-affine.

C) PFSP that are fully random in the sense that W is statistically independent of Λ with the measure $\propto w^{-\theta-1}dw$, where $0 < \theta < 1$; they are shown in [4] to be self-affine with $H = (1-\delta)/\theta$;

D) fractal sums of *micropulses* (FSM) which generate (FBM); see [2,3];

E) multifractal products of pulses [1]; and

F) concrete applications [5].

2 DEFINITIONS

2.1 Stationarity and self-affinity

The function $F(t)$ is said to have *stationary increments* if the translated function

$$F(t_0 + t) - F(t_0)$$

has the same distributions for all values of t_0. A function $F(t; C, \delta)$ is said to be *self-affine of exponent* $H > 0$ when the rescaled function

$$\rho^{-H}[F(t_0 + \rho t; C, \delta) - F(t_0; C, \delta)]$$

has the same distributions for all values of t_0 and $\rho > 0$. Some authors denote self-affinity by the term, self-similarity, which I proposed long ago but abandoned because it is inappropriate.

2.2 Pulse templates, pulses, affine convolutions, and fractal sums of pulses

The graph of $K(t)$, a one-dimension function of a one-dimensional variable, will be called *generator*, or *pulse template*, if $K(t)$ is constant outside an interval; we shall set the shortest such interval to be of length 1.

A *pulse* is a translated affine transform $K\left(\frac{t-t_n}{w_n}\right)$ of $K(t)$, where λ_n, t_n, and w_n are called the pulse's height, position, and width.

A *sum of pulses* is a function of the form

$$F(t) = \sum \lambda_n K\left(\frac{t-t_n}{w_n}\right).$$

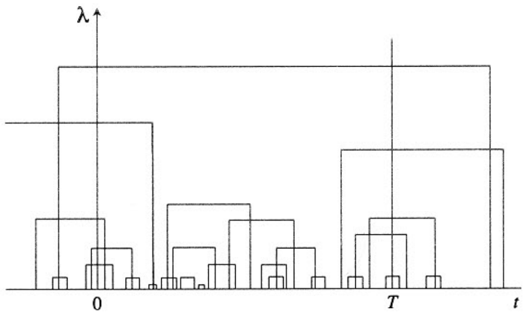


Fig. 1. Example of a fractal sum of pulses.

Figure 1 is an example of a sum of these pulses. When $F(t)$ is a self-affine function, it will be called a *fractal sum of pulses*, FSP.

An *affine convolution* of the sequence $\{\lambda_n, t_n\}$ by the kernel $K(\cdot)$ is obtained when the pulse heights and widths are linked by a relation $w_n = \sigma\lambda_n^\delta$, where $\sigma > 0$ and $\delta > 0$ are prescribed. Thus, the PFSP are affine convolutions of sequences $\{\lambda_n, t_n, w_n\}$.

2.3 The Lévy measure for the probability distribution of pulse height and position

The simplest template is a step function for all n. The distribution of $\{\lambda_n, t_n\}$ that insures self-affinity in that case was discovered by Paul Lévy and is classical. The same distribution continues to be required in all FSP. Let the plane of coordinates t and λ be called *address plane.* Take a rectangle $[\lambda, \lambda+d\lambda] \times [t, t+dt]$ in the address plane, such that $[\lambda, \lambda + d\lambda]$ does not contain $\lambda = 0$. Given an exponent $\delta > 0$ and two scale factors C' and C'', the Lévy measure is

$$C'\lambda^{-\delta-1}d\lambda dt, \text{ if } \lambda > 0, \text{ and } \mathrm{C}''|\lambda|^{-\delta-1}\mathrm{d}\lambda\mathrm{dt}, \text{ if } \lambda < 0.$$

Giving C' and C'' is, of course, equivalent to giving the overall scale $C' + C''$ and the skewness factor $C'/(C' + C'')$.

To define an FSP, the probability of finding an address point (λ, t) in the elementary rectangle is set equal to the Lévy measure. The number of address points in a domain $\mathcal{D}$ in the address space is taken to be a Poisson random variable whose expectation is the integral of the Lévy measure over $\mathcal{D}$. The total number of pulses is countably infinite.

2.4 Semi-random pulses and the probability distribution of pulse widths and position

The simplest pulses are the step functions used by Paul Lévy to generate the SLM. The pulses examined in this paper and illustrated in Table 1 and generalize the step functions. The height Λ is random with the Lévy measure, but $|\Lambda|$ fully determines the width W. The resulting processes are called *semi-random* and denoted as PFSP. To insure that the PFSP is self-affine, one must take

$$W = \sigma|\Lambda|^{\delta}, \text{ where } \sigma > 0.$$

The resulting joint measures of width and position are

$$C'w^{-1}d\lambda dw \text{ and } \mathrm{C''w^{-1}d\lambda dw}.$$

They are independent of δ. So is the probability distribution of W, namely,

$$Pr\{W > w\} \propto w^{-1}.$$

The units in which Λ and W are measured are arbitrary and unrelated. If those units are identical, either $C' + C''$ or σ can be normalized to 1 by changing the unit. However, up to scale, the distribution of an PFSP is determined by $(C' + C'')\sigma$ and the skewness factor $C'/(C' + C'')$. In the sequel, an important role is played by lateral limit theorems that are expressed most conveniently by fixing $C' + C''$ and allowing $\sigma \to \infty$. The consequences of $\sigma \to \infty$ are obvious when the pulses are "cylindrical:" made of a *rise* followed, after a span of $\Delta t = w$, by a *fall* of equal absolute value and opposite sign. Clearly, the contribution to $F(t + T; C, \delta) - F(t; C, \delta)$ from a pulse such that $w > T$ is not a pulse but a solitary rise or fall. Therefore, as $\sigma \to \infty$, each cylindrical pulse reduces to a rise or a fall, and the fact that a pulse has a bounded support becomes less and less significant.

3 SELF-AFFINITY AND THE EXPONENT $H = 1/\delta$; EXISTENCE OF GLOBAL DEPENDENCE

3.1 The self-affinity property of all PFSP

An *admissible* combination of a pulse shape and a value of δ is a combination such that the semi-random PFSP construction yields a well-defined random process $F(t; C, \delta)$. For other values of δ, the construction diverges and δ is called *excluded.* When the construction converges, it is easy to see that the resulting PFSP are self-affine with $H = 1/\delta$. This formula is familiar from the Lévy case when the pulses are step functions but with a striking novelty: $\delta > 2$ provokes an irreducible divergence for SLM. But for PFSP, there is no divergence, that is, δ is unrestricted.

In one case to be described elsewhere, the construction of $F(t + T; C, \delta) - F(t; C, \delta)$ diverges, but that of the second difference $[F(t+T; C, \delta) - F(t; C, \delta)] - [F(t; C, \delta) - F(t - T; C, \delta)]$ converges and is self-affine.

3.2 First corollary of self-affinity: Each PFSP defines a special domain of attraction, hence the standard limit problem concerning random processes becomes degenerate

In the study of random functions, the standard next issue is whether or not there exists an exponent H such that, setting $F(0; C, \delta) = 0$,

$$\text{weak} \lim_{\mathrm{T}\to\infty} \mathrm{T}^{-\mathrm{H}}\mathrm{F}(\rho\mathrm{T}; \mathrm{C}, \delta)$$

is a non-degenerate function of ρ, called the "attractor" of F. The most familiar attractions are WBM and SLM in the case of independence and FBM in the case of dependence.

When $F(t)$ is a PFSP, the standard limit problem does not arise, since in distribution $T^{-H}F(\rho T; C, \delta)$ is independent of T. That is, each PFSP defines its own domain of attraction of exponent $H = 1/\delta$.

Standard domains of attraction to an PFSP. Given a self-affine attractor $X(t)$, the next challenge is to describe its domain of attraction. This is the collection of r.f.'s $G(t)$ for which the rescaling (or renormalizing) function $A(T)$ can be selected so as to insure that

$$\text{weak} \lim_{\mathrm{T}\to\infty} \mathrm{A}^{-1}(\mathrm{T})\mathrm{G}(\rho\mathrm{T}) = \mathrm{X}(\mathrm{t}).$$

Thus far, my study of the domain of attraction of PFSP has limited itself to r.f.'s that are themselves sums of pulses, but involve a density other than Lévy's or a relation other than $w = \sigma\lambda^{\delta}$.

Clearly, weak $\lim_{T\to\infty} T^{-H}F(\rho T; C, \delta)$ is unchanged if the distribution of pulse heights is changed over a bounded interval, for example if pulse height is restricted to $\lambda > \varepsilon > 0$.

For the next obvious change, the example of SLM suggests replacing the constants C' and C'' in the Lévy density by functions $C'(\lambda)$ and $C''(\lambda)$ that vary slowly for $\lambda \to \infty$. And FBM suggests replacing the relation $w = \sigma\lambda^{\delta}$ by $w = \sigma(\lambda)\lambda^{\delta}$, where $\sigma(\lambda)$ is also slowly varying. These changes lead to a nonself-affine generalized PFSP. The questions is whether or not, as $T \to \infty$, there exists a rescaling $A(T)$ that makes $A^{-1}(T)G(T)$ converge weakly to a self-affine PFSP. I have been content with verifying (see Section 7.6) that a sufficient condition is that $C'(\lambda)\sigma(\lambda) \to 1$; in that case, $T^{1/\delta}A(T)$ is the inverse function of $T = \sigma(\lambda)\lambda^{\delta}$, implying that $A(T)$ is slowly varying. The condition $C'(\lambda)\sigma(\lambda) \to 1$ is demanding, resulting in a narrow domain of attraction, a notion discussed in Section 3.4.

3.3 Second corollary of self-affinity: all PFSP are globally dependent

A corollary of Section 3.2 is that if F is a PFSP, then $T^{-H}F(\rho T)$ fails to converge to a standard attractor relative to asymptotically independent increments, namely, either WBM or SLM. This implies that all semi-random PFSP must fail to satisfy the usual criteria that express that dependence between increments is local.

In fact, they are uniformly globally dependent. For example, define for each t the following two functions

- the rescaled finite past $T^{-1/\delta}[F(t) - F(t - Tp; C, \delta)]$
- and the rescaled finite future $T^{-1/\delta}[F(t + Tp; C, \delta) - F(t; C, \delta)]$.

Some pulses contribute to both past and future, hence these random functions of p are not statistically independent. Because of self-affinity, their joint distributions are independent of T. This means that strong mixing is contradicted uniformly for all T.

3.4 Thoughts on the role of limit theorems, given that, in the case of PFSP, the standard limit problem is degenerate

In the light of Section 3.3, let us compare the "attractands" with their attractor. The process of going to the limit to destroy idiosyncratic features of the attractand while preserving "universal" features is expected. This is the case for the most important attractors namely, the nonrandom attractor for the laws of large numbers, the Gaussian r.v. for the central limit theorems, and WBM for the functional central limit theorems. These attractors' domains of attraction are very broad, being largely characterized by the absence of global independence and of significant probabilities for large values. By contrast, when a basin of attraction is narrow, the attractor yields specific information about the attractands. Thus, there is a sharp contrast between broad universality with little information and narrow universality with extensive information.

Down to specifics. For SLM, assuming that the attractands are broadly dependent, their tails must be long and strictly constrained. For FBM, assuming that the tails are short, the dependence must be global and strictly constrained. Similarly, the sufficient condition $C'(\lambda)\sigma(\lambda) \to 1$ in Section 3.2 defines for each PFSP a (partial) domain of attraction that is tightly constrained. The novelty is that with *the tails and the global dependence must involve the same exponent.*

The resulting variety of forms creates a use for additional limit problems that would put order by destroying some of the PFSP's overabundant specifications. That is, the finding in Section 3.3 must spur the search for alternative limit problems, for which the domains of attraction are broader, therefore reveal more "universal" properties of the PFSP. Section 4 will advance one possibility.

4 THE CONCEPT OF "LATERAL LIMIT PROBLEM" AND THE EXPONENT α; FOR PFSP, THE LATERAL ATTRACTOR CAN BE EITHER UNISCALING ($\alpha = \delta = 1/H$) OR PLURISCALING ($\alpha = \min[2, \delta] \neq 1/H$)

4.1 Background of the new "lateral" limit problem

Neither the random walk nor the Poisson process of finite density is self-affine, but both are attracted in the usual way to the Wiener-Brownian motion $B(t)$, which is self-affine with $H = 1/2$. That is, writing $F(0) = 0$, it is true in both cases that $T^{-1/2}F(\rho T)$ depends on T for $T < \infty$, but not in the limit $T \to \infty$. Since the replacement of T by ρT transforms discrete time and F into real variables, rescaling time before taking a limit is necessary in the case of a random walk. But in the Poisson case the standard passage to the limit can be rephrased to avoid rescaling time. One imagines N independent and identically distributed Poisson processes $F_n(t)$, then one forms

$$\tilde{F}_N(t) = \sum_{n=1}^{N} F_n(t),$$

and one finds that,

$$\mathrm{weaklimit}_{\mathrm{N}\to\infty}\mathrm{N}^{-1/\alpha}\tilde{\mathrm{F}}_{\mathrm{N}}(\mathrm{t}) = \mathrm{B}(\mathrm{t}), \text{ with } 1/\mathrm{H} = \alpha = 2.$$

This rephrasing of the passage to the limit is not important in the Poisson case, but it has the virtue that it continues to make sense in the case of PFSP. The theory of the addition of independent identically distributed random variables tells us that, for fixed t, one must have $0 < \alpha \leq 2$ and the limit is a stable rv: Gaussian for $\alpha = 2$, and Lévy stable for $0 < \alpha < 2$. Table I lists the values of α for a selection of pulse shapes, and Section 7 gives an example of derivation.

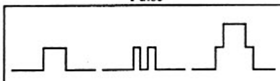
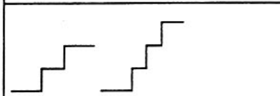
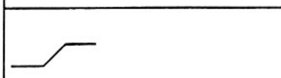
Pulse	$0 < \delta < 1$	$1 < \delta < 2$	$2 < \delta$
	$\beta_0 = 0$	$\beta_0 = 0$	FBM: $\alpha = 2$; $\nu = \delta$
	SLM: $\alpha = \nu = \delta$		
	$\beta_0 = \beta_i$	$\beta_0 = \beta_i = 0$	Excluded: Divergence for low λs is non renormalizable
		$\nu = \infty$	$\nu = \delta$
		FBM: $\alpha = 2$	
		$\nu = \infty$	Excluded: Divergence for low λs is non renormalizable
	Excluded	SLM $\alpha = \delta$ $1/2 < H < 1$	FBM: $\alpha = 2$; $\nu = \delta$ $H < 1/2$

Table 1

4.2 The lateral limit problem as applied to PFSP; reason for the term "lateral"

Let us now observe that forming $\tilde{F}_N(t)$ for a semi-random PFSP, and then letting $N \to \infty$ is equivalent to viewing $F(t)$ as a function of both t and $C' + C''$. Then we keep C'/C'' fixed and let $C' + C'' \to \infty$ and $C'' \to \infty$. One can think of the axis of $C' + C''$ as orthogonal to the axis of t, hence the term *lateral*.

For example, replacing C' by $C_1' + C_2'$ and C'' by $C_1'' + C_2''$ can be interpreted as follows. The pulses corresponding to C_1' and C_1'' can be called *red* and said to add up to $F_R(t)$, and the pulses corresponding to C_2' and C_2'' can be called *blue* and said to add up to $F_B(t)$; all pulses together yield $\tilde{F}(2,t) = F_R(t) + F_B(t)$. In order to represent $F(N,t)$ graphically on a page, one must rescale it by the factor $N^{-1/\alpha}$.

4.3 An important corollary of the results in Table I: global dependence can be "pluriscaling" ($H \neq 1/\alpha$), like in the limit case FBM; but it can also be "uniscaling" ($H = 1/\alpha$), like in the limit case LSM

Table I uses α to denote the usual exponent of stability. A first glance shows familiar r.f.'s among the lateral limits. The SLM has independent increments and satisfies $0 < \alpha < 2$ and $H = 1/\alpha$. The FBM, except for $H = 1/2$, has globally dependent Gaussian increments, so that $\alpha = 2$, but satisfies $H \neq 1/\alpha$.

In the nineteen sixties and for some time later, I carried out studies of global dependence that are reproduced in [7]. They concentrated on these two examples (broadened by fractional Lévy motion) and on the R/S statistic. Those examples made me think that global dependence can be tested and measured by a single exponent H. Given α, I thought that global dependence could be defined by $H \neq 1/\alpha$ and measured by $H - 1/\alpha$. It is a pleasure to note that I was prudent enough *not* to write of independence and dependence but rather of (R/S)-independence and (R/S)-dependence. But I confess having expected that any example beyond this latter distinction would belong to "mathematical pathology."

The PFSP show that I was thoroughly mistaken.

5 ADDRESS DIAGRAMS AND THE MECHANISM OF NON-LINEAR GLOBAL DEPENDENCE IN PFSP

5.1 Address function and diagram; characteristic function of F

Once again, each semi-random pulse is represented by a point in the address space $\{t, \lambda\}$. Therefore, when a pulse template includes one or more time-intervals, one can define an address function and an address diagram, as seen momentarily. Examples are provided by this paper's illustrations.

When there is one time interval it will be denoted by $[0, T]$. The *address* function of $[0, T]$, denoted by $\varphi(t, \lambda \mid 0, T)$, will be the contribution of the pulse represented by $\{t, \lambda\}$ to $F(T; C, \delta)$, where we set $F(0; C, \delta) = 0$, and - as may be the case - to other increments of F as well.

Since $\{t, \lambda\}$ has a Poisson distribution, and the contributions of the address points to F are additive, the logarithmic characteristic function of F is simply the integral

$$\psi(\xi) = C(C', C'') \int (e^{i\xi\varphi(t,\lambda|0,T)} - 1)|\lambda|^{-\delta-1} d\lambda$$

carried over the address space. Here,

$$C(C', C'') = C' \text{ for } \lambda > 0 \text{ and } \mathrm{C}(\mathrm{C}', \mathrm{C}'') = \mathrm{C}'' \text{ for } \lambda < 0.$$

When pulses are constant outside an interval, it follows that $\varphi = 0$ over a large *excluded* part of the address space. The analytic form of φ may depend on (λ, t), making it convenient to integrate $\psi(\xi)$ separately over a number of distinct domains. The boundaries of these domains will be said to form an *address diagram*. The case when the pulses are cylinders is illustrated in Figure 2. The address points in the dotted arrows have no effect.

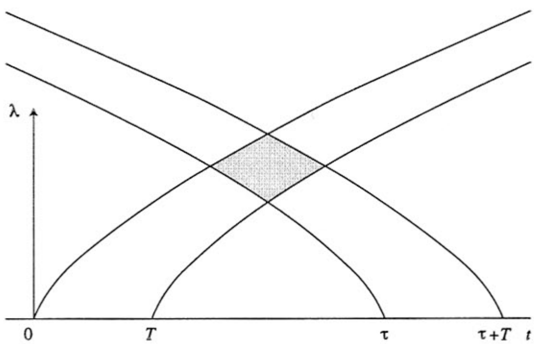


Fig. 3. Address diagram when pulses are cylinders.

5.2 Joint address diagrams and pictorial illustration of the reason for interdependence between the ΔF corresponding to two non-overlapping intervals; its strongly non-linear character.

The two intervals being $[0, T]$ and $[t, t+T']$, with $t \geq T$, is denoted by cross BUG. It it is important to fully understand the simplest case when the pulse reduces to two steps, either cancelling (up then down or down then up) or noncancelling (up and up or down and down). The case $0 < \delta < 1$ raises no convergence problem.

For both of these pulses $\varphi(t, \lambda \mid 0, T)$ and $\varphi(t, \lambda \mid t, t+T')$ are either $= \lambda$ or $= 0$. Therefore, dotted areas of Figure 3 have no effect and the domain of integration of φ splits into three parts: The first is denoted by horizontal hatching. It affects only $[0, T]$ and defines a r.v. Δ_{LL}. The second is denoted by cross-hatching. It affects both $[0, T]$ and $[t, t+T']$ and defines a r.v. Δ_O. The third is denoted by vertical hatching. It affects only $[t, t+T]$ and defines a r.v. Δ_{RR}.

That is,

$$\Delta_L = F(T; C, \delta) - F(0; C, \delta) = \Delta_{LL} + \Delta_O$$

$$\Delta_R = F(t+T; C, \delta) - F(t; C, \delta) = +\Delta_O + \Delta_{RR}$$

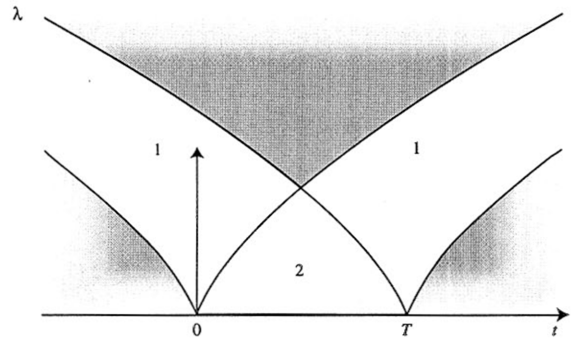


Fig. 4. Joint address diagrams and pictorial illustration of the reason for interdependence between the ΔF corresponding to two non-overlapping intervals.

Once again, Δ_{LL}, Δ_O and Δ_{RR} are independent and the symbol $\pm$ means $+$ when the pulse is noncancelling and – when the pulse is cancelling.

The origin and the nature of the resulting dependence between Δ_{LL} and Δ_{RR} are clearest when t is several times larger than T and easiest to follow if $C' = C''$ and their common value C is made to increase from 0.

The theory of Lévy stable variables correctly suggests the following: except for small C, the order of magnitude $\lambda(C)$ of the largest contribution to Δ_{LL} and Δ_{RR}, hence to Δ_L and Δ_R, satisfies $C[\lambda(C)]^\delta = 1$, i.e., $\lambda(C) \sim C^{1/\delta}$.

For small C, below the cross-over at $C \sim T$, both Δ_L and Δ_R continue to be dominated by $\lambda \sim \pm T^{1/\delta}$. To the contrary, the addend Δ_O comes exclusively from tail values of λ, and is small. The dependence is therefore small, except when $t = T$, which corresponds to neighboring intervals.

When C is very large, $\Delta_{LL} \propto C^{1/\delta}$ and $\Delta_{RR} \propto C^{1/\delta}$, while $\Delta_O \propto C^{1/2}$. Again, the dependence of Δ_L and Δ_R is small.

We are left with the midrange where C is of the order of t. There, both Δ_L and Δ_R are of the order of magnitude of the largest contributing λ. Once again, the dependence between Δ_L and Δ_R of the order of 1 if the same largest λ contributes to Δ_0, hence *both* Δ_L and Δ_R. Otherwise the dependence between Δ_L and Δ_R is small. Strong dependence has a probability of the order of 1, say 1/3.

As $C \to \infty$, the midrange where t is of the order of C also $\to \infty$, which shows that Δ_L and Δ_R become independent and do so in non-uniform fashion.

6 DISCUSSION OF TABLE I: EFFECTS OF PULSE SHAPE ON THE ADMISSIBLE δ, AND ON THE LATERAL ATTRACTOR

Changing the pulse shape greatly affects an PFSP. First, it affects the domain of admissible δ's. Next, it affects the attractor of $F(t)$ and in particular the dependence of α on δ. Several examples are summarized in Table I.

The left column of Table I illustrates a selection of pulse templates. Other templates are examined in follow-up papers. A striking immediate observation is that a discontinuous PFSP can have continuous lateral attractors, and continuous PFSP can have discontinuous lateral attractors.

The right column states that $\delta > 2$ is excluded when the pulse is non-cancelling but is admissible when the pulse is cancelling. In the latter case, the lateral attractor is FBM. In any event, $H < 1/2$ is incompatible with SLM.

The column second from the right shows that $1 < \delta < 2$ is admissible for all pulse templates and all values of δ, and that it yields varied and interesting results. Combining proven facts with inferences based on compelling heuristics, one is tempted to infer the following.

- For continuous pulses having left and right derivatives bounded away from 0 and ∞, the lateral attractor is FBM.
- For discontinuous or mixed pulses, the lateral attractor is SLM.

The column shows that the dependence of the limit on the template is far more complicated for $0 < \delta < 1$ than for $1 < \delta < 2$. When the pulse template is discontinuous, a formal extrapolation to $0 < \delta < 1$ of the results relative to SLM with $1 < \delta < 2$ is both meaningful and correct. When the pulse is continuous, the extrapolation to $0 < \delta < 1$ of the results relative to FBM with $1 < \delta < 2$ is meaningless, in fact, the construction diverges. As shown in a follow-up paper, one finds meaningful results by considering the second difference

$$\varDelta\varDelta F = [F(t+T; C, \delta) - F(t; C, \delta)] - [F(t; C, \delta) - F(t-T; C, \delta)].$$

The case $1 < \delta < 2$; *sensitive dependence of the lateral attractor on the pulse templates; a bridge between FBM and SLM.* Table I indicates that the attractor is FBM in the case of a smooth conical pulse, but it is SLM if the pulse has a discontinuity, however small, which occurs, for example, when the pulse is a stepped cone, namely a staircase made of many steps up followed by many steps down. "Intuition" suggests that, for finite C, these two pulse forms could not make much of a difference. This is confirmed by a comparative examination of the address diagrams. The diagram corresponding to the stepped cones approximates that of the smooth cone, except for large λ's. As the number of steps increase, so does the quality of the approximation, and it also spreads up to increasingly large λ's.

7 PROOFS OF THE CLAIMS IN TABLE I FOR THE CYLINDRICAL PULSES

The parameter σ is set to 1 in this Section. The address points with $\lambda > 0$ and with $\lambda < 0$ contribute to two independent parts of $F(T; C, \delta)$. The calculations leading to their characteristic functions are the same except for different values C' and C''. We restrict ourselves to the case $\lambda > 0$ and $C' = 1$.

7.1 The logarithms of the characteristic function (l.c.f.) of $F(T; C, \delta)$.

From Figure 1 it is clear that the address points that contribute to $F(T; C, \delta)$ fall into two domains described as left and right and denoted by $\mathcal{D}_L$ and $\mathcal{D}_R$. The strip $(\lambda, \lambda + d\lambda)$, where $\lambda > 0$, makes the following contribution to the l.c.f.

$$T[(e^{i\xi\lambda} - 1) + (e^{-i\xi\lambda} - 1)]\lambda^{-\delta-1}d\lambda \text{ if } \lambda > \mathrm{T}^{1/\delta},$$
$$\lambda^{\delta}[(e^{i\xi\lambda} - 1) + (e^{-i\xi\lambda} - 1)]\lambda^{-\delta-1}d\lambda \qquad \text{if} \quad \lambda < \mathrm{T}^{1/\delta}.$$

Integrating over λ and transforming to the rescaled variables $x = \lambda T^{-1/\delta}$ and $\mathrm{y} = \xi \mathrm{T}^{1/\delta}$ yields

$$\int_1^{\infty} [(e^{iyx} - 1) + (e^{-iyx} - 1)]x^{-\delta-1}dx + \int_0^1 [(e^{iyx} - 1) + (e^{-iyx} - 1)]x^{-1}dx.$$

This expression converges for all $\delta > 0$ and is the l.c.f. of a rescaled r.v. independent of T. Therefore, the rescaled increment $T^{-1/\delta}F(T; C, \delta)$ has a distribution independent of T. This is a property of self-affinity with $H = 1/\delta$. We know the mechanisms of self-affinity: in FBM, it is caused by global dependence without long-tailedness, in SLM it is caused by long-tailedness without global dependence, and in PFSP it is caused by both long-tailedness and global dependence, acting together with the same value of H. The next issue is to separate the long-tailedness and dependence aspects.

7.2 The attractor in the case $\delta < 2$. Lateral attraction to symmetric Lévy stable increments with $\alpha = \delta$.

Given N independent r.v.'s $F_n(T; C, \delta)$ with the above distribution, the behavior of $\Delta\tilde{F}_N = \sum_{n=1}^{N} F_n(T; C, \delta)$ depends sharply on the value of δ. When $\delta < 2$, the l.c.f. of $N^{-1/\delta}\Delta\tilde{F}_N$ can be written in the form

$$T \int_{(T/N)^{1/\delta}}^{\infty} (e^{i\xi u} + e^{-i\xi u} - 2)u^{-\delta-1}du$$
$$+N \int_0^{T^{1/\delta}} (e^{i\xi\lambda N^{-1/\delta}} + e^{-i\xi\lambda N^{-1/\delta}} - 2)\lambda^{-1}d\lambda.$$

When $N \to \infty$, the first term converges to the well-known l.c.f. of a symmetric Lévy stable r.v. with the stability parameter $\alpha = \delta$ and the scale parameter proportional to $T^{1/\delta}$. The second term is of order $N^{1-2/\delta}$, therefore converges to zero. Hence, for fixed T, $F(T)$ belongs to the symmetric δ-stable domain of attraction.

7.3 The dependence structure of $F(T; C, \delta)$ in the case $\delta < 2$. Lateral attraction to a Lévy stable r.f.'s with $\alpha = \delta$ and independent increments.

We proceed to the multidimensional structure of the PFSP. We show that the multidimensional distributions of the PFSP are attracted to that of a symmetric SLM. To prove it we need to find the limit in distribution of a linear combination

$$\sum_{i=1}^{k} \xi_i N^{-1/\delta} \sum_{n=1}^{N} F_n(T_i; C, \delta),$$

where $F_n(T_i; C, \delta), i = 1, 2, \ldots, k$, are nonoverlapping increments of an PFSP copy F_n, over (possibly different) time spans T_i. The limit should be the corresponding linear combination of independent δ-stable variables with scale parameters proportional to the respective $T_i^{1/\delta}$. Here, we consider increments over two disjoint time spans T_1 and T_2, i.e., $k = 2$. The general case is not mathematically more involved but requires an overload of notation. Our assertion follows from the same reasoning as in the one-dimensional case, if we show that the expression

$$N \int_D [(e^{i(\xi_1 - \xi_2)\lambda N^{-1/\delta}} - 1) - (e^{i\xi_1 \lambda N^{-1/\delta}} + e^{-i\xi_2 \lambda N^{-1/\delta}} - 2)]\lambda^{-\delta-1} dt d\lambda$$

converges to zero, where $\mathcal{D}$ is the dotted region depicted in Figure 3. But this expression is again of order $N^{1-2/\delta}$, which establishes the result.

7.4 The attractor in the case $\delta > 2$. Lateral attraction to Gaussian increments.

This case is very different, since the variance $EF^2(T; C, \delta)$ is finite, therefore the rescaled sum $N^{1/2}[\tilde{F}_N - E\tilde{F}_N]$ is asymptotically Gaussian, i.e. stable with $\alpha = 2$.

7.5 The dependence structure of $F(T; C, \delta)$ in the case $\delta > 2$. Lateral attraction to a fractional Brownian r.f.'s with $H = 1/\delta < 1/2$.

Examination of the characteristic function of a linear combination of nonoverlapping increments of the PFSP shows that it has the second derivative at 0, i.e. every linear combination has the second moment, and multidimensional distributions of the PFSP are attracted to multidimensional Gaussian distributions. Note that this Gaussian process in the limit must have stationary increments and be self-affine with the constant $H = 1/\delta$ since these properties are preserved under convolution and convergence in distribution. It is well-known that FBM with $H = 1/\delta$ is the unique Gaussian process satisfying these requirements.

7.6 Some semi-random PFSP belonging to the domain of standard attraction of a semi-random self-affine PFSP.

We replace the constants C' and σ with the slowly varying (at ∞) functions $\gamma(\lambda)$ and $\sigma(\lambda)$ such that the function $w = \sigma(\lambda)\lambda^{\delta}$ is monotonically increasing. Writing the inverse function of $w(\lambda)$ as $\lambda = w^{1/\delta}L(w)$ yields the identity

$$L(w) = \sigma^{-1/\delta}(w^{1/\delta}L(w)),$$

which is an implicit equation for $L(w)$ and will be used momentarily without having to be solved.

When $\lambda > 0$, the strip $(\lambda, \lambda + d\lambda)$ makes the following contribution to the l.c.f.

$$\begin{aligned} &T[(e^{i\xi\lambda}-1)+(e^{-i\xi\lambda}-1)]\gamma(\lambda)\lambda^{-\delta-1}d\lambda \qquad \text{if} \quad \lambda > \mathrm{T}^{1/\delta}\mathrm{L}(\mathrm{T}),\\ &\sigma(\lambda)\lambda^{\delta}[(e^{i\xi\lambda}-1)+(e^{-i\xi\lambda}-1)]\gamma(\lambda)\lambda^{-\delta-1}d\lambda \qquad \text{if} \quad \lambda < \mathrm{T}^{1/\delta}\mathrm{L}(\mathrm{T}).\end{aligned}$$

Integrating over λ and transforming to the rescaled variables $x = \lambda T^{-1/\delta} L^{-1}(T)$ and $y = \xi T^{1/\delta}L(T)$ yields

$$\begin{aligned} &\int_1^{\infty}[(e^{iyx}-1)+(e^{-iyx}-1)]x^{-\delta-1}\{\gamma[xT^{1/\delta}L(T)]L^{-\delta}(T)\}dx\\ +&\int_0^1[(e^{iyx}-1)+(e^{-iyx}-1)]x^{-1}\{\gamma[xT^{1/\delta}L(T)]\sigma[xT^{1/\delta}L(T)]\}dx.\end{aligned}$$

The integral over $(1,\infty)$ converges for all $\delta > 0$. Assuming that the functions γ and σ are such that also the second integral is finite (e.g. γ and σ are both bounded in the neighborhood of zero), the above expression gives the l.c.f. of a rescaled r.v. $T^{-1/\delta}L^{-1}(T)F$.

This l.c.f. may depend on T. If so, the following question arises: under what conditions on $\gamma(\lambda)$ and $\sigma(\lambda)$, and hence on $L(\lambda)$, does this l.c.f. converge to that prevailing in the PFSP case $\gamma(\lambda)\sigma(\lambda) = C'\sigma$? (We know that the product $C'\sigma$ determines the type of an PFSP.) Because of the identity that links $\sigma(\lambda)$ and $L(w)$ the two factors written between braces are identical (asymptotically, when $T \to \infty$); therefore the two halves of the l.c.f. yield the same condition on convergence. It is

$$\lim_{\lambda\to\infty}\gamma(\lambda)\sigma(\lambda) = C'\sigma.$$

In other words, the functions $1/\gamma(\lambda)$ and $\sigma(\lambda)$ must vary slowly with λ and asymptotically proportionally to each other.

The question concerning whether or not these conditions are also necessary has not been addressed yet.

Acknowledgments. In 1977-78, I studied semi-random PFSP with a one-dimensional t and a multi-dimensional F: early simulations for the second row of Table I were performed by M.R. Laff, and I conjectured that the closure of the

set of values of $F(t)$ remains of Hausdorff-Besicovitch dimension $D = \delta$. Soon later, J. Hawkes proved this conjecture. In the mid-1980s, I studied an application of semi-random PFSP with a multi-dimensional t and one-dimensional F: early simulations performed by S. Lovejoy are reported in [5]. I also made conjectures concerning the random PFSP; in due time, they were proved in [2] and [4]. Earlier versions of this paper was discussed at length with R. Cioczek-Georges and M. Frame. Diagrams were drawn by H. Kaufman.

References

1. J. Barral and B.B. Mandelbrot: Probability Theory and Related Fields **124**, 409 (2002)
2. R. Cioczek-Georges and B.B. Mandelbrot: Stochastic Processes and Their Applications **60**, 1 (1995)
3. R. Cioczek-Georges and B.B. Mandelbrot: Stochastic Processes and their Applications **64**, 143 (1996)
4. R. Cioczek-Georges, B.B. Mandelbrot, G. Samorodnitsky and M. Taqqu: Bernoulli **1**, 201 (1995)
5. S. Lovejoy and B.B. Mandelbrot: Tellus **37A**, 209 (1985)
6. B.B. Mandelbrot: Zeitschrift für Wahrscheinlichkeitstheorie **22**, 145 (1972)
7. B.B. Mandelbrot: The Fractal Geometry of Nature (W.H. Freeman, New York, 1982)
8. B.B. Mandelbrot: In Lévy Flights and Related Phenomena in Physics (Nice, 1994). Ed. G. Zaslawsky, M. F. Schlesinger & U. Frisch (Lecture Notes in Physics) (Springer, New York, 1995)
9. B.B. Mandelbrot: Gaussian Self-Affinity and Fractals: Globality, the Earth, $1/f$, and R/S (Springer-Verlag, New York, 2002)
10. B.B. Mandelbrot: (to appear)
11. M.S. Taqqu and G. Samorodnitsky: Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance (Chapman and Hall, New York, London, 1994)