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THE FRACTAL RANGE OF THE DISTRIBUTION OF GALAXIES; CROSSOVER TO HOMOGENEITY, AND MULTIFRACTALS

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ABSTRACT. This paper takes as established that the large scale distribution of galaxies includes a self-similar fractal range. The astonishing power of simple fractal algorithms to generate rich form is recalled and illustrated, then two issues are tackled. A) Does the fractal range stop at 5 Mpc, as asserted by Peebles et al.? Or does it continue beyond? Does it stop before the limits of observation? That is, should one believe the conventional statistical arguments in favor of 5 Mpc? B) The simplest fractal distributions are "fractally homogenous," that is, homogeneous over a fractal set, and zero outside of this set. However, the distribution of galactic and inter-galactic mass is non homogenous to the extreme. It can be self-similar, in which case it follows a "multifractal measure," as discussed by the author in 1974. This paper is concerned with questions of method, and analysis of data is not included.

1. Introduction and Summary

1.1. THE FRACTAL RANGE

The existence of a fractal range in the distribution of galaxies does not appear to be questioned, though appearances sometimes suggest the contrary. Thus, some writers prefer to make use of terms such as "self similar range" or "scaling range," which have the same meaning, but sound less affirmative. Other writers say that they have proof that *the* fractal model fails, but they willfully restrict the scope of "fractal model" to the "pure mathematical" case of fractals without cross-over.

The evidence in favor of fractals is basically as follows.

On the one hand, the author's models (as restated in Section 2) are easily fitted to have not only the observed correlation properties, but also the observed visual appearance. That is, the prevalence of voids and filaments was not known when these fractal models were developed, yet these features do not have to be put in separately, but appear to be an unavoidable consequence of the fractal character of the distribution. This last fact is an empirical observation that is not fully understood, and deserves further examination, but cannot be dismissed.

On the other hand, suppose that a geometric model of distribution fails to be a fractal. Could it fit the observed power law correlation functions, and have the correct appearance, while invoking only a small number of parameters. There may be no mathematical proof this cannot be done, but there is no known example, either.

Therefore, a prudent student of the large-scale structure will find it worthwhile to keep in mind the properties of the fractals. He will attempt to know more about the unexpectedly rich variety of pattern allowed by simple fractal constructions, and to know better the techniques required for their study. Some references to fractals could be said to imply that fractal geometry reduces to a few simple arguments about how different quantities scale with respect to each other. Today, this is very far from being the case. The substantial body of knowledge that exists cries out to be tapped further in the study of the Universe.

1.2. A LIST OF POSSIBLE GEOMETRIC MODELS OF DISTRIBUTION

In the following list by increasing complication, the last possibility is phrased to cover every possible views of the relevance of fractals to the description of the Universe, a description that should be the first step towards understanding.

— **1**. *Homogeneity*. The Universe is homogeneous, save for very local effects. This would be the simplest Possibility, but of course it fails very grossly to fit the data.

— **2**. *Pure fractal*. Save for very local effects, the Universe is a self-similar fractal up to infinite scales. More precisely, the galaxies are of equal masses, and the Universe is fractally homogenous. This would be the next simplest Possibility. In this case, the appropriate statistical substitutes for the correlations follow power laws throughout, the principal parameter being one number called fractal dimension.

Comment. Possibility 2 predicts a vanishing overall density of matter, a property that is known to elicit fierce a *priori* opposition. Let us, therefore, comment immediately that other fractal models are available, as will be seen momentarily. Therefore, to disprove Possibility 2 is NOT to disprove the relevance of fractals.

— **3**. *Pure multifractal*. The Universe is a self-similar multifractal throughout. The precise meaning of this term will be explained in Sections 4 and 5, but implies unequal galaxy masses and an interglacting medium. In this case, the principal parameter is not one number, but a probability distribution. The appropriate substitutes for the correlations follow power laws throughout, but with different exponents.

— **4**. *One crossover*. The Universe is a self-similar fractal only up to a crossover scale that satisfies $0 < R_{cross} < \infty$, and the crossover between the fractal and the homogeneous ranges occurs very sharply. That is, there is no transient range of significant width, whose structure would require additional parameters in order to be specified.

Comment. To be the distance beyond which the large-scale structure of the Universe becomes homogeneous, R_{cross} should be the size of the largest significant structures (such as the voids). To understand the contrast between Possibilities 2 and 4, the telling background example is that of a structure very familiar in statistical physics: percolation clusters. Percolation clusters at criticality fall under Possibility 2. In percolation clusters above criticality, R_{cross} is finite, namely is the size of the largest observed patterns. These percolation patterns at criticality are "voids," whose shape would not surprise the astronomer of 1988.

— **5**. *One crossover*. There is a crossover R_{cross} like in Possibility 4, but the Universe is multifractal.

— **6**. *Two crossovers*. The Universe involves a fractal range for "small" scales, an ultimate homogenous range for very large scales, and, in addition, involves a broad intermediate range characterized by significant structures. In this case, the single measure of scale R_{cross} of Possibility 2 is replaced by *at least two* measures of scale.

Comment. Insofar as we can tell, this Possibility is embraced by J.P.E. Peebles who invokes an intermediate range with negative correlations to explain the size of the observed voids. This is why the smaller measure of scale will be denoted by $R_{peebles}$, and the larger one will be denoted by R_{upper}. Clearly, $R_{peebles}$ is a lower bound to R_{upper}, but the Universe is *not* homogenous beyond $R_{peebles}$; in fact $R_{peebles}$ is far below the size of confirmed voids.

— **7**. *Two crossovers*. The Universe is like in Possibility 6, but with *fractal* replaced by *multifractal*.

— **8**. *Anything that does not fit in 1 to 7*. One possibility is that there are successive "rings" of different fractal dimension. An extreme possibility is that the distribution is ruled by a chaos without order.

Comment. We subscribe to the notion that the best models are the simplest. Therefore, in cases of doubt, we advocate the Possibility with the lower number.

1.3. THE ISSUE OF CROSSOVER AND THE RELIANCE UPON STATISTICS

Let us now elaborate. We first rephrase the above list of Possibilities by extracting two important issues. We assume that the fractal range, is not questioned for small distances. The horizon of observation will be denoted by R_{max}.

The first issue is whether this fractal zone crosses over sharply to a homogenous zone for some $R_{cross} < R_{max}$, or ends for some $R_{peebles} < R_{max}$ to be followed by a transient zone, or is a least as deep as R_{max}. In the latter case, we shall say that $R_{cross} = \infty$.

Let us recall that our book *The Fractal Geometry of Nature (FGN)* was clearly partial to Possibility 2, yet open-minded on this issue, primarily for lack of actual empirical checks. The recent analysis of the data in Pietronero 1987 and in Coleman et al. 1988, together with the rejoinder in Davis et al. 1988, have convinced us that, *either* Davis et al. is wrong in its criticism, *or* this issue *cannot* be resolved on purely statistical grounds. On the other hand, the visual analysis of the observation of increasingly large filaments and voids continues to encourage us to be open-minded, with clear partiality towards Possibilities 2 or 4.

The main fact is that to perform a statistical test between the alternatives $R_{cross} < R_{max}$ and $R_{cross} = \infty$ turns out to be a very delicate matter. Most important, it is necessary to limit statistics exclusively to tests that avoid prejudging a priori against the possibility $R_{max} = \infty$. This happens to require fresh statistical thinking, instead of blind reliance on previously "proven" techniques. The reason is that all the existing methods of statistics make explicit or implicit assumptions that happen to become invalid when $R_{cross} = \infty$. In particular, customary and usually innocuous normalizations, like those involved in the definition of the usual correlation and of the pseudo correlation used by Peebles 1980, yield reported results that are processed to excess, and have been made practically impossible to interpret by the reader.

Section 3 will analyze some statistical tools, step by step. One neutral summary of the evidence is the mass radius function $M(R)$. Its derivative, divided by $4\pi R^2$, is a conditional occupation probability; it is better than $M(R)$ in some ways, but less acceptable in other ways. Arguments will be given against replacing it by the normalized pseudo correlation function of Peebles et al.. Normalization is proper in more conventional statistics, but in the case when the fractal range is significant normalization is risky and unacceptable.

Comment. The seasoned statistician knows that, when a theory is submitted to a sufficiently wide battery of tests, it often happens that every hypothesis fails at least one test. This is the case even when the tests themselves have proven their applicability. When the statistical techniques are new and unproven, one individual negative test *cannot* suffice to eliminate an otherwise attractive possibility.

Statistics is little used in the hard sciences, and we must confess puzzlement at the importance it has achieved in the present context. The facts that filaments and voids of increasing size continue to be observed when one reaches the deepest levels of observation is a very clear-cut symptom of an underlying fractal distribution; indecisive statistics involving arguable corrections to the data carry little weight in comparison.

1.4. THE ISSUE OF MULTIFRACTALITY

Granting that there is a fractal range, the second basic issue is whether (in this range) the distribution of mass is homogeneous on a fractal set, or whether instead it involves galaxies of varying mass, as well as interstellar matter. If either or both is the case, and the distribution is self-similar, it must be multifractal. Examples of fractally homogeneous measure are obtained by placing a uniform measure on either of the fractal dusts defined by the tales in Section 2. The resulting models can now be called *unifractal*. But it is nearly as easy to study fractal but non uniform measures. As a matter of fact, it is easy to construct a distribution of mass that combines self similarity, very high peaks (to be interpreted as galaxies of variable mass), and a low background (to be interpreted as a very variable interstallar mass). To fulfill this aim (in the analogous case of intermittent turbulence), we have introduced and developed the notion of *multifractal measure*, in 1968 then mostly in 1972-1976, but it has not acquired a large following until recently. The most widely known approach to multifractals is, unfortunately, quite unnecessarily complicated and artificial. Our original approach, in a recently completed form, is much more straightforward, and a survey is included as Section 5, in order to make it readily available to the student of large scale structure of the Universe.

1.5. THE ISSUES IN SECTION 1.3 AND 1.4 INTERACT

It will very soon become necessary to face them simultaneously.

2. Two Fancy Tales of How the World Began. Demonstration by Examples of Power of Simple Fractal Models to Generate Unexpectedly Rich Structures.

This section begins by restating in fanciful style our two basic models of ga clustering using random fractals.

2.1. "THE SEEDING OF THE HEAVEN"

"In the beginning, the heaven was a void. And the Master of Matter, Light and Life proclaimed, Let there be matter: and matter was. It was one point. And the Master proclaimed, Let matter be seeded over the heaven, and Let every small part of the heaven be just like every other small part and like every large part. And two archangels set forth hopping; wherever they alighted, they left a pinch of matter and then resumed their journey as in its beginning. And the parts of the heaven were all made just alike. And the Master was everywhere, dwelling in every pinch of matter; and the heaven looked the same from every point where the Master dwelt."

2.2. "THE PARTING OF THE HEAVEN"

"In the beginning, the heaven was filled with matter. And the Master of Matter, Light and Life proclaimed, Let matter part away. Let it remove itself to form voids without number, and Let every small part of the heaven be just like every other small part and like every large part. And matter removed itself, and the Master was everywhere, dwelling in every place that was not in a void: and the heaven looked the same from every point where the Master dwelt."

2.3. COMMENT ON THE GENERATIVE POWER OF THE FRACTALS

In Chapters 33 to 39 of my book, *The Fractal Geometry of Nature (FGN)*, the above fancy tales are translated into sober fractal models, involving very simple statistical algorithms one can easily simulate on the currently available computers. A priori, one expects simple algorithms to generate nothing much of interest. In the specific case of galaxy modeling, this low expectation may be related to the natural but quite incorrect identification of all fractal models with the very early but extraordinary crude ones. For example, in the model of Fournier d'Albe, little is put in, and nothing more is obtained as output. Therefore, there is a strong tendency to use highly specified algorithms, in which every one of the features one wishes to see in the output (e.g. large voids) has been knowingly introduced in the input. An example of a needlessly over-specified fractal model is the one advanced by Soneira and Peebles. Similarly, Peebles 1980 proposes to correct in advance for presumed inadequacies of our "Seeding" model, by adopting multiple "Seeding centers," distributed uniformly. This introduces a finite R_{cross}, but is far too hasty a solution.

Given the a priori fear that simple models *must* be inadequate, it is a surprise to see that the simulations of our "Seeding" and "Parting" models look far more realistic than anticipated by anyone . . . even by us. Since "to see is to believe," many examples are shown in my book, *FGN*.

Yet another fractal construction, recently put forward by Szalay and Vicsek, has become famous as the cover of Audouze et al. 1988. It may or may not be physically realistic, but it strengthens further our belief that almost any sufficiently "natural" fractal dust would be reminiscent of the actual distribution of the galaxies.

These examples are meant to underline the power of simple fractal models to generate rich form. This power is the geometric facet of a fact that is growing in public awareness, namely of the power of simple dynamical systems to generate rich and seemingly chaotic orbits. Our hope is that the fractals' power will cease to be underrated, and that they will no longer be dismissed casually when it happens that some single statistical test appears to encounter difficulties.

3. On Diverse Statistical Summaries of the Data, and on the Pitfalls of "Normalizing" them

3.1. NON-TRUNCATED FRACTALS AND THE MASS-RADIUS FUNCTION

In order to simplify, this Section makes the "unifractal" assumption that all galaxies carry the same mass. This makes it possible to define the *mass-radius function*

$M(R)$ = mass within a radius R around a fixed (randomly selected), galaxy.

Under Possibility 2 of Section 1.2, on has

$$M(R) = F(R)R^D.$$

This is reminiscent of the formula valid when the distribution is completely homogeneous, but there is a *fundamental* difference. In the homogeneous case, the mass in a sphere is $M(R) = (4\pi/3)\delta R^3$, which is the power R^3 with a *numerical* multiplicative prefactor. To the contrary, non-truncated fractals involve the very important fractal prefactor $F(R)$, which is *not* a constant. It is a stationary random function of $\log R$, and it can vary greatly.

The variability of $F(R)$ or of $\log F(R)$ around their expectations is an interesting characteristic of a fractal, independent of its D and recommended for further empirical and theoretical study. It is one aspect of a fractal's "lacunarity." For example, as variability measured by the usual variance increases, the fractal's lacunarity also increases (*FGN*, Chapters 34 and 35). We have investigated diverse families of fractals, in which different members differ solely by the value of the fractal dimension D. As D is varied from its highest possible value down to its lowest possible value (which is usually 0), the variability of F increases very sharply. The works of de Vaucouleurs and Peebles 1980 suggest $D \sim 1.2$, which is a small value for a dust in 3 dimensional space. Hence, our impression is that we should expect $F(R)$ to have a large fluctuation. This impression deserves to be subjected to a hard critical study.

In order to appreciate what happens as we move from $M(R)$ to the covariance and the pseudo-correlation, it is best to make the move in several distinct steps, and to introduce notation gradually.

3.2. NON-TRUNCATED FRACTALS AND THE LOCAL CONDITIONAL DENSITY, WITHIN THE VOLUME BOUNDED BY THE SPHERES OF RADII R AND $R + \Delta R$

Now consider the local conditional density at distance R from the point P, that is the mean density within the volume bounded by the spheres of radii R and $R + \Delta R$ centered at the point P. It is

$$\Delta M/4\pi R^2 \Delta R = (1/4\pi)R^{D-3}[DF(R) + R\Delta F(R)/\Delta R].$$

The expectation $\langle F(R)\rangle$ is positive and finite, and is independent of R. And the expectation $\langle \Delta F(R)\rangle R/\Delta R = \langle \Delta F(R)\rangle/\Delta \log R$ vanishes, because we saw that the prefactor $F(R)$ is a stationary random function of $\log R$. Therefore, the plot of $\log \Delta M$ versus $\log R$ would run around a straight "trend line" of slope $D - 3$, with superposed fluctuations throughout. Several plots corresponding to different samples would have the same trend line, but entirely distinct fluctuations.

3.3. NON-TRUNCATED FRACTALS. THE USUAL NOTION OF "REPRESENTATIVE SAMPLE" IS NOT APPLICABLE. CONDITIONAL DENSITIES MUST NOT BE RENORMALIZED.

Statisticians have almost always dealt with situations where samples of sufficiently large size can be viewed as "representative" of the whole population, but this is not the case for non-truncated fractals. In particular, evaluate

$$\frac{\Delta M/4\pi R^2 \Delta R}{M(R_{\text{sample}})/4\pi R^3_{\text{sample}}}$$

This is the ratio between the density at distance R and the overall density in a sample of size R_{sample}. For a standard distribution, the denominator is hardly random at all if the sample is sufficiently large, and the above renormalization is useful. But the fractal case is totally different. The ratio of the expectations would simply be

$$\left(\frac{R}{R_{\text{sample}}}\right)^{D-3},$$

and would provide marvelous material for the estimation of D. However, the ratio of non-averaged quantities is

$$\frac{DF(R) + R\Delta F(R)/\Delta R}{F(R_{\text{sample}})}\left(\frac{R}{R_{\text{sample}}}\right)^{D-3}.$$

Its overall trend is $(R/R_{\text{sample}})^{D-3}$, as expected. However, there is a random prefactor. Its numerator duly depends upon the randomness at the distance R, but its denominator mixes in the randomness at the distance R_{sample}. Again, plot log (ratio) against $\log R$. There is still a straight trend line of slope $D-3$. But the prefactor is reflected in a translation log (denominator of the prefactor), which depends on the overall sample. Thus, different samples have parallel but distinct trend lines. Any sensible scientist will push these line together in order to estimate their common slope, but the affect of renormalization upon the estimation of D is not helpful at all.

3.4. NON-TRUNCATED FRACTALS. THE COVARIANCE (OR "CORRECT CORRELATION FUNCTION") I.E., THE CONDITIONAL DENSITY AVERAGED OVER ALL ORIGINS.

Let us return to the "raw" prefactor $DF(R) + R(\Delta)F(R)/\Delta R$ of section 3.2. In order to average out its wild fluctuations, the best is to average for fixed R over all the spheres centers P. The resulting quantity is called "correct correlation function" by Pietronero 1987, but I prefer to continue to use the standard probabilistic term, which is *covariance*. Write $n(P)$ and $n(P+\overline{R})$ for the number of galaxies in a small sphere around P and around a point $P+\overline{R}$ whose distance to P is of length R. One has

$$\Gamma(R) = \frac{\langle n(P)n(P+\overline{R})\rangle}{\langle n\rangle}.$$

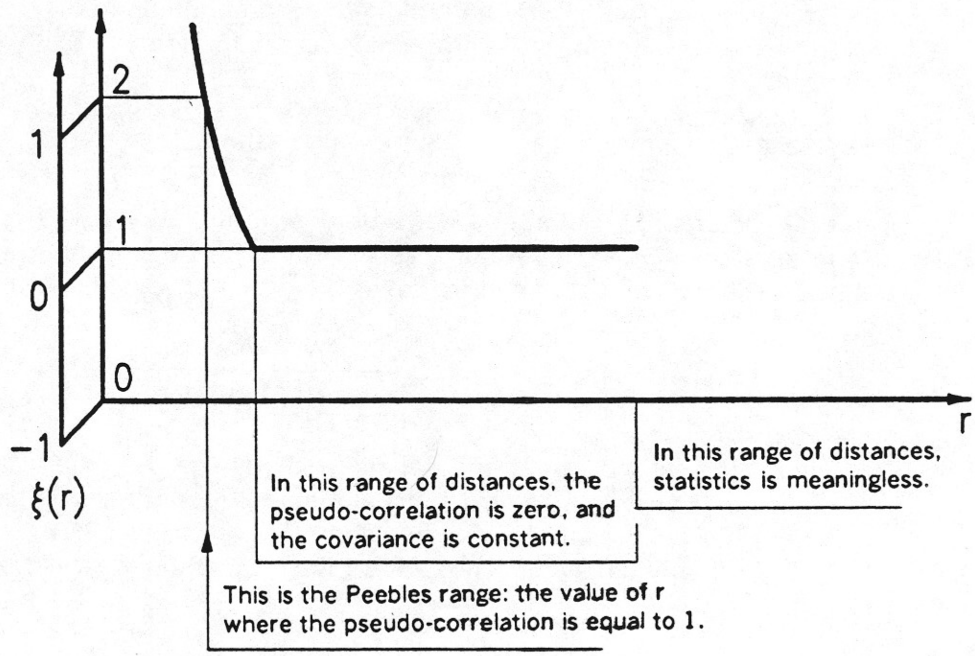

Figure 1. The relation between the covariance and the normalized "correlation" for a fractal universe with a sharp crossover to uniformity.

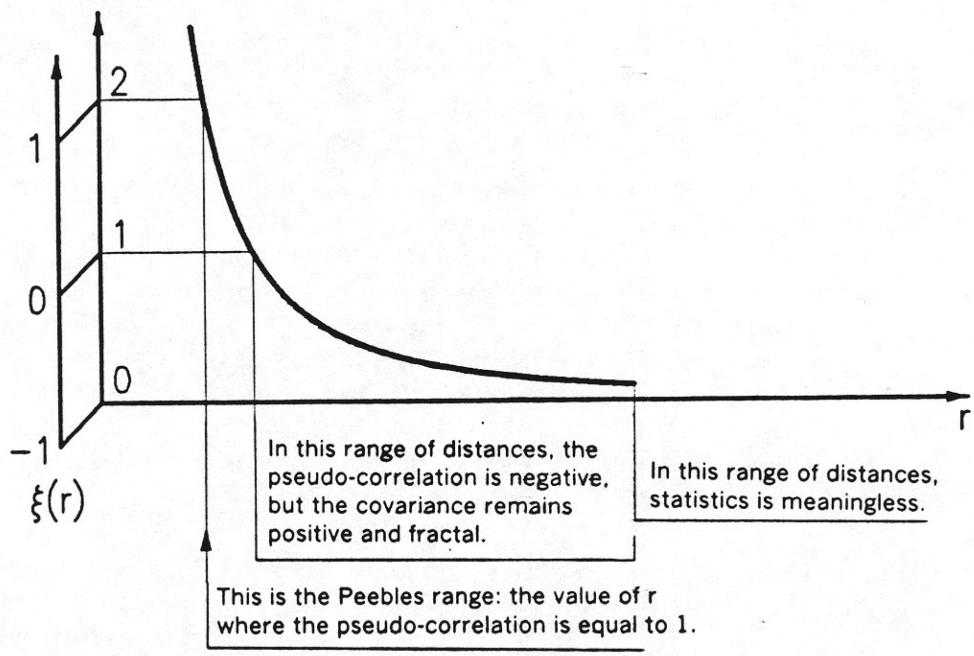

Figure 2. A typical renormalized correlation. The normalization does not use the value of $\langle n \rangle$ that is given by the last point of the sample covariance of the sample under investigation, but rather a value obtained by a larger but different sample.

Comment. It is an unavoidable feature that any given portion of space of radius R_{max} (e.g., the volume covered by a catalogue carried to a uniform depth) includes a small number of spheres of large radius, but a large number of spheres of small radius. When the sample average is carried over the origins of all the spheres within the catalog, the sample values for different R's fail to be independent. Hence, the fluctuations in F and ΔF average out even more poorly than if the samples had been independent. One deals with a reduced "effective sample size," whose value depends upon R. More precisely, when R is well below R_{max}, the effective sample size is near the actual sample size, and the conditional density is reliable. But for R close to R_{max}, the effective sample size becomes small, and the conditional density becomes greatly affected by the fluctuation of $F(R_{max})$. For example, while the expected conditional density decreases up to $R = R_{cross}$, the sample density may well actually increase in the range below R_{cross}.

3.5. NON-TRUNCATED FRACTALS. THE (PSEUDO) CORRELATION.

Let us now turn to the renormalized ratio of Section 3.3. By averaging over all the sphere centers P, one obtains the quantity

$$\frac{\Gamma(R)}{\langle n \rangle} = \xi(R) + 1.$$

Again, the renormalization via the division by $\langle n \rangle$ introduces entirely spurious effects.

The quantity $\xi(R)$ is called *correlation* by astronomers, but the statisticians' correlation is a different expression that ≤ 1 in absolute value, while $\xi(R)$ certainly can exceed 1, and the value where it reaches 1 is given a special standing.

3.6. THE DISTRIBUTIONS OF GALAXIES AND OF CLUSTERS, COMPARED

This section brings two remarks together. First, it was predicted in Section 3.3 and 3.5 that renormalization should be expected to introduce a spurious prefactor, hence a spurious translation in doubly logarithmic coordinates. Second, it can be shown, under a wide range of definitions of the notion of "galaxy clusters," that the functions Γ should be identical for galaxies and for their clusters.

The two remarks in the preceding paragraph make us expect that the $\xi(R) + 1$ function of galaxies and of galaxy clusters should differ by a factor, i.e., by a translation on doubly logarithmic coordinates. This is indeed the situation that is observed, as seen, e.g., in Szalay and Schramm 1985. This discrepancy has been viewed as a real empirical fact to be explained, and also as a counter argument to "the" (that is, the simplest) fractal description. To the contrary, we have always expected that this discrepancy will turn out to be almost certainly spurious. The data analysis in Pietronero 1987 confirms our expectation.

3.7. FRACTALS WITH ONE CROSSOVER

A wide class of random fractals with sharp crossover at R_{cross}, are covered by Possibility 4 of Section 1.2. They include variants of the Parting model of Section 2. For these distributions, the fractal non homogeneity is described by the behavior of the mass-radius function $M(R)$. One has

$$M(R) = F(R)R^D \qquad \text{for} R < R_{\text{cross}}$$
$$M(R) = (4\pi/3)\delta R^3 + \text{ (a fluctuation } \sim \sqrt{(4\pi/3)\delta R^3}\,) \qquad \text{for} R > R_{\text{cross}}$$

In the homogenous range $R > R_{\text{cross}}$, things are classical: there is a non random factor involving a density δ, and a Poisson fluctuation factor, which is *additive,* and whose form is familiar to everyone.

To the contrary, the fluctuation factor in the fractal range $R < R_{\text{cross}}$ is multiplicative, like in Section 3.1. The local density described in the title of Section 3.2 is

$$\Delta M/4\pi R^2 \Delta R = (1/4\pi)R^{D-3}[DF(R) + \Delta F(R)R] \qquad \text{for } R < R_{\text{cross}}.$$
$$\Delta M/4\pi R^3 \Delta R = \delta + \text{ (a fluctuation term)} \qquad \text{for } R > R_{\text{cross}}.$$

On a plot of log (density) as function of log R, the fractal and the homogenous regimes give straight trend lines of respective slopes $3 - D$ and 0. The crossover occurs where these straight lines intersect each other.

In truncated fractal models I know well (and also for percolation clusters a little above criticality), the crossover between these two regimes is quite sudden, which is a great asset. (The corresponding expected mass-radius plot is the plot of an integral, hence the crossover is far more gradual, which is a drawback.)

What about the effect of renormalization on log (density) where there is a single crossover $R_{\text{cross}} < R_{\max}$ we are in the standard statistical world, in which renormalization is perfectly legitimate, as seen on Figure 1.

For $R < R_{\text{cross}}$, the function $\xi(R) + 1$ stabilizes at 1, and the function $\xi(R)$ stabilizes at 0.

The radius where $\xi(R) = 1$ and $\xi(R) + 1 = 2$ has been singled out by P.J.E. Peebles in the analogous context to be tackled in Section 3.8. Let this value be denoted by R_{peebles}. Clearly, when $\xi(R) + 1 \sim R^{D-3}$, and $R < R_{\text{cross}}$, with $D = 1.2$, one has $R_{\text{peebles}} = 2^{\ 1/1.8}\, R_{\text{cross}} = .68\, R_{\text{cross}}$.

3.8. FRACTALS WITH AT LEAST TWO CROSSOVERS. THE (PSEUDO) CORRELATION AND THE DEFINITION OF R_{peebles}.

Section 3.7. shows that $\xi(R)$ has a useful meaning when there is a single sharp crossover $R_{\text{cross}} < R_{\max}$. Could $\xi(R)$ continue to be of use in other cases? This cautious writer expects little from statistics, and would not trust any expression "in the wild" before it has been "tamed" on a well-understood explicit example. However, something very much like $\xi(R)$, with one essential modification to be distributed shortly, has been extensively used in many works by P.J.E. Peebles, both those summarized in Peebles 1980 and more recent ones. This use is bold, in fact, it is reckless in our opinion, because the underlying model is never fully described — to our knowledge. Implicitly, Peebles does not believe in Possibility 4, since he describes a function he calls "correlation" behaves like on Fig. 2: it is $= 1$ for $R = 5$ Mpc; it is > 0 for $R < 1.46 \times 5$ Mpc and it is $= 0$ over some ultimate homogeneous range, but it is < 0 over an unspecified intermediate range beyond 1.46×5 Mpc. In this framework, the existence of the intermediate range, where $\xi(R)$ is most often negative, is indispensable to account for the existence of large voids and of all the other interesting structures observation keeps revealing.

Now to the essential modification Peebles brings to $\xi(R)$. In his view, in order to evaluate the global density of galaxies, it is not only permissible, but is desirable, to use counts that are broader than the counts that lead to $\Gamma(R)$.

Granted this modification, let us go beyond formal manipulation, and ponder the implications of Fig. 2, without questioning the validity of the essential modification. Given the list of Possibilities drawn in Section 1.2, it is clear that the simplest interpretation of Figure 2 involves Possibility 6, with R_{peebles} being (within some factor of the order of 1) the radius where one leaves the fractal range. However, this is *not* the way R_{peebles} is ordinarily presented. There is a widespread perception that R_{peebles} (within some factor of the order of 1) is a measure of R_{upper}. We see that this perception is not warranted. As a matter of fact, there is invariably a gap in the graph of $\xi(R)$, between the largest R for which a correlation is reported, and the presumed asymptotics. The result is that given Figure 2, no numerical value can conceivably be inferred for R_{upper}. In words, the value of R_{peebles} gives no hint of where the asymptotic homogeneity begins to prevail.

Let us now dig deeper, by questioning the assumption that one can plug into $\xi(R)$ a global density estimated on the basis of independent very deep data. This equality assumes that on the scale of the largest existing catalogue, the Universe is already homogeneous. Since this is what our task is to either confirm or contradict, assuming it in advance is *not permissible*. We also observe that deep surveys involve drastic corrections of uncheckable validity. The cosmologists feel that the density of visible matter is smaller than it "should" be, so that their corrections cannot inspire full confidence. This writer has attended the Seminars that followed the 1987 Balantonfüred Symposium of the IAU. While several speakers asserted that the global density is a well-defined number, its value did not lead to any consensus.

3.9. CONCLUSION OF SECTION 3

Many statistically dubious steps enter in the procedures customarily used to reach the widely accepted conclusion that the Universe becomes homogenous at comparatively short distances. Therefore, this conclusion is not persuasive. By its definition, R_{peebles} is at best concerned with the range of significant fractal correlation, and is likely to underestimate it. More important, the value of R_{peebles} gives no inkling whatsoever on the distances beyond which the universe is homogeneous.

Our feeling, therefore, is that, in the present state of knowledge, it is imperative *not* to renormalize the observed covariance $\Gamma(R)$. On Figure 2, a bold line represents the typical non normalized covariance $\Gamma(R)$. This line's overall shape suggests that these data are entirely compatible with Possibilities 2 and 4 of Section 1.2.

Analogous criticisms, combined with a fresh analysis of the data, are made in Pietronero 1987 and in Coleman et al. 1988. The counter analysis by Davis et al. 1988 involves unconvincing corrections. The strongest conclusion one may draw from this counter analysis is that the unquestioned data do not suffice to decide between the thesis of Peebles and Possibility 2 in the list in Section 1.2.

4. Spatial Variability Beyond the Fractally Homogenous Model

More detailed versions of Sections 4 and 5 are found in Mandelbrot 1989 and in our forthcoming *Selecta*.

In the models described in Section 2, mass is concentrated on a definite portion of space. Using the appropriate technical terms, mass is "supported by a closed fractal set." Furthermore, the mass distribution can be called "fractally homogenous." The rough meaning is that all galaxies are given the same mass, and the more precise meaning is that when two portions of the supporting set are identical except for translation, they support the same mass. These assumptions contradict two unquestionable facts.

First is the great inequality that prevails between galaxy masses.

Second (to quote from *FGN*, p. 376) is "our knowledge of the existence of interstallar matter. Its distribution is doubtless *at least* as irregular as that of the stars. In fact, the notion that it is impossible to define a density is stronger and more widely accepted for interstellar than stellar matter. To quote deVaucouleurs, 'it seems difficult to believe that, whereas visible matter is conspicuously clumpy and clustered on all scales, the invisible intergalactic gas is uniform and homogenous . . . [its] distribution must be closely related to . . . the distribution of galaxies. . . .

"[Thus, in the models of Section 2, parts of space] of less immediate interest were artificially emptied to make it possible to use *closed* fractal sets, but eventually these areas must be filled. This can be done using a fresh hybrid [now called *multifractals*. A multifractal] mass distribution in the cosmos will be such that no portion of space is empty, but, [given two] small thresholds θ and λ, a proportion of mass at least $1 - \lambda$ is concentrated on a portion of space of relative volume at most θ."

5. The Principal Ideas Underlying Multifractal Measures

5.1. AN OLD BUT GOOD ILLUSTRATION

Multifractals are old, insofar as we had first developed the basis of this technique in the years 1968 to 1976, in order to study different aspects of the intermittency of turbulence. Since "to see is to believe," Fig. 3 reproduces the earliest illustration of a multifractal, as it first appeared in our earliest full paper on this topic, Mandelbrot 1972. We expect it to inspire astronomers to many applications.

The horizontal axis shows "time" divided into small boxes of width Δt, and the vertical axis shows the sequence of the masses within these boxes. If the total integral measure over the total time span $[0,T]$ is set to 1, one can think of *the measure in a box* as *the probability of hitting this box*. If an analogous diagram were drawn for a measure having a density, it would be an approximation to this density — and a first characterization of our measure would be provided by the distribution of this approximate density along the horizontal.

In the present instance, however, the situation is extremely different. By design, the measure is approximately self-similar, in the sense to be discussed in Section 5.3. It follows that this measure grossly fails to have a density, nor is it discrete. For example, if the Δt is halved, the sharing of the measure in an original Δt between the two halves is usually very unequal. There is no such thing as a notion of "distribution" for the values of this measure. Fortunately, there is a very useful substitute.

5.2. THE NOTIONS OF LIMIT PROBABILITY DISTRIBUTION $\rho(\alpha)$, AND OF $f(\alpha)$.

Take different values of Δt, and, for each value of Δt, plot the corresponding measure distributions on doubly logarithmic coordinates. The measures we want to call multifractal have the following property. When both logarithmic coordinates of the plots drawn for different Δt's are reduced by the same factor log Δt, the reduced plots of the distribution converge to a limit as $\Delta t \to 0$. This property can be turned around, and used to *define* the notion of multifractal. (But one must realize that the convergence to the limit may be slow.)

The reduced horizontal logarithmic coordinate is denoted by α, and will be seen to be a quantity called Hölder exponent. The reduced vertical logarithmic coordinate corresponding to the limit will be denoted by $\rho(\alpha)$. It will be seen that it is negative for all α, except where $\rho(\alpha)$ reaches its maximum, which is 0.

As has been first pointed out by Frisch and Parisi 1985, it is convenient to also introduce a quantity denoted by $f(\alpha) = \rho(\alpha) + 1$. When $f(\alpha) \geq 0$, one can interpret $f(\alpha)$ as being one fractal dimension of a suitable set. This is the only case Frisch and Parisi consider, and this is also the only case relevant to the application to astronomy. The replacement of $\rho(\alpha)$ by $f(\alpha)$ has virtues in some cases, but our feeling is that, fundamentally, it hides the nature of the multifractals.

Until the preprint of Frisch and Parisi was distributed in 1983, multifractals had continued to develop only in the sense that the mathematics was very much extended (see Kahane & Peyrière 1976). But they did not receive new applications, nor were they mentioned in *Astrophysical Journal Letters*. Their spread is a recent phenomenon, and most readers who have heard of them are likely to know presentations that follow the approach common to Frisch and Parisi and to Halsey et al.. Unfortunately, the algebra of these presentations is needlessly complicated, artificial, and of limited applicability and the terminology of Halsey et al. hides the extremely simple and almost familiar nature of the underlying structure. We shall, therefore, adopt the nota-

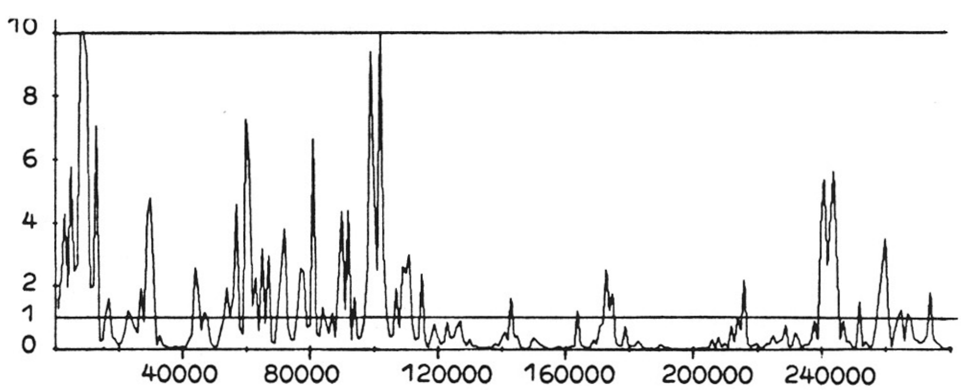

Figure 3. The earliest simulation of a sample from a multifractal measure. This measure is called limit lognormal in Mandelbrot 1972 (others call it "M's 1972 measure").

tion of Halsey et al., but follow our original approach in the form into which it has lately developed.

The multifractal formalism centers around the functions $\rho(\alpha)$ or $f(\alpha)$. In the simplest case, this paper obtains $f(\alpha)$ via the Lagrange multipliers procedure of statistical thermodynamics, which has long been familiar to every physicist. Later on, a full mathematical justification of the formalism, valid in a broader context in which $f(\alpha)$ can very well be negative, is provided by reference to existing (but little-known) limit theorems of probability due to Harald Cramèr, and concerned with "large deviations".

Mandelbrot 1974 considers two distinct kinds of random self-similar multifractals, respectively called *conservative* (or *microcanonical*) and *canonical*. This distinction is crucial to the study of low-dimensional cuts of multifractals embedded in a high dimensional space. But in astronomy this issue is not important.

5.3. SELF-SIMILAR MEASURES AND BEYOND

We need some definitions concerning measures $\mu(S)$. These $\mu(S)$ will be positive; therefore again, those not familiar with measure can think of μ as being the probability of hitting the set S. The multifractal measures obtained as a result of multiplicative cascades are the closest analog among measures of the exact self-similar fractal sets. Recall that a fractal set is exactly self-similar, if it can be decomposed into parts, each of which is obtained from the whole by a contracting similitude, $\mathscr{C}$. Such a set if fully determined by a collection of contractions. For example, each third of a basic fractal called Sierpinski gasket is obtained from the whole by a contracting similitude of ratio $r = 1/2$. Starting with any triangle in a "prefractal collection of triangles," the interpolation of the shape itself continues without regard to the "past" construction steps.

Now suppose that a (positive) measure $\mu(P)$ is defined for each third of the gasket, for each third of a third etc.. When the part P' is obtained from the part P by the contracting transformation $\mathscr{C}$, so that $P' = \mathscr{C}(P)$, the conditional measure of P' in P is defined just like a conditional probability, that is, by the ratio $\mu(P')/\mu(P)$ of the measure $\mu(P')$ to the conditioning measure $\mu(P)$. Now the idea strict of self-similarity for a measure is that the interpolation of the measure carried by a triangle in a prefractal collection of triangles also continues without regard to the "past" steps. That is as the parts contract, the measures they carry contract proportionately. To express this idea, take a second contracting transformation $\mathscr{L}$, and compare $\mu(P')/\mu(P)$ with $\mu[\mathscr{L}(P')]/\mu[\mathscr{L}(P)]$. If these conditional measures are identical, the measure μ will be called a *strictly self-similar multifractal*.

A random measure is called *statistically* self-similar if, given one or a finite collection of non overlapping parts $P_\gamma = \mathscr{C}_\gamma(P)$, the distribution or the joint distribution of the quantities $\mu(P_\gamma)/\mu(P)$ depends only on the contractions $\mathscr{C}_\gamma$.

Side remark. In a more general mathematical fractal set, the parts are obtained from the whole by transformations that are *non linear*. Examples where the contractions are in some sense *near linear* include the Julia sets of polynomial maps. The corresponding multifractals include the harmonic measures on these sets. Other examples of multifractal measures concern the limit sets of groups based upon inversions in circles (*FGN*, Chapters 18 and 20). A case when the limit set itself is a straight line is examined by Gutzwiller & Mandelbrot 1988. Finally, the *"fat fractals"* (new term for the fractals in *FGN*, Chapter 15) and the *Mandelbrot set* involve essentially non-linear transformations.

5.4. THE BASIC NON RANDOM SELF-SIMILAR MULTIFRACTALS

5.4.1. *Basic background: the binomial multifractal measure.* To construct this measure, given m_0 satisfying $\frac{1}{2} < m_0 < 1$ and $m_1 = 1 - m_0$, we spread mass over the halves of every dyadic interval, with the relative proportions m_0 and m_1. If $t = 0.\eta_1\eta_2 \ldots \eta_k$ is the development of t in the binary base 2, and φ_0 are φ_1 the relative frequencies of 0's and 1's in the binary development of t, the binomial measure assigns to the dyadic interval $[dt] = [t, t + 2^{-k}]$ of length $dt = 2^{-k}$ the mass

$$\mu(dt) = m_0^{k\varphi_0} m_1^{k\varphi_1}.$$

Adapting the classical notion of Hölder exponent to apply to the interval $[dt]$, we write

$$\alpha = \log[\mu(dt)] / \log(dt) = -\varphi_0 \log_2 m_0 - \varphi_1 \log_2 m_1,$$

and $0 < \alpha_{min} = -\log_2 m_0 \leq \alpha \leq \alpha_{max} = -\log_2 m_1 < \infty$. (The Hölder exponent has been given many new names. For example, it has been relabeled as "dimension" by Hentschel and Procaccia 1983, or as "pointwise dimension," but the term "dimension" *must* be reserved to sets.)

The number of intervals leading to φ_0 and φ_1 is $(k\varphi_0)!(k\varphi_1)!/k!$, giving the box fractal dimension

$$\delta = \log[(k\varphi_0)!(k\varphi_1)!/k!] / \log(dt).$$

For large k, the Stirling approximation yields

$$\delta = -\varphi_0 \log_2 \varphi_0 - \varphi_1 \log_2 \varphi_1.$$

Thus, α determines φ_0, hence $\delta = f(\alpha)$.

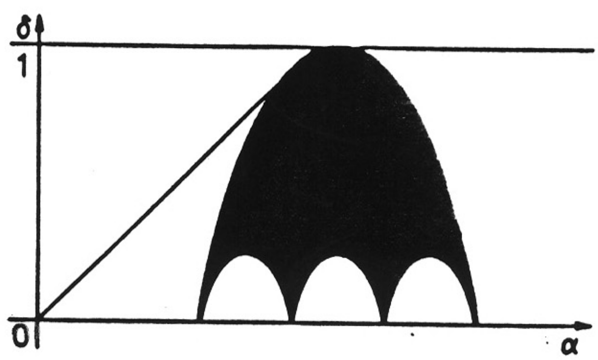

Figure 4. Rough idea of the domain of (α, δ) for a multinomial multifractal with $b = 4$, when the m_β differ from one another. The upper boundary defines the function $f(\alpha)$. Here, $\alpha_{min} = \min(-\log_b m_\beta) > 0$, and $\alpha_{max} = \max(-\log_b m_\beta) < \infty$.

A theorem by Eggleston relates δ to the Hausdorff dimension (Billingsley 1967).

5.4.2. *The multinomial measure.* To construct a multinomial measure of base $b > 2$, we require b masses m_β $(0 \leq \beta \leq b-1)$. The b-adic intervals characterized by the frequencies φ_β of the digits β in the base-b development $0.\eta_1\eta_2 \ldots \eta_k$ yield

$$\alpha = -\sum \varphi_\beta \log_b m_\beta \quad \text{and} \quad \delta = -\sum \varphi_\beta \log_b \varphi_\beta .$$

Now, the points (α, δ) cover a domain shown in black on Figure 4.

5.4.3. *The Lagrange multipliers argument, and the Legendre and inverse Legendre relations of the Gibbs theory.* The sets of φ_β's yielding the same α are dominated by the highest dimension term. This term maximizes $-\sum \varphi_\beta \log_b \varphi_\beta$, given $-\sum \varphi_\beta \log_b m_\beta = \alpha$, and $\sum \varphi_\beta = 1$. The classical method of Lagrange multipliers introduces a multiplier q, with $-\infty < q < \infty$, and yields

$$\varphi_\beta = \frac{b^{q \log_b m_\beta}}{\sum b^{q \log_b m_\beta}} = \frac{m_\beta^q}{\sum m_\beta^q} .$$

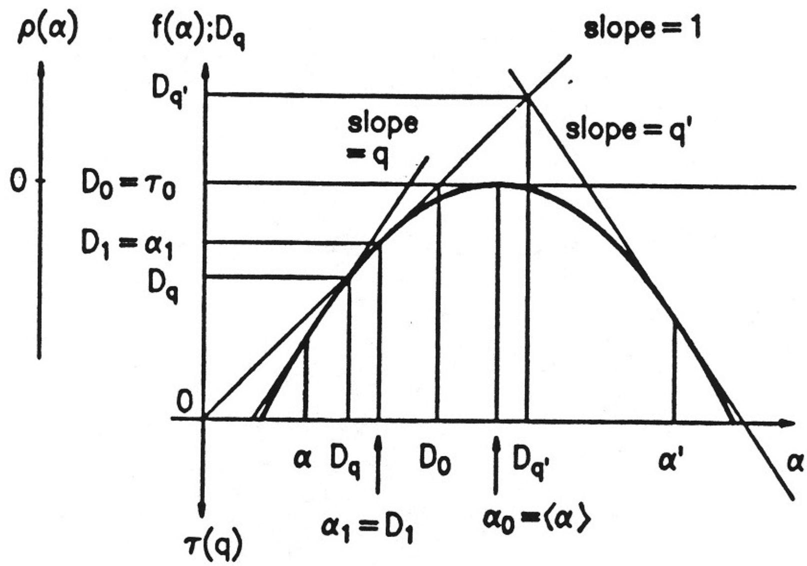

Figure 5. A multifractal diagram. The ordinate scale to the left shows $\rho(\alpha) = \lim_{k\to\infty}(1/k)\log_B$ (probability) versus the Hölder $\alpha = -(1/k)\log_b$ (measure). The ordinate scale to the right shows the function $f(\alpha) = \rho(\alpha) + D_0$. It is well-known that $q = f'(\alpha)$ and $-\tau(q)$ is the ordinate of the intercept of the tangent of slope q by the vertical axis. Let us add the observation that D_q is coordinate of the intercept of this tangent by the main bisector of the axes.

Define the quantity $\tau(q) = -\log_b \sum m_\beta^q$, which the mathematical statisticians call "cumulant generating function." In terms of τ, the Lagrange multipliers determine q and $f(\alpha)$ from α by

$$\alpha = -\sum \varphi_\beta \log_b m_\beta = -\frac{\partial}{\partial q} \log_b \sum m_\beta^q;$$

$$\max \delta = f(\alpha) = -\frac{\sum (q \log_b m_\beta - \log_b \sum m_\beta^q) m_\beta^q}{\sum m_\beta^q}.$$

That is,

$$\alpha = \frac{\partial \tau(q)}{\partial q} \text{ and } f(\alpha) = q\frac{\partial \tau}{\partial q} - \tau = q\alpha - \tau.$$

Figure 4 is now replaced by its upper boundary, which is the graph of a function $f(\alpha)$. Clearly, $\alpha > 0$ and $\delta \geq 0$, hence $f(\alpha) \geq 0$, $\alpha_{\min} > 0$, $f(\alpha_{\min}) \geq 0$ and $f'(\alpha_{\min}) = \infty$, and $\alpha_{\max} < \infty$, $f(\alpha_{\max}) \geq 0$ and $f'(\alpha_{\max}) = -\infty$. Multifractals that are not multinomial, yet posses these properties we call "pseudo-multinomial."

Formally, q = inverse temperature, τ = Gibbs free energy, and f = entropy.

5.4.4. *The term "multifractal formalism" and the question of actual computation.* The equations $\alpha = \tau'$ and $f = q\alpha - \tau$ are the "multifractal formalism." Frisch and Parisi 1985 and Halsey et al. 1986 obtain the same result via a steepest-descent argument, which experts will recognize as identical to the Darwin-Fowler justification of the Lagrange multipliers procedure. This is *not* the right way to proceed, just as *no one* will think of teaching thermodynamics by describing the Darwin-Fowler method directly, without first presenting the Lagrange multipliers. Therefore, Section 5.4.3 has taken the path towards the same formalism that involves the least effort and the fullest understanding. Section 5.5 describes the next simplest generalization.

A considerable literature has developed around ways of using this formalism. This literature is, obviously, unaffected by a change in the foundations.

5.5. THE RANDOM 1974 MULTIFRACTAL MEASURES

5.5.1. *Generalization of the multifractal formalism by an application of Harald Cramèr's theorem on large deviations.* Now we proceed to the exactly renormalized "1974 multiplicative multifractals" introduced in Mandelbrot 1974. First observe that the $f(\alpha)$ of a multinomial measure is unchanged if the indexes of the masses m_β are shuffled at random before each stage of the cascade that distributes mass. Next, suppose that $b = B^E$, with positive integers B and E. With no change in the algebra, the multinomial measure with random weight assignment can be interpreted as spread on cells in a E-dimensional cube of base B. The weights in the cells inject a random multiplier M that can take the values m'_β, with the probability $1/b$ for each value. In a cost-free generalization, consider a random multiplier satisfying only $M \geq 0$ and $\langle M \rangle = 1/b$.

Clearly, the mass $\mu(d\mathbf{x})$ in the b-adic cube of side B^{-k}, starting at $\mathbf{x} = 0.\eta_1\eta_2 \ldots \eta_k$, is

$$\mu(d\mathbf{x}) = M(\eta_1)M(\eta_1, \eta_2) \ldots M(\eta_1, \ldots \eta_k) \ldots .$$

Here, the successive M are identically distributed and independent. Hence

$$\alpha = -(1/k)[\log_b M(\eta_1) + \log_b M(\eta_1, \eta_2) \ldots]$$

is the average of k independent random variables. To tackle the distribution of α, "large deviations theorems" of H. Cramèr are available "off-the-shelf" (a pleasant surprise); see Book 1984, Chernoff 1952, Daniels 1954, 1987. These theorems establish that, as $k \to \infty$,

$(1/k) \log_b$ (probability density of α_L) converges to a limit, to be denoted as $\rho(\alpha)$.

The quantities $(1/k) \log_B$ (probability of values $> \alpha > \langle\alpha\rangle$) (resp., of values $< \alpha < \langle\alpha\rangle$) converge to the same limit. It is easily shown that

$$f(\alpha) = \rho(\alpha) + E = \rho(\alpha) + \text{dimension of the measure's support.}$$

It is a noteworthy fact that in the generalized Gibbs formalism resulting from the Cramèr theorem, different M's yield different $f(\alpha)$'s, and conversely.

Obviously, Cramèr-type theorems extend to the case when the factors M are weakly dependent or weakly non-identical.

5.5.2. *Comments concerning lognormality.* Section 5.5.1 may surprise those many readers who know the literature to the effect that $\log_b M(\eta_1) + \log_b M(\eta_1, \eta_2) + \ldots$ is asymptotically Gaussian, so that α is asymptotically lognormal. These assertions result from the application of a different renormalization, one that leads to the classical central limit theorem. It is indeed correct that the central limit theorem yields some information about the multiplicative multifractals. Also, this information is universal, but it is of very limited scope. It only implies that $\rho(\alpha)$ and $f(\alpha)$ are parabolic near their maximum. Away from the maximum, the behavior of $\rho(\alpha)$ and $f(\alpha)$ is *not* universal.

There is a seeming paradox here. On the one had, the probability outside of the central bell tends to 0 as $k \to \infty$, meaning that the tails become thoroughly insignificant. In the limit $k \to \infty$, the most probable value, the expectation and the other usual parameter of location all converge to each other. On the other hand, those "negligibly" few values in the tails are so huge that their contributions to all the moments of order $q \neq 0$, and to $\tau(q)$, are predominant. Moreover, the moments and $\tau(q)$ depend on the exact $f(\alpha)$, that is, are *non universal.*

In any event, the functions $(\rho)\alpha$ and $f(\alpha)$ are *not* like those from the lognormal M, except if the M are lognormal. Furthermore, lognormal M's require very special precautions. (Mandelbrot 1974 shows that the "principle of lognormality" claimed in 1962 by Kolmogorov is logically untenable. Frisch and Parisi 1985 have noted that, while their approach stemmed from Mandelbrot 1974, it was less general because it did not accommodate the lognormal.)

5.5.3. *The meaning of negative dimensions.* The special "pseudo-multinomial" situation requires $Pr\{M_{\max}\} \geq b^{-1}$ and $Pr\{M_{\min}\} \geq b^{-1}$. But a first feature of our 1974 multiplicative measures is that they allow $\alpha_{\min} = 0$ and $\alpha_{\max} = \infty$ and $f(\alpha_{\min}) = \log_b[Pr\{M_{\max}\}] + 1 < 0$ and $f(\alpha_{\max}) < 0$.

When $f(\alpha)$ is viewed as a fractal dimension, $f(\alpha) < 0$ is impossible. However, in numerous applications, the restriction to $f(\alpha) \geq 0$ leads to self contradiction, or is otherwise not acceptable. However these applications involve either of the following two possibilities: a) the possibility of investigating the measure within the intersection a multifractal by a line (i.e., a thin cylinder) or by a plane (i.e., a flat pancake), and b) the possibility of taking successive independent samples from a population. Neither of these possibilities occurs in astronomy, hence it is safe to keep to $f(\alpha) \geq 0$

5.5.4. *The quantities* $D_q = \tau(q)/(q-1)$ *are "critical dimensions."* For 1974 measures, D_q is a *critical dimension* for the moments of order q, as shown in Mandelbrot 1974. However, the task of proving this property exceeds the space available here. It suffices to point out that the proof relies upon on of the most interesting aspects of the 1974 theory, namely on the study of low dimensional cuts of higher dimensional multifractals; see Mandelbrot 1989.

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Note. Mandelbrot 1988 is superseded by Section 3 of the present text. The old text is far too brief to make its point, and there are typographical errors.

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Question by Valérie de Lapparent: Analysis of the recent CFA redshift survey shows that the galaxy distribution is characterized by well defined scale lengths: for example the thickness of the sheets and the mean separation of the galaxies with sheets are nearly constant throughout the observed region. This result seems difficult to reconcile with a self-invariant description of the galaxy distribution.

Reply by Benoit Mandelbrot: Thank you for the information you report, and allow me to respond by making two points.

The first point concerns method. Nowhere do my papers or books claim that a fractal description of a phenomenon in real space applies in *pure* mathematical rigor. To the contrary, I have always stressed that every real space model is an approximation that requires corrective terms, for example, cross-overs or cut-offs. (Let me add also that "pure fractals" are indeed often found in phase space.)

I have also stressed that the model one should prefer as a best first-order approximation should be the model that requires the smallest amount of second-order corrections. Therefore, even if the discrepancy you describe were to be confirmed, it would not *by itself* suffice to discard the fractal model as a first approximation valid within range of distances that is sufficiently broad to matter. Your earlier data suggest that its limit may not have been reached.

The second point concerns data analysis. The prowess of the observational astronomers never cease to amaze the mere theoreticians. But I am on the record as being less uniformly impressed by the methods used to analyze the resulting data. In fact, successive analyses of the same body of observations have often yielded contradictory results. New analyses will surely be made of the most remarkable empirical findings that you have reported a few years ago. Let us wait for the results before we rush to conclusions.

Question by P.J.E. Peebles: As you know, I argued in my book, *Large-Scale Structure of the Universe*, that the observations contradict the assumption that the galaxy distribution in the visible part of the universe is a pure scale invariant fractal. Nothing in the observational developments since then has caused me to change my mind.

Reply by Benoit Mandelbrot: Thank you for this opportunity to state my side in the friendly but inconclusive dispute we have been carrying on for a long time.

As I say in response to Valérie de Lapparent, "pure fractals" can be found in phase space, but I would not be completely surprised if they can *never* be found in any phenomenon in real space. Therefore, a lawyer may argue that we agree.

In any event, I am pleased that you *do not* specifically disagree with my minimal thesis, that an "impure fractal" model applies to galaxies, at least up to some crossover.

Now to the statement of some of our disagreements. In order to be used to contradict a theoretical model, observations must be subjected to careful analysis. This includes statistical analysis, and we certainly disagree on, a) which statistical tools are appropriate for the galaxy data and, b) what to do when one finds oneself beyond the range of applicability of statistics. Statistics tends to be a boring subject, but we agree that it does matter (though I have less confidence in it than you may have). By using the standard methods of statistical analysis, as you do, it may be true that one must indeed reach your conclusions, but I believe that your statistical methods grossly prejudge the issue, and are *not* applicable here. You are doubtless aware of the dispute that has recently pitted Pietronero et al. against Davis et al.. I tend to side with Pietronero, or – at worst – from our viewpoint to conclude that if the data are so poor as to demand brutal correction before analysis, one cannot conclude much on the basis of these data.

Second point. My criticism of your measure of the crossover has been countered by friends of yours, who assert that it does not matter whether the figure is 5 or 50, as long as the same definition is used throughout. I might perhaps have agreed, if the model being tested were characterized by a single crossover. (Possibility 4 in the terminology of Section 1.2.) But it appears that your account of the existence of the voids must accept the existence of an intermediate range of distances in which the correlations are negative. Therefore, you make an assumption that requires at least two crossovers. (Possibility 6 in the terminology of Section 1.2.) If so, the lower crossover, which I call R_{peebles} to avoid ambiguity, might conceivably describe the point when one moves beyond the fractal range. But homogeneity would only be established after a much longer distance R_{upper}, which need not bear any simple relation to R_{peebles}. This R_{upper} is by far the more relevant notion, but it remains to be estimated. The evidence you provide does not exclude that $R_{\text{upper}} = \infty$, which would mean that the distribution beyond R_{peebles} is neither fractal nor uniform, but a mess. Maybe this is the case, but we should do our best to avoid this conclusion, and find order in the large scale structure.