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Multifractality of the harmonic measure on fractal aggregates, and extended self-similarity

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We show that DLA follows a surprising new scaling rule. It expresses that the screened region, in which the harmonic measure is tiny, increases more than proportionately as the cluster grows. This scaling rule also gives indirect evidence that the harmonic measure of lattice DLA follows a hyperbolic probability distribution of exponent equal to 1. This distribution predicts that sample moments behave erratically, hence explains why the common restricted multifractal formalism fails to apply to DLA.

The simplicity of the growth rules in DLA, diffusion limited fractal aggregation [1], and their basic role in understanding the fractal aspects of many physical phenomena [2–6], have motivated extensive quantitative studies. However a full understanding of the resulting complex structures is still lacking. One reason, in our opinion, lies in the incompleteness of the description. This letter describes two new scaling properties of the harmonic measure, μ, of plane DLA. The need for two distinct scalings implies that the notion of self-similarity splits into several distinct sub-notions. Our more significant new scaling property of μ is unusual, and indicates that DLA satisfies an "extended form" of self-similarity, but not the form that is ordinarily postulated. We compare small DLA clusters and sized down and coarse-grained clusters, and find them to differ from each other in a systematic and unexpected way. In rough terms, the "screened" region, in which the harmonic measure is tiny, increases far more than proportionately as the cluster grows.

Plane DLA is generated by allowing an "atom" to perform Brownian motion until it hits an initial "seed". At that instant, the seed is modified by embedding this atom, and a fresh Brownian atom is launched against the enlarged target. By a classic result of Kakutani [7], the distribution of the hitting points is the "harmonic measure". In the dielectric breakdown model (DBM) [8] the growth rules are based explicitly on this measure. Overwhelming evidence from computer simulations [2–6] shows that the arrival of very many atoms transforms the seed into a cluster with about the same degree of complication at all scales of observation.

That is, DLA is nearly self-similar, but departures from simple self-similarity are questionable. Quantifying their statistical nature has proven to be a dauting task. It was hoped for a time that the harmonic measure can be represented within a restricted form of the notion of multifractal (Frisch and Parisi [9] and Halsey et al. [10]). This would have implied that μ is self-similar in a strong sense. With local irregularities of μ being characterized by the classical Hölder exponent α, the structure of strong self-similarity is characterized either by a function $\tau(q)$ defined for $-\infty < q < +\infty$, or by a function $f(\alpha)$, whose graph is shaped like an asymmetric form of the symbol $\cap$. Unfortunately, except perhaps for the sites with the highest growth probability, the results of this multifractal analysis of DLA have been mutually contradictory or otherwise unacceptable. Together with many other authors, we interpret these difficulties as strong evidence that the power-law scaling relations that characterize the restricted multifractals fail to apply to DLA [11–15]. These observed "anomalies" are intimately related to the behaviour of small harmonic measures.

However, as we have recently shown [16] by explicit examples, the failure of $\tau(q)$ to exist for all q does *not necessarily* imply that a measure is not self-similar. The measures in ref. [16] fit in the context of the more general theory of multiplicative multifractals [17,18], yet the right-hand side of the $f(\alpha)$ is altogether absent, so that such a μ can be called a "left-sided multifractal". This letter's principal point, however, goes beyond the evaluation of $f(\alpha)$. We argue that a one sided $f(\alpha)$ fails to describe adequately the distribution of a left-sided measure μ. In addition to the scaling relation that is needed to define $f(\alpha)$, additional scaling relations are necessary for the characterization of other, significant aspects of μ.

The present letter attacks the problem of the distribution of the harmonic measure μ using a direct probabilistic method [18,17]. A first reason is that the attempts to fit the harmonic measure on DLA within the restricted multifractal formalism seem to fail because of the high statistical scatter characteristic of small μ. One consequence is that small μ's cannot be trusted. Thus, the exponential decay of the minimum probability (as postulated in ref. [13]) would be hard, either to confirm by direct methods, or to confront with other alternatives. Another consequence is that statistical techniques such as the partition function become unreliable at best.

Our numerical work was done for both circular and cylindrical DLA. Our figures concern circular geometry, but cylindrical geometry gives similar results. We grew 10 clusters of $N=50000$ particles using a random walker algorithm [#1]. The potential was computed by solving the discrete Laplace equation iteratively on the square lattice underlying the growth, with boundary conditions 0 on the cluster and 1 on a circle with radius equal $\frac{3}{2}$ the overall size of the clusters. The harmonic measure μ at a site on the boundary of the cluster is theoretically proportional to the gradient of the potential, but we approximate it by the potential at the nearest neighbors of the cluster, and then normalize it [8]. The value of μ was evaluated for every point of every

[#1] We used an algorithm by Y. Hayakawa, which grows the more common site version of DLA.

cluster. This was done at the successive stages of growth $N=781, 3125, 12500, 50000$, i.e., for $N_k=4^k N_0$, with $N_0=781$ and $k=0, 1, 2, 3$. At each stage k, we estimated the probability density $p_k(\alpha)$ of the Hölder exponent $\alpha=\log \mu/\log N_k$, by determining the sample frequency of α for each cluster with this value of k and then averaging over the 10 clusters.

If the harmonic measure on DLA had been a restricted multifractal, one would expect [18,16] the quantities $C_N(\alpha)=(1/\log N)\log p_N(\alpha)$ to converge to the better known quantity $f(\alpha)/D_0-1$ as $N\to\infty$. The plots of $C_N(\alpha)$ would therefore provide an approximation of $f(\alpha)$. (These rescaling rules are usually expressed in terms of the size L of the cluster, but $L\sim N^{1/D}$.) Also note that the actual convergence to $f(\alpha)$ may be extremely slow and the approximation of $f(\alpha)$ by $C_N(\alpha)$ may be poor. Nevertheless, fig. 1 shows a high level of bunching for low α. But the right sides show no sign of converging as $N\to\infty$. The difficulties encountered by Legendre estimation of $f(\alpha)$, which starts with $\tau(q)$, appear to originate solely with these tails.

Fig. 2 shows the same data replotted in a different fashion (highly nonstandard in physics), which we call *positive Cauchy rescaling*. The abscissa is taken to be $\log \mu/\log N-\log\log N$, and the ordinate is taken to be $\log p_N(\alpha)$ itself, without any renormalization. Now, it is the right sides of the resulting graphs that collapse into one.

The seemingly peculiar rescaling used to obtain fig. 2 was neither picked by chance, nor by trial and error, but was inspired by a limit theorem that is little known in physics, in which the limit is called the asymmetric or positive Cauchy law. This theorem enters via the theory of multiplicative multifractals [18]. Before we restate the

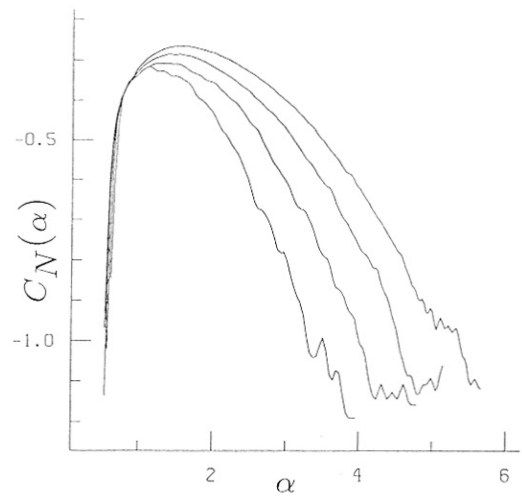

Fig. 1. Cramèr rescaling of the densities of the distributions of α's, as estimated from 10 clusters of masses $N=781$, 3125, 12500, 50000. The ordinate is $C_N(\alpha)=(1/\ln N)\ln p_N(\alpha)$ and the abscissa is $\alpha=-\ln\mu/\ln N$. For a restricted multifractal, the limit of $C_N(\alpha)$ for $N\to\infty$ would be $f(\alpha)-1$. Here, to the contrary, the right tails of the distributions fail to converge.

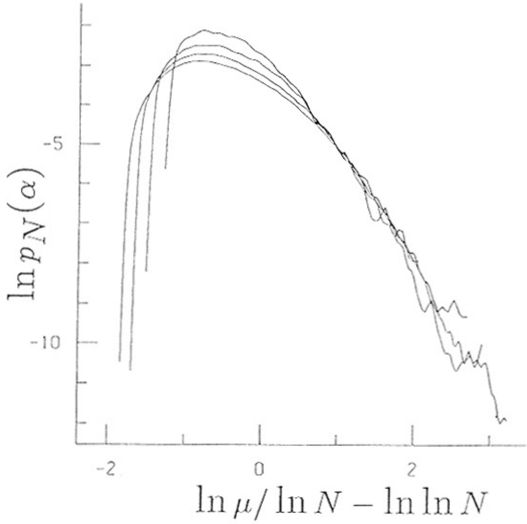

Fig. 2. The same estimated densities as in fig. 1, but with positive Cauchy rescaling. Here the left sides fail to converge, but the right sides do converge.

main ideas of that theory, let us acknowledge that in the case of DLA, the successive stages we shall describe are conjectural.

Consider a regular grid of base 2 on the unit interval [0, 1] and denote by $\mathrm{I}(\beta_1, \beta_2, ..., \beta_k)$ the interval $[t, t+\epsilon]$, where $\epsilon = 2^{-k}$ and $t = 0, \beta_1, \beta_2, ..., \beta_k$ (in binary representation), with β_i equal to 0 or 1. The structure of a multiplicative multifractal is determined by a random variable M, whose expectation satisfies $\mathsf{E}M = \frac{1}{2}$ (to insure conservation) and other conditions [17]. The first stage of the multiplicative cascade begins with mass equal to 1 on [0, 1] and redistributes it by giving to the subinterval $\mathrm{I}(\beta_1)$ the mass $M(\beta_1)$. The second stage gives to the interval $\mathrm{I}(\beta_1, \beta_2)$ the mass $M(\beta_1) \times M(\beta_1, \beta_2)$. After k stages, the interval $\mathrm{I}(\beta_1, \beta_2, ..., \beta_k)$ contains the mass

$$\mu_t(\epsilon) = M(\beta_1)\, M(\beta_1, \beta_2) \ldots M(\beta_1, \beta_1, ..., \beta_k) .$$

Each realization of the cascade (i.e., each seed in the "random" generator) yields a different "sample". The rules of the multiplication are statistically the same at each step of fine-graining, in the sense that, given t, the multipliers $M(\beta_1)$, $M(\beta_2)$, etc., are independent and identically distributed (i.i.d.). Therefore the measures resulting from these infinite cascades are *statistically* self-similar and they are multifractals in the more general sense [17,16,18].

For convenience, we now introduce the random variable $V = -\log_2 M$, whose probability density, $p(v)$, is obtained from that of M, and we write

$$H_k = \log \mu(\epsilon) / \log \epsilon \tag{1}$$

$$= \frac{1}{k} [V(\beta_1) + V(\beta_1, \beta_2) + \ldots + V(\beta_1, ..., \beta_k)] = \frac{1}{k} \sum_{n=1}^{k} V_n . \tag{2}$$

This H is a random variable whose sample value, denoted by α, is simply the Hölder exponent ("singularity strength"). Here, it is simply the sample average of k i.i.d. random variables $V_1, ..., V_k$, with probability density $p(v)$, a topic extensively studied in probability theory [19].

By construction the above multiplicative cascades yield statistically self-similar measures. Hence one may hope that the probability densities $p_k(\alpha)$ of H_k, corresponding to successive prefractal levels , can be renormalized (or collapsed) in such a way that a suitably renormalized version of the density p_k converges to a limit other than 0 or ∞. If a limit exists, it can be used to characterize the fractal properties of the multifractal measure.

First suppose that the first and second moments of the distribution $p(v)$ exist. This seems almost obvious, but turns out to be a special case. Then two familiar rules of probability theory are valid: the law of large numbers and the De Moivre–Laplace central limit theorem [19,20] tell us that the sample average $\sum V_n/k$ converges to the population (ensemble) expectation $\mathsf{E}V$ and that the distribution of $\sum(V_n - \mathsf{E}V)/\sqrt{k}$ converges to the Gaussian of variance $\mathsf{E}(V^2) - (\mathsf{E}V)^2$. However, a description of multiplicative multifractals requires far more detailed knowledge. The appropriate limiting distribution is given by the little known rule from *Harald Cramèr's theorem on large deviations* [18,17]. It asserts that the probability density $p_k(\alpha)$ of H_k is such that $C_k(\alpha) = (1/k)\log_2 p_k(\alpha)$ converges to a limit $C(\alpha)$. In the case that $C(\alpha)$ is neither 0 nor ∞, this function provides a characterization of the fractal properties of the measure. In the case of restricted multifractals [16], the function $f(\alpha)$ [9,10] is equal to $f(\alpha) = D_0 + C(\alpha)$ [18,17], with $D_0 = 1$ on the interval.

This terminates our exposition on the basics in ref. [17]. However, as ref. [16] points out, all the above rules may either fail, or yield trivial results. As a preliminary illustration that will prove significant, let us suppose that the random variables V_n are Cauchy distributed, that is, have the "Lorentzian" probability density $p_1(v) = 1/[\pi(1+v^2)]$ [17]. Probabilists are familiar with the following easily verified fact: in the Cauchy case, $\sum V_n/k$ has precisely the same distribution as each of its addends V_n, i.e., $p_k(\alpha) = p_1(\alpha)$. On the other hand $\mathsf{E}V = \infty$ and $\mathsf{E}(V^2) = \infty$, so that the expressions that enter into the law of large numbers and the Gaussian central limit are both meaningless. As to $C_k(\alpha) = (1/k)\log p_k(\alpha)$, it takes the form $C_k(\alpha) = -(1/k) \times \log[\pi(1+\alpha^2)]$. Therefore, $C(\alpha) = \lim_{k\to\infty} C_k(\alpha) \equiv 0$, hence $f(\alpha) \equiv D_0$. In other words, not only do the usual (Gaussian) scaling properties fail altogether, but the Cramèr rescaling yields a degenerate result. Whenever such is the case, one hopes to find a new renormalization scheme that would yield an alternative to the functions $C(\alpha)$ or $f(\alpha)$. It may even occur (like in a left-sided fractal measure [16] and – according to our results – for DLA) that, in order to characterize more completely the fractal properties of μ, one must consider more than one normalization. For DLA, the Cramèrian $f(\alpha)$ normalization produces a collapse for low α's (see fig. 1), while the positive Cauchy rescaling produces a collapse for high α's, shown in fig. 2.

For Cauchy distributed V's, the fact that the distribution of H_k is independent of k is in itself an unexpected alternative scaling property. It implies that plots of the densities $p_k(\alpha)$, corresponding to different levels of coarse graining, automatically collapse back on to a Cauchy distribution. This is a very strong property, because it is known to identify the Cauchy distribution uniquely, among all possible limits of sums of i.i.d. random variables.

In our context, however, the quantities V_n cannot be Cauchy distributed, because $M<1$ implies $V=-\log M>0$. But suppose that $\Pr(V>v)\sim v^{-1}$, like in the Cauchy case. If so, the variable V is in the domain of attraction of the positive Cauchy law [19], which can be shown to be intricately related to the case $\lambda=1$ of the multifractals we have examined earlier [16]. The main implication is that the densities of H_k can be collapsed by subtracting a quantity proportional to $\log k$ [19]. In our clusters, $k=\log N$, therefore we recover the rescaling $\alpha-\log\log N$ which is used in fig. 1. Conversely, if a distribution is known to be a limit under the rescaling procedure leading to fig. 1, that distribution is perfectly determined. Furthermore, one can show that the tail of $f(\alpha)$ for $\alpha\to\infty$ behaves like in the $\lambda=1$ case in ref. [16]. That is, $f(\alpha)\simeq D_0-c\exp(-c'\alpha)$, with c and c' being some positive constants.

Individual errors in a sample of a Cauchy random variable are of the same order of magnitude as the average error over many samples. The same is also nearly the case for the positive Cauchy. This is why sample moments for the harmonic measure on DLA behave erratically, and thus why the method of moments fail in estimating $f(\alpha)$.

It is known that $\exp(-L^2)$ is an absolute lower bound for the behaviour of the smallest growth probability in lattice DLA [21,15]. Assume, as the above findings suggest, that H_k is in the domain of attraction of the positive Cauchy. Then $\Pr(H_k>\alpha)\sim\alpha^{-1}$. The largest value $\alpha_{\max}(N)$ in a sample of N such random variables is expected to satisfy $\Pr[H_k>\alpha_{\max}(N)]\approx 1/N$, i.e., $\alpha_{\max}(N)\approx N$. In a cluster of N sites, one thus expects the smallest probability to behave like $\exp(-L^D)$, D being the fractal dimension of the cluster. This behaviour was assumed for DLA in ref. [13].

When moments are helpless to describe a distribution, statisticians work with "quantiles". The tail quantile Ω_r of order r [20] of H_k (with $0<r<1$) is defined by $\Pr(H_k>\Omega_r)=r$. In the positive Cauchy case, all Ω_r behave like $\log k$. Therefore, imagine that the observed μ's have been "censored" systematically, by erasing the lower values up to a proportion r. Then the censored minimum $\mu_{\min}(r)$ would satisfy $\log[\mu_{\min}(r)]\sim-\log L\log\log L$. If censorship is unsystematic, one expects "$\mu_{\min}$" to fall somewhere between the quantile's decay and that of the absolute minimum.

The above limiting distribution has been obtained by varying cluster size. One can also study the limiting behavior of the density of the probabilities obtained by coarse-graining the harmonic measure on a single given large cluster. We have coarse grained 10 clusters of mass $N=12\,500$ and size $L\sim N^{1/D}$, with square boxes of sizes 2^k with $k=0, 1, 2, \ldots$, and have determined the densities $p_k(\alpha)$. Fig. 3 shows the results of

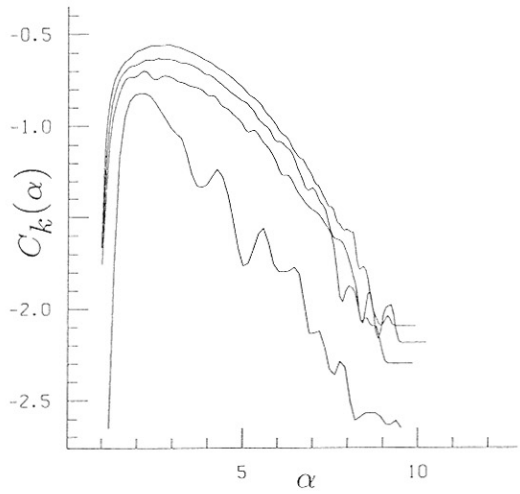

Fig. 3. Cramèr rescaling for the densities of the distributions of the α's as estimated after coarse-graining the harmonic measure on 10 clusters of mass $N=12\ 500$ and radius ≈ 180, with square boxes of sizes 2^k, $k=1, 2, 3, 5$. The Cramèr rescaling plots $C_k(\alpha)=(1/k)\log_2 p_k(\alpha)$ versus $\alpha=-(1/k)\ln_2 \mu$.

Cramèr rescaling of $p_k(\alpha)$ for $k=1$, 2, 3 and 5. The collapse is very good for $k=1$, 2, 3, while a positive Cauchy rescaling (which need not be shown here) would yield no collapse at all. Furthermore, comparing the density p_k with the density $\tilde{p}_0$ of clusters of size $2^{-k}L$, one finds that the p_k has a much longer right-tail than $\tilde{p}_0$. Thus, the screened regions in which the α's of the harmonic measure are huge seem to increase more than proportionately as the cluster grows.

We conclude that the harmonic measure on DLA can be described as self-similar, both from the point of view of growth and from the point of view of coarse-graining. However, two notions of self-similarity must be involved. Under coarse-graining, one may be content with the standard notion, which underlies the restricted multifractals. But to study growth one needs an alternative rescaling rule, and an extended notion of self-similarity.

Of course, our experimental discovery that DLA satisfies an extension of self-similarity is not contingent on the theoretical argument that led us to test for positive Cauchy rescaling. However, this letter raises but does not answer a challenging question concerning the applicability to DLA of the theory of random multiplicative processes. Since, as already noted, the cascade postulated by this theory is still conjectural, why does this theory prove so effective? The same question also arises in the context of turbulence, where restricted multifractals are also insufficient.

This text reproduces IBM research report RC-16595 of August 1990, with minor editorial corrections. The same experiment has now been redone with off–off lattice DLA. The resulting graphs are practically indistinguishable from those included in this paper.

References

[1] T.A. Witten and L.M. Sander, Phys. Rev. Lett. 47 (1981) 1400.

[2] L. Pietronero and E. Tosatti, eds., Fractals in Physics (North-Holland, Amsterdam, 1986).

[3] J. Feder, Fractals (Plenum, New York, 1988).

[4] P. Meakin, in: Phase Transitions and Critical Phenomena, C. Domb and J. Lebowitz, eds., vol. 12 (1988) p. 335.

[5] T. Vicsek, Fractal Growth Phenomena (World Scientific, Singapore, 1989).

[6] A. Aharony and J. Feder, eds., Fractals in Physics (North-Holland, Amsterdam, 1989), Physica D 38 (1989) 1–398.

[7] S. Kakutani, Proc. Imp. Acad. (Tokyo) 20 (1944) 706.

[8] L. Niemeyer, L. Pietronero and H.J. Wiesmann, Phys. Rev. Lett. 52 (1984) 1033.

[9] U. Frisch and G. Parisi, in: Turbulence and Predictability of Geophysical Fluid Dynamics and Climate Dynamics, Proc. Enrico Fermi Int. School of Physics, M. Ghil, R. Benzi and G. Parisi, eds. (North-Holland, Amsterdam, 1985) p. 84.

[10] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia and B.I. Shraiman, Phys. Rev. A 33 (1986) 1141.

[11] J. Lee and H.E. Stanley, Phys. Rev. Lett. 61 (1988) 2945.

[12] S. Schwarzer, J. Lee, A. Bunde, S. Havlin, H.E. Roman and H.E. Stanley, Phys. Rev. Lett. 65 (1990) 603.

[13] R. Blumenfeld and A. Aharony, Phys. Rev. Lett. 62 (1989) 2977.

[14] B.B. Mandelbrot and C.J.G. Evertsz, Nature 348 (1990) 143.

[15] C.J.G. Evertsz, P.W. Jones and B.B. Mandelbrot, J. Phys. A 24 (1991) 1889.

[16] B.B. Mandelbrot, Physica A 168 (1990) 95;
B.B. Mandelbrot, C.J.G. Evertsz and Y. Hayakawa, Phys. Rev. A 42 (1990) 4528.

[17] B.B. Mandelbrot, Pure Appl. Geophys. 131 (1989) 5; in: Fluctuations and Pattern Formation (Cargèse, 1988), H.E. Stanley and N. Ostrowsky, eds. (Kluwer, Dordrecht, Boston, 1988) pp. 345–360.

[18] B.B. Mandelbrot, J. Fluid Mech. 62 (1974) 331.

[19] B.V. Gnedenko and A.N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables (Addison-Wesley, Reading, MA, London, 1968).

[20] B.V. Gnedenko, The Theory of Probability (Chelsea, New York, 1967).

[21] J. Lee, P. Alström and H.E. Stanley, Phys. Rev. A 39 (1989) 6545; Phys. Rev. Lett. Comm. 62 (1989) 3013.