Physica A 168 (1990) 95–111
North-Holland

NEW "ANOMALOUS" MULTIPLICATIVE MULTIFRACTALS: LEFT SIDED $f(\alpha)$ AND THE MODELLING OF DLA

Benoit B. MANDELBROT

Physics Department, IBM T.J. Watson Research Center, Yorktown Heights, NY 10598, USA and Mathematics Department, Yale University, New Haven, CT 06520, USA

A sharp distinction is drawn between general multiplicative multifractals, as originally introduced by the author, and the more familiar but more restricted class defined by Frisch and Parisi and by Halsey et al. All the general multiplicative multifractals are exactly renormalizable, by design. However (this is important and perhaps surprising), it is shown that the scaling relations that serve to define the restricted class fail to extend to certain multiplicative multifractals. As a result, in addition to the familiar restricted class, the general multifractals are shown to allow for a wide variety of "anomalous" behaviors, several of which are described. We believe that suitable examples of these anomalies are of the most direct practical relevance. In particular, there is evidence that they characterize the very important concrete applications to fully developed turbulence and to diffusion limited aggregates (DLA). This paper concerns several new classes of multifractals we have recently proposed as tools to model the anomalies of DLA. They are "left sided", that is, characterized by $\alpha_{\max} = \infty$ with $f(\alpha_{\max}) = D_0$. Related multifractals characterized by $\alpha_{\max} = 0$, and even by $\alpha_{\max} = \infty$ and $\alpha_{\min} = 0$, are encountered along the way.

1. Introduction to two distinct meanings of multifractality

The term "multifractal" may have at least two meanings, depending upon the role given to *multi*. At one time, this distinction seemed to lack practical bite, but we feel that recent work on fully developed turbulence and on diffusion limited aggregates (DLA) has made it become essential.

The earlier and more general meaning comes from the notion of "*multiplicative cascade* that generates non-random or random measures", and describes "measures that are multiplicatively generated". The virtue of *all* multiplicatively generated measures is that they are exactly renormalizable (just like the simplest self-similar fractals, such as the Sierpinski gasket). This general meaning of multifractality is investigated in early [1–3] and recent papers of the author and co-workers. (Ref. [1] and other early papers are to be reprinted in ref. [4].) Note that our papers of 1974 have laid the ground for a general notion (the word *fractal* had not been coined yet!) that went beyond the original specific application to fully developed turbulence.

In addition, a second meaning has since been introduced by Frisch and Parisi [5] and by Halsey et al. [6]. Let space be subdivided into small boxes of side ε, and let $\mu_j(\varepsilon)$ denote the measure within the jth box. A multifractal can be defined as "a non-random measure for which it is true for all $-\infty < q < \infty$ that the *partition function* $\chi(q, \varepsilon) = \Sigma\ \mu_j^q(\varepsilon)$ scales like a power of the form $\varepsilon^{\tau(q)}$". An alternative form of this definition involves the Holder exponent $\alpha = \log \mu_j / \log \varepsilon$. Thus, it is useful to represent such a multifractal as a *multiplicity* of intertwined measures, each of which is supported by a Cantor set and is characterized by a uniform α. The proportions of different α's in the mixture are characterized by a function $f(\alpha) \geqslant 0$, whose graph is shaped like the sign $\cap$, perhaps asymmetric and leaning to the side, but always standing above $f = 0$". Applying the method of steepest descents, one shows that this function $f(\alpha)$ is related to the function $\tau(q)$ by the Legendre transform. The basic facts about this approach are restated in these Proceedings in ref. [7].

We now come to one of the central points of this paper. In *restricted cases* of our multiplicative multifractals, the scaling relation that defines $\tau(q)$ is indeed valid for all q. As a matter of fact, this was precisely the point of departure of ref. [5]. On the other hand, we shall give explicit examples of multiplicative multifractals with the property that the scaling relation *fails to hold*, either for small enough negative q's or for high enough positive q's. Insofar as they differ *qualitatively* from the familiar restricted class, these multiplicative measures are "anomalous". Nevertheless, they preserve the desirable property of being exactly renormalizable. As already implied, some of the examples we have constructed fit fully developed turbulence. We propose to provide further examples that we believe fit DLA, the diffusion limited aggregates.

Notations. In the non-random case, the bounds on "normal" q's will be denoted by q_{bottom} and q_{top}. In the canonical random case, mentioned in Section 6, yet another cause of anomaly may restrict q to lie below a different threshold, denoted by q_{crit}, which may satisfy $1 \leqslant q_{\text{crit}} \leqslant q_{\text{top}}$.

The reader may be reassured to know that, in our more general multiplicative multifractals, counterparts of the function $f(\alpha)$ and $\tau(q)$ continue to be needed and continue to be cap-convex (like $-x^2$). Do they continue to be linked by "the" Legendre transforms? Yes, but only at the cost of introducing (here again) a fine distinction. It happens that it continues to be true that $f(\alpha) = \min_q[\alpha q - \tau(q)]$, which implies that the graph of $f(\alpha)$ is straight for $q > \min(q_{\text{top}}, q_{\text{crit}})$, and for $q < q_{\text{bottom}}$. But the alternative Legendre transforms that express both α and f as functions of q only apply in the restricted range $q_{\text{bottom}} < q < \min(q_{\text{top}}, q_{\text{crit}})$.

The first major novel consequence of using the notion of multiplicative multifractal is that the function $f(\alpha)$ of a multiplicative multifractal need not, in general, be $\geqslant 0$ and shaped like $\cap$. Instead, it can take one of several alternative overall shapes, each of which corresponds to a *qualitatively* distinct behavior for the corresponding multiplicative process.

The new multiplicative multifractals introduced in this paper also raise a second important new issue. We shall show that even a fairly detailed empirical knowledge of $f(\alpha)$ gives unexpectedly incomplete information about such a measure, and we shall claim that an acceptable description of DLA requires information that goes beyond the $f(\alpha)$ that one can hope to obtain empirically. In the simplest case, yet another form of scaling comes in, characterized by a new scaling exponent λ.

Could one further generalize our multiplicative multifractals, without abandoning renormalizability? We prefer not to try until the need arises. However, the end of section 6 refers to a small step in that direction.

2. "Anomalies"

Let us elaborate on our claim that, from the viewpoint of the steepest descents formalism of Frisch–Parisi and Halsey et al., the important application to DLA turns out to present deep "anomalies". Section 6 tackles briefly the related anomalies associated with certain dynamical systems, and the very different anomalies associated with fully developed turbulence.

In the example of DLA, the growth probabilities are known to be ruled by the harmonic measure. One observes the experimental "anomaly" that, for $q<0$, the partition function $\chi(q, \varepsilon)$ *fails to scale* like $\varepsilon^{\tau(q)}$. Therefore, a blind application of existing computer programs yields wildly disagreeing $f(\alpha)$'s, and even $f(\alpha)$'s with cusps. The restricted theory of multifractals *does not* allow for these baffling and contradictory results, which is why the harmonic measure has been described as being "non-multifractal". This term appears in the very title of R. Blumenfeld and A. Aharony [8], which also includes an extensive bibliography of the rich recent literature on the multifractal study of DLA. (We hope to be forgiven if this bibliography is not repeated here). One may be tempted to elaborate on this "non-multifractality", and conclude that this harmonic measure is not renormalizable. If true, this would be a very serious setback.

However, the very same anomalies can immediately be fitted by the new classes of our multiplicative multifractals presented in sections 4 and 5. This shows that, after all, despite its anomalies, the harmonic measure may well be renormalizable. We view this conclusion as very reassuring. The price to pay is that, in these examples, the steepest descents method is altogether invalid for $q<0$, and the validity of the broader Legendre transform is something of a coincidence: this is an issue we hope to discuss in detail in the near future.

Combining these conclusions with those of several recent publications [9–13], we venture to claim that the known multifractal anomalies basically disappear if one follows the open ended approach of our general multifractals, and if one accepts the need for a separate discussion for each of several qualitatively distinct basic categories, each illuminated by well chosen special cases.

Note incidentally that the binomial and finite multinomial measures, which are non-random and are studied in detail in sections 1 to 6 of ref. [9], exhibit essentially every feature of the restricted multifractals, as defined above. In particular, the thermodynamic interpretation of the multifractal formalism comes near-unavoidably through the Lagrange multipliers borrowed from the most elementary statistical thermodynamics. Our impression is that this path shows best the simplicity of this interpretation.

3. Non-random multifractals with an infinite base that can readily be made to exhibit a new "anomaly" needed to account for DLA

One reason why many measures are expected to be multifractal (e.g., the harmonic measure on DLA) is because they are supported by a fractal set (e.g., a Julia set, or the boundary DLA). However, the basic ideas behind multiplicative fractals are best understood if this complication is postponed, and if one first examines measures supported by the interval [0, 1] (e.g., linear cuts through developed turbulence, or developed turbulence itself). Thus, using a widely known notation, we assume $D_0=1$.

This paper (which has a large overlap with ref. [12]) provides several families of multiplicative measures on [0, 1], having by and large the properties claimed in ref. [8]. Randomness brings in genuine complications, but it is possible to avoid them in the present discussion, by working with *non-random* measures. This section describes a general idea, then sections 4 and 5 provide two basic illustrative examples.

A construction. We start with the interval [0, 1] carrying a mass equal to 1. Hence, the measure we shall obtain will be usable as a probability measure. At each stage of construction, [0, 1] is divided into an infinity of (necessarily unequal) sub-intervals, which we index from right to left by the unbounded integer $\beta \geqslant 1$. The βth sub-interval of [0, 1] is taken to be of length r_β, with $\Sigma_{\beta=1}^{\infty} r_\beta = 1$, and to contain the mass m_β, with $\Sigma_{\beta=1}^{\infty} m_\beta = 1$. Next, the sub-intervals of lengths r_β are subdivided into sub-sub-intervals of lengths $r_{\beta_1} r_{\beta_2}$, for all combinations of β_1 and β_2, and these sub-sub-intervals are made to contain the respective masses $m_{\beta_1} m_{\beta_2}$. The process is allowed to continue ad infinitum. It is clear that the measure it generates is *self-similar*, in the sense that the *relative* distribution of mass is the same in all $(\text{sub})^k$-intervals.

When $m_\beta \equiv r_\beta$, this measure is, of course, uniform, but in all other cases, it is multiplicative multifractal.

To describe a multifractal, it is the custom to evaluate its function $\tau(q)$. We must postpone to a later occasion a critical discussion of the meaning of $\tau(q)$ when the subdivision is infinite. We shall be content to explore a blind generalization of a formula that is familiar in the restricted theory. There, $\tau(q)$ is known to be given implicitly by the following relation due to Hentschel and Procaccia, which we call the "generating equation":

$$\sum_{\beta=1}^{\infty} m_\beta^q r_\beta^{-\tau(q)} = 1 .$$

A special case. Take $m_\beta = m^{\beta-1} - m^\beta$, with $0 < m < 1$, and $r_\beta = r^{\beta-1} - r^\beta$, with $0 < r < 1$. A moment's thought shows that the resulting multiplicative measure reduces to the "skew binomial" multifractal, which is defined by assigning the masses m and $1-m$ to sub-intervals of $[0, 1]$ having the lengths r and $1-r$. The graph of the resulting $f(\alpha)$ is well known to be shaped like $\cap$. Also, $\alpha_{\min}$ and $\alpha_{\max}$ are, respectively, the smaller and the larger of the two quantities $\log m/\log r$ and $\log(1-m)/\log(1-r)$.

Generalization. The reason for introducing the new construction is, of course, that it also allows a variety of more interesting outcomes. The nature of their anomalies will be seen to depend on the following functions:

$$M^*(\beta) = -\log_2 \sum_{u=\beta}^{\infty} m_u , \quad R^*(\beta) = -\log_2 \sum_{u=\beta}^{\infty} r_u$$

$$\text{and} \quad \alpha^*(\beta) = \frac{M^*(\beta)}{R^*(\beta)} .$$

A criterion. Suppose that $\lim_{\beta\to\infty} \alpha^*(\beta)$ exists. (The case when it does not exist will be examined elsewhere.) The novel and interesting features that motivate our new construction are the following. We show momentarily that $\lim_{\beta\to\infty} \alpha^*(\beta) = \infty$ is a sufficient condition for $\alpha_{\max} = \infty$, hence for $q_{\text{bottom}} = 0$. We show in section 7 that $\lim_{\beta\to\infty} \alpha^*(\beta) = 0$ is a sufficient condition for $\alpha_{\min} = 0$, hence for $q_{\text{top}} = 1$. The condition that $\lim_{\beta\to\infty} \alpha^*(\beta)$ exists and is >0 and $<\infty$, is sufficient to obtain the "usual" situation, in which $0 < \alpha_{\min} < \alpha_{\max} < \infty$, hence $q_{\text{top}} = \infty$ and $q_{\text{bottom}} = -\infty$.

The case $\alpha_{\max} = \infty$. A sufficient condition for $\alpha_{\max} = \infty$ is that one can identify at least one point where $\alpha = \infty$. When $\alpha^*(\beta) \rightarrow \infty$, such is indeed the case at the left most point of $[0, 1]$. It is easy to see that in that case $0 < \alpha_{\min} < \infty$.

A consequence of $\alpha_{\max} = \infty$ is that $q_{\text{bottom}} = 0$, meaning that for $q < 0$, the function $\chi(q, \varepsilon)$ *fails* to scale like $\varepsilon^{\tau(q)}$, and the function $\tau(q)$ fails to be defined. To prove this, observe that the largest addend in the sum $\Sigma\, \mu_j^q(\varepsilon)$ always comes from the interval where $\mu(\varepsilon)$ is smallest. In this instance, this interval goes from 0 to ε. When $\varepsilon = \Sigma_{u=\beta}^{\infty} r_u$ for some β, this interval contains the mass $\Sigma_{u=\beta}^{\infty} m_u$. Thus,

$$\min_j \mu_j(\varepsilon) \sim \varepsilon^{\alpha^*(\beta)} \quad \text{and} \quad \sum \mu_j^q(\varepsilon) \geqslant \varepsilon^{q\alpha^*(\beta)} .$$

Does there exist a finite $\tau(q)$ such that $\Sigma\, \mu_j^q(\varepsilon) = \varepsilon^{\tau(q)}$ for $\varepsilon \rightarrow 0$? The answer is *no*, because assuming this behavior would yield a contradiction. [*Proof*: A little algebra would yield the result that, for every q, $|\tau(q)/q| \geqslant \alpha^*(\beta)$ for $\varepsilon \rightarrow 0$. But $\beta \rightarrow \infty$ as $\varepsilon \rightarrow 0$, and $\lim_{\beta=\infty} \alpha^*(\beta) = \infty$ is true by hypothesis. Hence $\tau(q)/q$ would be infinite for all q.] Observe that a non-defined $\tau(q)$ enters in the form of $\tau(q) \rightarrow -\infty$.

Conclusion. When $\alpha^*(\beta) \rightarrow \infty$, our infinite base multiplicative multifractal satisfies $\alpha_{\max} = \infty$, hence is not a restricted multifractal.

The very special role of $m_\beta = 2^{-\beta}$. Note that, in the preceding discussion, the algebra can be greatly simplified by simplifying the dependence of either $M^*(\beta)$ or $R^*(\beta)$ on the parameter β. Given that $M^*(1) = R^*(1) = 0$, this parameter may simply be set to be $\beta - 1$. It is easier to set $M^*(\beta) = \beta - 1$, i.e., $\Sigma_{u=\beta}^{\infty} m_u = 2^{1-\beta}$. Then the expressions for m_β and for the criterion function $\alpha^*(\beta)$ become

$$m_\beta = 2^{-\beta} \quad \text{and} \quad \alpha^*(\beta) = (\beta - 1)/R^*(\beta) .$$

It is easy to see that one may redefine $M^*(\beta)$ and $R^*(\beta)$ with logarithms of base $1/m \neq 2$. This would suggest selecting $m_\beta = m^{\beta-1} - m^\beta$ with $m \neq 2^{-1}$, but $0 < m < 1$. However, the change would bring no significant generalization, only the replacement of $\log 2$ by $-\log m$ in most formulas.

Section 7 extends the above considerations to the case $\alpha_{\min} = 0$, which is not needed in sections 4–6.

4. A first family of new multifractals: one-sided $f(\alpha)$'s yielded by $m_\beta = 2^{-\beta}$ and $r_\beta = \beta^{-\lambda} - (\beta + 1)^{-\lambda}$, where $\lambda > 0$ is a parameter

Now, having set $m_\beta = 2^{-\beta}$, we proceed in this section and the next to examine in detail two simple examples of sequences r_β.

When $r_\beta = \beta^{-\gamma} - (\beta+1)^{-\gamma}$, our criterion function of section 3 becomes

$$\alpha^*(\beta) = (\beta - 1)/R^*(\beta) = (\beta - 1)/\lambda \log_2 \beta .$$

As $\beta \to \infty$, this quantity tends (rapidly) to ∞, so that $\alpha_{\max} = \infty$.

Minimal measure. We have

$$\min_j \mu_j(\varepsilon) = 2^{1-\beta} .$$

Since $\varepsilon = \beta^{-\gamma}$ and $L = 1/\varepsilon = \beta^{\gamma}$, elimination of β yields

$$\min_j \mu_j(\varepsilon) = \mathrm{e}^c \exp(-c\varepsilon^{-1/\lambda}) \quad \text{and} \quad \min \mu_j(L) = \mathrm{e}^c \exp(-cL^{1/\lambda}) .$$

The constant c is non-intrinsic: here, $c = \log 2$, but taking $m_\beta = m^{\beta-1} - m^\beta$ with $m \neq \frac{1}{2}$ would yield $c = -\log m$.

This logarithmic behavior is postulated in ref. [8], and has strongly contributed to our writing the present paper.

The function $\tau(q)$ for $q < q_{\text{bottom}} = 0$. We see that, irrespective of the choice of $\tau(q)$, one has $2^{\beta|q|}\beta^{(\lambda+1)\tau(q)} \to \infty$. Therefore, the left-hand side of the generating equation diverges for all $\tau(q)$. This confirms that the implicit equation for $\tau(q)$ has no solution. That is, $\tau(q)$ is not defined. To give further evidence that $\Sigma\, \mu^q(\varepsilon)$ *fails* to scale like $\varepsilon^{\tau(q)}$, note that

$$\chi(q, \varepsilon) \geqslant [\min_j \mu_j(\varepsilon)]^q \sim \exp(|q|c\varepsilon^{-1/\lambda}) .$$

On the basis of certain limit theorems of probability (whose scope is beyond the present discussion), we believe that the symbol $\geqslant$ can be replaced by the symbol $\sim$. In any event, the present multiplicative measure is not a restricted multifractal.

The function $\tau(q)$ for $q > q_{\text{bottom}} = 0$, The situation is very different. Lengthy but conceptually straightforward arguments [12] show that for non-integer λ one has for small $q > 0$:

$$\tau(q) = -1 + (c_1 q - c_2 q^2 + \cdots) + c_\lambda q^\lambda + \cdots$$

The function $f(\alpha)$. Now let us move (again, formally) to the function $f(\alpha) = \min_q[\alpha q - \tau(q)]$. For small α, $f(\alpha)$ is ruled by the large positive q's. Hence, $f(\alpha)$ is non-anomalous: it is positive with a readily obtained value of $\alpha_{\min}$. But larger α's are ruled by the small positive q's. Here, the leading terms of $\tau(q)$ are

$$\tau(q) = \begin{cases} -1 + c_\lambda q^\lambda & \text{for } \lambda < 1\,, \\ -1 + c_1 q + c_\lambda q^\lambda & \text{for } 1 < \lambda < 2\,, \\ -1 + c_1 q - c_2 q^2 & \text{for } \lambda > 2\,. \end{cases}$$

Hence, the Legendre transform defined by $f(\alpha) = \min_q[\alpha q - \tau(q)]$ exhibits one of the following extreme anomalies for sufficiently large α. Write $\kappa = \min[2, \lambda/(\lambda - 1)]$.

For $\lambda < 1$, one has $\kappa < 0$ and one finds that $\alpha_0 = \infty$ and that $f(\alpha) \sim 1 - c_\kappa(\alpha - \tilde{\alpha})^\kappa$ as $\alpha \to \infty$. Here c_k and $\tilde{\alpha}$ are constants that could be "tuned" by changing r_β for small β.

For $\lambda > 1$, one has $\kappa > 0$ and one finds that $\alpha_0 < \infty$ and that $f(\alpha) \sim 1 - c_\kappa(\alpha_0 - \alpha)^\kappa$ for $\alpha \sim \alpha_0 - \varepsilon$ and $f(\alpha) = 1$ for $\alpha > \alpha_0$.

Let us add that, when λ is an integer, there are terms in $q^\lambda \log(1/q)$. For $\lambda = 1$, one has $\tau(q) \sim -1 + c_1 q \log(1/q)$, which yields $f(\alpha) \sim 1 - c_\kappa \exp[-c'(\alpha - \tilde{\alpha})]$ for $\alpha \to \infty$.

For $\lambda > 1$ and small α, $f(\alpha)$ is shaped like the left half of $\cap$. Thus, for $\lambda \leqslant 1$ the right side of $f(\alpha)$ is nonexistent and for $\lambda > 1$ the right side reduces to the horizontal. For all λ, $f(\alpha)$ can be said to be "left sided". The three behaviors of $f(\alpha)$ are illustrated schematically by the bold curves of fig. 1.

Recall that one can also determine $f(\alpha)$ by giving f and α as functions of the exponent q taken as parameter. There would no change for $\lambda \leqslant 1$. But for $\lambda > 1$, the function $f(\alpha)$ would only be defined for $\alpha \leqslant \alpha_0$, which would be incorrect, and would hide an extremely unexpected and important aspect of a complicated reality.

Sampling and the definition of an "effective" $f(\alpha)$. Now let a large number N of points be chosen independently on [0, 1] following the probabilities given by

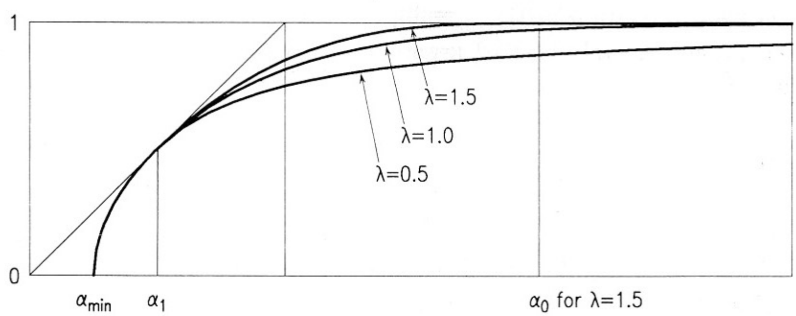

Fig. 1. The three basic shapes of $f(\alpha)$ for a left sided multifractal. The low α portions are completely schematic. Precise graphs are found in ref. [12].

our measure μ. A box of measure $\mu_j(\varepsilon)$ then will include $N\phi_j(\varepsilon, N)$ points, with $\phi_j(\varepsilon, N) \sim \mu_j(\varepsilon)$, except that most boxes such that $N\mu_j(\varepsilon) < 1$ will be empty.

In restricted multifractals, the number of empty boxes decreases rapidly as N is made to increase faster than $\varepsilon = 1/L$. The frequencies $\phi_j(\varepsilon, N)$ are then combined into a partition function, which yields a sample estimate of $\tau(q)$. Finally, $f(\alpha)$ is *defined* as the Legendre transform of $\tau(q)$. In left sided multifractals, to the contrary, we know that the same approach leads to a meaningless estimate of $f(\alpha)$. One would like, however, to be able to define for each ε a notion of "effective" $f_\varepsilon(\alpha)$. Unfortunately, the restricted theory has no room for such a notion.

The quantities $\phi_j(\varepsilon)$ also enter into our general theory of multiplicative multifractals, but in a very different fashion, which does happen to involve an effective $f_\varepsilon(\alpha)$. The difference is very important, but in this paper the issue can only be sketched. The point is that our approach does not *define* $f(\alpha)$ as the Legendre transform of $\tau(q)$. Instead, given that $D_0 = 1$ in the present case, our approach introduces the quantity $\rho(\alpha) = f(\alpha) - 1$ as a limit of the form $\rho(\alpha) = \lim_{\varepsilon\to 0} \rho_\varepsilon(\alpha)$, where $\rho_\varepsilon(\alpha) = -(1/\log \varepsilon) \log p_\varepsilon(\alpha)$, and where the quantities $p_\varepsilon(\alpha)$ are the probability densities of α for given ε.

To form $\rho_\varepsilon(\alpha)$ is to take the logarithm of a probability density, and to renormalize it by dividing by $\log \varepsilon$. This is precisely how the Holder exponent α is obtained from μ, namely, by taking the logarithm and renormalizing it through division by $\log \varepsilon$. To plot a function on doubly logarithmic coordinates is a cliché in the study of fractals. But here the plot *is not* a straight line! The fact that the sequence $\rho_\varepsilon(\alpha)$ does indeed have a limit is a remarkable feature of the main tool of our theory, which is Harald Cramer's theory of large deviations [9]. As may have been expected, the Cramer theory does use the Legendre transform, but only to *evaluate* $f(\alpha)$, not to *define* it.

Now we come to an important distinction. For the restricted multifractals, $f(\alpha)$ is useful even for relatively large ε. The reason is that the convergence of $\rho_\varepsilon(\alpha)$ to $\rho(\alpha)$ is acceptably rapid and uniform, and the "finite ε" corrections $\rho(\alpha) - \rho_\varepsilon(\alpha)$ have not yet been found to be needed in physics. (They probably will !)

For the special multipliers that yield left sided multifractals, to the contrary, the convergence of $\rho_\varepsilon(\alpha)$ to $\rho(\alpha) = f(\alpha) - 1$ turns out to be extraordinarily slow and the shape of the approximant, even if ε is small, gives a totally misleading idea of the actual limit $f(\alpha)$. We have even seen that, if $\lambda > 1$ (hence $\alpha_0 < \infty$), the limit of $\rho_\varepsilon(\alpha)$ is identical to 0 for $\alpha > \alpha_0$. Therefore, this limit hardly matters at all when $\varepsilon > 0$.

Instead what does matter greatly is the "preasymptotic" behavior of $\rho_\varepsilon(\alpha)$. This paper gives us no room to go beyond asserting that, denoting $-\log \varepsilon$ by k, one has

$$p_k(\alpha) \sim ck^{1-\lambda}\alpha^{-\lambda-1} \quad \text{and} \quad \rho_k(\alpha) \sim (1/k)\log(ck^{1-\lambda}\alpha^{-\lambda-1}) .$$

Our fig. 2 compares very schematically the shapes of $\rho_k(\alpha)+1$ and of $f(\alpha)$. In a different context, fig. 2 of ref. [14] had illustrated the same effect (but without explaining it) for data from a multifractal that a recent study [15] has shown to be closely related to our case $\lambda = 1$ (see section 6).

The expression $f_k(\alpha) = \rho_k(\alpha) + 1$ *is not the Legendre transform of any function* $\tau_k(q)$. One of many reasons is that all Legendre transforms are cap-convex, while our $\rho_k(\alpha)$ is cup-convex for large α.

In a rough way, $f_k(\alpha)$ is a modified box dimension that would only concern boxes of size b^{-k}. But there is no Hausdorff–Besicovitch dimension behind this box dimension.

Pre-asymptotics and the search for an "effective" $\tau(q)$. Given that

$$\log \chi(q, \varepsilon)/\log \varepsilon = |q|(\log 2/\lambda)\varepsilon^{-1/\lambda}/\log \varepsilon ,$$

it is tempting to view the right-hand side as a "variable" $\tau(q)$. This amounts to defining, for each ε and $q<0$, an "effective" function $\tau(q, \varepsilon)$ that satisfies $\tau(q, \varepsilon) \to -\infty$ as $\varepsilon \to 0$. When λ is small, this effective $\tau(q, \varepsilon)$ will vary greatly with ε, and it will be difficult to fit the data by a single straight line. But when λ is large, $\tau(q, \varepsilon)$ will vary more slowly, and tolerant curve fitting algorithms will manage to fit a single straight line. Alternatively, a physicist overzealous to find a straight fit will decide that one may discard the data for small ε, or perhaps for largish ε, and then fit a straight line to the remaining data. This is the procedure that seems to have produced many of the published $f(\alpha)$'s relative to DLA; no wonder that different statistical "fixes" yield different results.

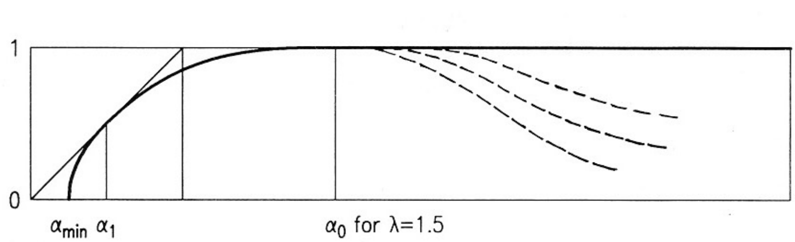

Fig. 2. This schematic view of the convergence of $\rho_k(\alpha)+1$ to $f(\alpha)$ shows that the left-sidedness of the asymptotic $f(\alpha)$ fails to be reflected in the shape of finite sample approximants to $\rho_k(\alpha)+1$. The value of k increases from bottom to top. Detailed study of other classes of multifractals suggests that the form of $f(\alpha)$ to the left of α_0 is likely to differ markedly from $\rho_k(\alpha)+1$ near its maximum. The difference or "bias" is likely to depend upon details or r_β for moderate β, and we prefer to disregard it here.

Truncation. One can truncate our multiplicative multifractals by replacing all the sub-intervals with $u > \beta$ by a single sub-interval with the mass $m'_\beta = 2^{-\beta}$ and the length $\varepsilon(\beta) = R^*(\beta) = \beta^{-1}$. Denote the truncated measure by μ_β. When inspection stops at scales $\geqslant \varepsilon(\beta)$, the true μ and the truncated μ_β cannot be told apart. Moreover, the truncated measure seems to be a restricted multifractal. Therefore, one may hope to describe it by standard functions $\tau(q, \beta)$ and $f(q, \beta)$, with the property that, as $\beta \to \infty$, one has $\tau(q, \beta) \to \tau(q)$ and $f(q, \beta) \to f(\alpha)$.

In practice, this hope is unfortunately unfulfilled. It is indeed true that $\chi_\beta(q, \varepsilon) \sim \varepsilon^{\tau(q,\beta)}$ for small enough ε, but "small enough" means smaller than $\varepsilon(\beta) = \beta^{-1}$. However, these scales are beyond the reach of observation, which is (as a result of truncation) restricted to $\varepsilon > \beta^{-\lambda}$.

Screening. Thinking in terms of the hoped-for use of these measures to model DLA, values $\alpha < \alpha_1 - \varepsilon$ could correspond to "unscreened sites" and values of $\alpha > \alpha_1 + \varepsilon$ could correspond to "screened sites". Thus, in the case $\lambda > 1$ (but not for $\lambda \leqslant 1$), the screened sites subdivide further according to whether $\alpha < \alpha_0$ or $\alpha > \alpha_0$.

Mathematical digression and query. In the case of restricted multifractals, it is known that $f(\alpha)$ is a Hausdorff–Besicovitch dimension. For the binomial measure, this property follows from difficult but standard theorems of Eggleston and also of Volkmann (see ref. [9], p. 24). Turning to the present generalization and supposing that $\alpha_0 < \infty$ and $f(\alpha) = 1$ for $\alpha > \alpha_0$, consider the Hausdorff–Besicovitch dimension of the set of points where α satisfies $\alpha_0 < \alpha' < \alpha < \alpha''$, for given $\alpha' > \alpha_0$ and $\alpha'' > \alpha'$. It is tempting to conjecture that this set's dimension is 1 for all α' and α''. Nevertheless, this set's linear measure (that is, its Lebesgue measure and its Hausdorff measure in the dimension 1) is necessarily 0, because the Holder alpha equals α_0 almost surely.

Smooth or arbitrary-order phase transitions. Our model demonstrates the possibility of a smooth "phase transition" in the $f(\alpha)$ curve, characterized by $\kappa < 1$, as well as of transitions of all finite orders $\geqslant 2$.

5. A gradual cross over from restricted to left sided multifractals. Second family of left sided $f(\alpha)$'s, and their extension

The input $r_\beta \sim \beta^{-\lambda-1}$ used in section 4 was knowingly "tailored" for very specific goals. Recalling the very special example by which we had started in section 3, we also know already that our new infinite base construction is allowed to lead to a $\cap$-shaped $f(\alpha)$. It remains to input alternative sequences r_β, in order to span the wide gap between the skew binomial and the family $r_\beta \sim \beta^{-\lambda-1}$ in section 4.

A first multiparameter family is based on $m_\beta = 2^{-\beta}$ *and* $\Sigma_{u=\beta}^{\infty} r_u = \exp[-\lambda(\log\beta)^\eta]$, *where* $\lambda > 0$ *and* $\eta > 0$. There is nothing new to the value $\eta = 1$, which brings us back to $r_\beta = \beta^{-1} - (\beta+1)^{-\lambda}$.

For all η, the criterion function of section 3 is $\alpha^*(\beta) \sim \beta/\lambda(\log\beta)^\eta$. Therefore, $\alpha^*(\beta) \to \infty$ as $\beta \to \infty$, and we deal with a left sided multifractal. When $\eta > 1$, this measure is even more "anomalous" than any measure obtained via $r_\beta \sim \beta^{-(\lambda+1)}$. And for $\eta < 1$, this family does not reached down to the skew binomial.

A far more interesting second multiparameter family is based on $m_\beta = 2^{-\beta}$ *and* $\Sigma_{u=\beta}^{\infty} r_u = \exp[-\lambda(\beta-1)^\eta]$, *where again* $\lambda > 0$ *and* $\eta > 0$. Again, the value $\eta = 1$ brings us back to $r_\beta = r^{\beta-1} - r^\beta$, with $r = \mathrm{e}^{-\lambda}$, i.e., to the measure in section 4. For very small η, β^η is "like" $\log\beta$, and r_β is "like" $\beta^{-(\lambda+1)}$. This suggests that this family can be said to include the range from the multifractal in section 4 to the skew binomial of section 3.

The criterion function of section 3 becomes

$$\alpha^*(\beta) = (\beta-1)^{1-\eta}/\lambda .$$

When $\eta < 1$, we find $\alpha_{\max} = \infty$, but the power $(\beta-1)^{1-\lambda}$ increases far less rapidly than the ratio $\sim\beta/\log\beta$ in section 4. When $\eta > 1$, we find $\alpha_{\min} = 0$, a possibility discussed in section 7.

The value of $\mu_j(\varepsilon)$ *at the left most point of* $[0, 1]$. We have

$$\mu_j(\varepsilon) = \mathrm{e}^c \exp(-c\lambda^{-1/\eta}|\log\varepsilon|^{1/\eta}).$$

Again, the constant c is non-intrinsic: here, it is $c = \log 2$, but taking $m_\beta = m^{\beta-1} - m^\beta$ with $m \neq \frac{1}{2}$ would yield $c = -\log m$.

Remark. A preprint we have received since the rest of the paper has been written [16] proposes this last expression to describe the harmonic measure of DLA, in preference to the exponential dependence feature in section 4. This is not the proper place to discuss which of our several analytic expressions gives a better fit to the data. But one reason for including the present example is to show that all the behaviors that have been proposed so far can fit very well in the framework of our multiplicative multifractals.

The extremal $\alpha_j(\varepsilon)$. The values of α is either largest (if $\eta < 1$), or smallest (if $\eta > 1$) at the right end point of $[0, 1]$, where one has

$$\text{extremal } \alpha_j(\varepsilon) = (\log 2)\lambda^{-1/\eta}(|\log\varepsilon|)^{-1+1/\eta} .$$

As expected, extremal $\alpha_j(\varepsilon) \to \infty$ if $\eta < 1$, and $\to 0$ if $\eta > 1$.

The function $\tau(q)$. In the generating equation for $\tau(q)$, the addend is

$$2^{-q\beta} r_\beta^{-\tau(q)} = \lambda\eta\beta^{\eta-1} \exp[-(\log 2) q\beta + \lambda\tau(q)\,(\beta-1)^\eta]\,.$$

When $\eta < 1$ and $q < 0$, the term $2^{|q|\beta}$ predominates. Therefore, no choice of $\tau(q)$ (of either sign) can prevent the addend from tending to infinity with β. A fortiori, the left-hand side of the generating equation diverges for all $\tau(q)$. The generating equation has no solution for $q < 0$. But for $\eta < 1$ and $q > 0$, no problem is expected, and none arises, because $2^{-q\beta}$ predominates. Hence $q_{\text{bottom}} = 0$, and we deal with a left sided multifractal.

When $\eta > 1$, to the contrary, no problem is expected if $q < 0$, and no problem arises. In that case, $\tau = 0$ yields $\Sigma\, 2^{-q\beta} > 1$, while $\tau \ll 0$ yields $0 < \Sigma\, 2^{-q\beta} \ll 1$. The sum $\Sigma\, 2^{-q\beta} r_\beta^{-\tau}$ being monotone increasing as function of τ, its value necessarily crosses 1 for some negative value of $\tau(q)$. But if $\eta > 1$ and $q > 1$, all $\tau < 0$ yield $\Sigma\, 2^{-q\beta} r_\beta^{-\tau} < 1$, and for all $\tau > 0$, one has $2^{-q\beta} r_\beta^{-\tau} \to \infty$ as $\beta \to \infty$. Hence, $q_{\text{top}} = 1$.

This leaves $\eta = 1$ as yielding the only restricted multifractal in this family.

The shapes of $\tau(q)$ *and* $f(\alpha)$. Without being able to dwell on details, let us say that for all η and λ the graph of $f(\alpha)$ is made of two parts: for $\alpha < \alpha_0$, one has a left side with a second order maximum, and for $\alpha > \alpha_0$, one has a horizontal right side. (Hence, we deal with a "phase transition" of order 2.) For the purposes of the following paragraph, we shall write this $f(\alpha)$ as $f_{\text{true}}(\alpha)$.

Important discontinuities. A most interesting sharp discontinuity arises between $\eta = 1$ and η just below 1. Below $\eta = 1$, one must define two distinct $f(\alpha)$'s. The left sided "true" $f_{\text{true}}(\alpha)$ only concerns the far-out asymptotics. That is, $f_{\text{true}}(\alpha)$ only matters when ε is below a threshold that converges to 0 with $1 - \eta$. When ε is not small, the mass in an interval of length ε is for all practical purposes skew binomial, i.e., it is ruled by an $f(\alpha)$ that is $\cap$-shaped.

What about intermediate and decreasing values of ε? To describe them, it is best again to use our probabilistic definition of $f(\alpha)$ as a limit, a definition that is already touched upon in section 4. It turns out that one should expect the histogram of α to be shaped like $\cap$, except for the added presence of an "anomalously" large number of "anomalously" large values that form a "tail" for large α. Using the statisticians' vocabulary, one would be tempted to consider these large values to be "outliers" generated by a "contaminating" mechanism unrelated to the mechanism that generates the $\cap$-shaped portion. For data in our spanning family, however, this would be a totally incorrect interpretation.

A symmetric sharp discontinuity arises between $\eta = 1$ and η just below 1, but now the apparent "outliers" are expected to be found to the left of the $\cap$-shape relative to $\eta = 1$.

6. Miscellaneous remarks

For details of the multifractals described in this paper, see refs. [12, 13] and forthcoming references. The connections between the construction of sections 4 and 5 and the physics of DLA begin to be understood [13].

An issue of rigor. Sections 3 and 5 have started with a formal equation for $\tau(q)$ and have continued by using diverse formal manipulations that are known to be valid for restricted multifractals. But the function $f(\alpha)$ obtained by these manipulations falls beyond the restricted class. This raises an issue of rigor that we shall have to discuss elsewhere.

A genuine underlying complication is that $\tau(q)$ can be defined in either of two different ways: The partition function [5, 6] can be used when μ is a restricted multifractal, but in other cases one must use the earlier definition we had advanced in ref. [1].

Randomness and its necessity in modeling. One point of the present paper is that, even in the non-random case, the multiplicative multifractals introduced in ref. [1] are more general than the multifractals introduced in refs. [5, 6].

A different advantage of the approach in ref. [1] is that it applies immediately and rigorously to random multifractals. Of course, in the hands of many researchers, the theory in refs. [5, 6] has been informally extended to the random case by resorting to averaging. The reason we call these extensions *informal* is because they are not based on any theory. This is why they involve murky discussions of which of seveal methods of averaging is "the best". Our original theory of multiplicative multifractals tackles these issues in advance.

One feature of our random multiplicative multifractals, is that they involve a different definition of $\tau(q)$.

The anomalies of $f<0$ and $\alpha<0$ in random multifractals, and energy dissipation in fully developed turbulence. This paper has discussed the anomalies that consist either in $\alpha_{\max}=\infty$ with $f(\alpha_{\max})=1$, or in $\alpha_{\max}=f(\alpha_{\min})=0$. We wish to stop for just a moment to draw attention to refs. [10, 11], which discuss very different anomalies, which are needed in the study of turbulence.

One of these anomalies is due to the presence of negative $f(\alpha)$'s. When one starts with the usual formalism and then superposes diverse methods of averaging upon it, one finds that negative $f(\alpha)$'s do occur with certain methods, but not with others, and their origin and meaning is totally obscure.

In our approach, to the contrary, they are perfectly well understood and essential. In broad outline, the positive $f(\alpha)$'s can serve to define a *typical* distribution of a random fractal measure, and the negative $f(\alpha)$'s describe *fluctuations* one may expect in a finite size sample.

Ref. [11] discusses yet another anomaly, which allows $\alpha<0$ and gives rise to the bound q_{crit} on q, first introduced in ref. [1]. One has $1 \leq q_{\text{crit}} \leq q_{\text{top}}$ and $\tau(q_{\text{crit}}) = 0$; when $q_{\text{crit}} = 1$, one also has the further relation $\tau'(1) = 0$. This definition of q_{crit} only makes sense with our definition of $\tau(q)$. However, q_{crit} also entes in the study of non-random multifractals, where it is necessary to justify calling $D(q) = \tau(q)/(q-1)$ a form of dimension. (Note that the treatment of H in ref. [9], hence fig. 5, are in error; the correct result is given in ref. [11]. The journal issue in which ref. [9] appears is also reproduced as a book [17]. This reference is summarized in ref. [18].)

On left sided multifractals that arise in dynamical systems. Let us mention that the "anomaly" of left sided $f(\alpha)$, which is described in this paper, is not limited to multifractals in real space, like DLA, but does extend to certain dynamical systems. This fact broadens the impact of our criticism of the restricted multifractals as being a tool of inadequate generality for the needs of physics. The new application, described in ref. [15], involves a non-linear multifractal cascade already discussed in ref. [14]. (Ref. [14] came very close to identifying left sided multifractals and the anomaly $\alpha_{\min} = f(\alpha_{\min}) = 0$ of section 7, but it stopped before the critical last steps.)

It is shown in ref. [15] how the multifractal in section 4 was originally obtained form the multifractal in ref. [14]: by a linearization that yielded $\lambda = 1$, followed by a generalization that allowed other values of λ. Thus, the argument in section 3, which may seem to have come a priori, was only developed after the fact.

7. Continuation of section 3

Let us add a little to the considerations in section 3.

The case $\alpha_{\min} = 0$. First, a warning. In order that $\alpha_{\min} = 0$, a *sufficient* condition is that some point carries an "atom" of positive measure. But this condition is *not* necessary: $\alpha_{\min} = 0$ also holds when $\mu[t, t+\varepsilon] \sim L(\varepsilon)$, where $L(\varepsilon)$ is any function (such as $1/|\log \varepsilon|$) that $\to 0$ with ε, but more slowly than any positive power ε^{α}.

Now, let us study $\alpha_{\min} = 0$. The argument proceeds roughly as in section 3 for $\alpha_{\max} = \infty$. A sufficient condition for $\alpha_{\min} = 0$ is that one can identity at least one point where $\alpha = 0$. When $\alpha^*(\beta) \to 0$, such is indeed the case for the left-most-point of $[0, 1]$. It is easy to see that, in that case, $0 < \alpha_{\max} < \infty$.

A consequence of $\alpha_{\min} = 0$ is that $q_{\text{top}} = 1$, meaning that for $q > 1$, $\chi(q, \varepsilon)$ *fails* to scale like $\varepsilon^{\tau(q)}$, and the function $\tau(q)$ *fails* to be defined. [*Proof*: Observe that we now have $\max_j \mu_j(\varepsilon) \sim \varepsilon^{\alpha^*(\beta)}$.] Hence, the existence of $\tau(q)$ would, again, require the inequality $\varepsilon^{q\alpha^*(\beta)} \geq \varepsilon^{\tau(q)}$. When $q > 0$, this becomes $\tau(q) \leq q\alpha^*(\beta)$, which would require $\tau(q) \leq 0$. Since $\tau(q) \geq 0$ for $q > 1$, we have proved that $\tau(q)=0$ for $q>1$, if $\tau(q)$ exists. But, if $q>1$, it would follow from $\tau(q)=0$ that $\chi(q, \varepsilon)=$ constant for all ε. This last relation only holds if the measure concentrates at $\alpha=0$, which we shall see is not the case.] Hence, $\tau(q)$ is not defined for $q>1$.

Conclusion. When $\alpha^*(\beta)\rightarrow 0$, our infinite base multiplicative multifractal satisfies $\alpha_{\min}=0$, hence is not a restricted multifractal.

Reciprocity between the anomalies $\alpha_{\max}=\infty$ and $\alpha_{\min}=0$. It happens that much about our new multiplicative multifractals does not depend on the functions m_β and r_β, i.e., on $M^*(\beta)$ and $R^*(\beta)$ taken separately. They depend solely on the behavior of the functions $R^*(M^*)$ and $M^*(R^*)$, obtained by eliminating β between the functions $M^*(\beta)$ and $R^*(\beta)$. The fact that two alternative functions are involved expresses that our input quantities m_β and r_β obey exactly the same constraints. It also follows that one can exchange their roles. This will exchange the roles of M^* and R^*, and replace the anomaly $\alpha_{\max}=\infty$ by the anomaly $\alpha_{\min}=0$, or conversely.

The two measures obtained in this fashion, call them reciprocal and denote them by μ and $\tilde{\mu}$, must be closely related. Indeed, one can verify that the functions $\mu([0, \tilde{\mu}])$ and $\tilde{\mu}([0, \mu])$, which are monotone increasing, are the inverse of each other. Graphically, the relation between μ and $\tilde{\mu}$ is as illustrated in the case of a very analogous measure by figs. 1 and 6 of ref. [14], where the graph of μ as function of $\tilde{\mu}$ is called a "slippery staircase". Its apparent horizontal steps are in fact "not quite" horizontal, which is why the measure $\tilde{\mu}$ does not quite include atoms. We shall publish elsewhere the proof that the functions f and $\tilde{f}$ that characterize μ and $\tilde{\mu}$ are linked by $\tilde{f}(\alpha)=\alpha f(1/\alpha)$.

Combining the anomalies $\alpha_{\max}=\infty$ and $\alpha_{\min}=0$ through a bilateral generalization of our construction. Now make the parameter β an integer between $-\infty$ and ∞. This generalization allows the relations $\lim_{\beta\rightarrow\infty}\alpha^*(\beta)=-\infty$ and $\lim_{\beta\rightarrow-\infty}\alpha^*(b)=0$ to be both true. It is even easy to satisfy the identity $\tilde{f}=\alpha f(1/\alpha)$, which is a way to insure that the reciprocal measure $\tilde{\mu}$ is identical to μ. An example when this goal is fulfilled is when $m_{-\beta}=r_\beta$ and $r_{-\beta}=m_\beta$. The example shows that, even after the behavior of $\alpha^*(\beta)$ at $\beta\rightarrow\infty$ has been fixed, there exist an infinity of different "self-reciprocal" measures. When $\alpha_{\max}<\infty$, and $\alpha_{\min}=1/\alpha_{\max}>0$, these measures are restricted multifractals.

Acknowlegements

Many discussions with C.J.G. Evertsz (over many months at Yale) and with A. Aharony (before during and after the Cyril Domb meeting) have been invaluable. Aftereffects of an earlier collaboration with M.C. Gutzwiller and inputs from R. Stinchcombe, D. Coppersmith and Y. Hayakawa have been most helpful. The author also acknowledges the hospitality of the Sackler Institute for Solid State Physics at Tel-Aviv University, where much of this paper was written. This research has been financially supported in part by the Office of Naval Research, Grant N00014-88-K-0217.

References

[1] B.B. Mandelbrot, J. Fluid Mech. 62 (1974) 331; also Compt. Rend. (Paris) A278 (1974) 289, 355.
[2] B.B. Mandelbrot, in: Statistical Physics 13, Int. IUPAP Cont., Haifa, 1977, D. Cabib, C.G. Kruper and I. Riess, eds., Ann. Israel Phys. Soc. 2 (1) (1978) 225.
[3] B.B. Mandelbrot, The Fractal Geometry of Nature (Freeman, New York, 1982).
[4] B.B. Mandelbrot, Fractals and Multifractals: Noise, Turbulence and Galaxies (Selecta, vol. 1) (Springer, New York), forthcoming.
[5] U. Frisch and G. Parisi, in: Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics, M. Ghil, ed., International School of Physics Enrico Fermi, Course 88 (North-Holland, Amsterdam, 1985) p. 84.
[6] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia and B.I. Shraiman, Phys. Rev. A 33 (1986) 1141.
[7] A. Aharony, Physica A 168 (1990) 479, these Proceedings.
[8] R. Blumenfeld and A. Aharony, Phys. Rev. Lett. 62 (1989) 2977.
[9] B.B. Mandelbrot, Pure Appl. Geophys. 131 (1989) p 5.
[10] B.B. Mandelbrot, Physica A 163 (1990) 306; Developed in: Fractals, Proc. Erice Meeting, 1988, L. Pietronero, ed. (Plenum, New York, 1989).
[11] B.B. Mandelbrot, in: Frontiers of Physics, Landau Memorial Conference, Tel Aviv, 1988, E. Gotsman, ed. (Pergamon, New York, 1989) p. 91.
[12] B.B. Mandelbrot, C.J.G. Evertsz and Y. Hayakawa, Phys. Rev. A (1990).
[13] B.B. Mandelbrot and C.J.G. Evertsz, Nature, submitted.
[14] M.C. Gutzwiller and B.B. Mandelbrot, Phys. Rev. Lett. 60 (1988) 673.
[15] B.B. Mandelbrot, in: Chaos in Australia, G. Brown, ed. (World Scientific, Singapore, 1990).
[16] S. Schwartzer, J. Lee, A. Bunde, S. Havlin, H.E. Roman and H.E. Stanley, Novel logarithmic singularity in . . . DLA (preprint dated 12/1989).
[17] C. Scholz and B.B. Mandelbrot, eds., Fractals in Geophysics (Birkhäuser, Basel, Boston, 1989).
[18] H.E. Stanley and N. Ostrowsky, eds., Fluctuations and Pattern Formation, Cargèse, 1988 (Kluwer, Dordrecht, Boston, 1988) p. 345.