References

Style of reference and books referenced by italic capitals

References to serials are scattered through the paper at the proper places. The books are denoted by letters suggested by various mnemonic devices: initials of the author or the titles or (in the case of *Selecta*) the initials of economics, noise, Hurst, and chaos.

FM Frame, M. and Mandelbrot, B.B. 2002, *Fractals, Graphics and Mathematics Education.* Mathematical Association of America and Cambridge University Press.

FGN Mandelbrot, B. B. 1982, *The Fractal Geometry of Nature*, W. H. Freeman and Co., New York and Oxford. The second and later printings include an Update and additional references. Earlier versions were *Les objets fractals: forme, hasard et dimension*, Flammarion, Paris, 1975 (fourth edition, 1995) and *Fractals: Form, Chance and Dimension*, Freeman, 1977. There are innumerable translations, for example, the 1975 book was translated into Basque.

SE Mandelbrot, B. B. 1997E, *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk* (Selecta, Volume E) Springer-Verlag, New York.

SN Mandelbrot, B. B. 1999N, *Multifractality and 1/f Noise: Wild Self-Affinity in Physics.* (Selecta, Volume N). Springer-Verlag, New York.

SH Mandelbrot, B. B. 2002H, *Gaussian Self-Affinity and Fractals: Global Dependence, R/S,* $1/f$, *Rivers & Reliefs.* (Selecta, Volume H). Springer-Verlag, New York.

SC Mandelbrot, B. B. 2004C, *Fractals and Chaos: the Mandelbrot Set and Beyond.* (Selecta, Volume C). Springer-Verlag, New York.

ST Mandelbrot, B. B. 2004T, *Thermometry and Thermodynamics: Foundations and Generalization.* Webbook.

quantitative measure for it was a challenge that defied an easy answer. Science was powerless to tackle roughness until I found that in many cases it obeys diverse geometric scaling rules that can be accounted for by a dilation invariance. Fractures of metals are iconic from that viewpoint, as pointed out in Section 1.4.

Three forms of dilation invariance stand out. A fractal whose detailed structure is a reduced-scale image of the overall shape (perhaps statistically deformed), is called "self-similar." When the reduction ratios are different in different directions, the fractal is "self-affine." When the reduction ratios vary from point to point, one deals with "multifractality" (Section 3.3)

A first key continuing part of fractal geometry consists in identifying and classifying cases ruled by some form of dilation invariance.

A second key continuing part of fractal geometry results from the fact that dilation invariance provides the study of roughness with an increasing number of intrinsic quantitative tools — beginning with several distinct flavors of fractal dimension. That is, dilation invariance is the ingredient that makes roughness manageable. This is also why fractal geometry is a very broadly useful first approximation. Rough aspects of mathematics, nature, and culture come together because they can be studied by closely related tools, and progress in each aspect benefits from progress in the others. But unity stops at a certain point: each example has specific features that must eventually be acknowledged.

In 1975, having conceived and began to develop systematically a nascent geometry of roughness, I turned to the Latin adjective for "rough and broken up," namely *factus*, and coined for this geometry the term *fractal*.

Let me now restate the key scientific claim I put forward increasingly forcibly and continue in buttressing. A workable path towards rational rugometry has now been identified as being made of rough shapes that are dilation invariant. They are the fractals.

1.3 Explanatory background in older sciences that study other sensations of Man

It is good to keep in mind that the earliest sciences started as ways to organize substantial collections of messages that Man receives from the various "senses." The complexity of most messages is such that a science can take off only after it identifies "representative" special cases to be studied first.

For acoustics, an important step consisted in recognizing that chirps or drums are very difficult to handle, but idealized vibrating strings or pipes lead to periodic sums of sinusoids. That is, acoustics became quantitative when it managed to define "pure sounds" and measure their pitch by a frequency. As had to be the case, this quantitative measure is consistent with "intuition" and the extensive earlier knowledge manifested, for example, in music. The limitations of acoustics continue to be notorious, but do not prevent it from being extraordinarily useful.

Similarly, the theory of heat became quantitative when Galileo devised the thermometer and measured hotness by a temperature. Here too, a limitation must be recognized: far from equilibrium, the theory of heat continues to struggle.

In the same vein, the examples of real rough curves or surfaces that are usefully close to being self-similar or self-affine allowed me to define "pure" or "perfect" roughness as analogous to the classic concept of "perfect gas in equilibrium." The latter is invariant by translation of time, the fractals — once again — by dilation.

Like pure sound or pure elliptic motion under gravitation, pure roughness is an abstraction and fractal geometry cannot address roughness that is far from being dilation-invariant. But dilation-invariant roughness is useful: its scope is considerable and must be expanded before facing further tasks.

1.4 Fractal dimension as the first intrinsic and quantitative measure of roughness; metal fractures and a conjectured fundamental universality

As first measures of pure roughness, I proposed notions that were known but viewed as esoteric: fractal dimension or Hölder exponent or codimension. It was necessary to first reinterpret these notorious concepts as being numerical characteristics of an invariance (self-similarity, self-affinity, or multifractality) and then expand their study, both concretely and intuitively.

From the preceding viewpoint, particular iconic importance attaches to a study by myself, Passoja & Paullay (*Nature,* 308 (1984) 721-2). We found metal fractures to be dilation invariant with a dimension that exceeds 2 — the dimension of smooth surfaces — by 1/3. This property has been confirmed by extensive later work that went beyond metals to glasses and covered sizes covering five decades at least. The range is sometimes even broader, but may be limited by the nature of the data. Fractality is the special ingredient making it possible to measure roughness intrinsically by what is now often called the "roughness exponent."

This discovery of the "universal" excess dimension 1/3 has provided the nascent rational rugometry of metal fractures with a broad and fundamental observation. It defines a "macroscopic" aspect of the study of fracture that must be added as conjecture to the more prevalent "microscopic" approaches.

An invidious claim one hears is that fractal geometry has solved or advanced no existing problem in physics. This claim is, among others, contradicted in the contexts of metal fractures and turbulence. But it may be true that the more visible role of fractals in physics has not been directed to what already existed but to the future. The very fact of proposing a quantitative measure of roughness has raised *thoroughly new problems* of all kinds. Several have already been solved, for example problems concerning the fractal dimensions of two very distinct kinds of physical clusters, examined in Sections 2.6 and 6.3. Other new problems remain wide open and there is no reason to expect them to be easy.

1.5 A fundamental formal kinship between the nascent "rational rugometry" and thermometry

The suitable measure of roughness having been found in previously esoteric notions of mathematics, rugometry might have developed in ways quite distinct from the sciences based on previously quantified "sensations." But in important cases fractal dimension takes the form $\Sigma p \log p$, which is an "information" hence a further link with thermodynamical entropy. This resemblance is far from complete but brings a high level of formal unity and suffices to allow many questions concerning roughness to

1.6 *Regrettable "centrifugal" tendencies splitting the fractal synthesis. The many historically separate notions of "scaling"*

Today — to my great regret — "centrifugal" developments affect several "chapters" of my work that arose in the 1950s and 1960s. All had been slow in acquiring a broad following until they were empowered by being subsumed in fractal geometry. Now they have taken off and tend to develop on their own. Some are commented upon in suitable sections of this paper. One is the study of Zipf's and other "power laws" and Lévy stable distributions, which I began in the 1951. Another is "econophysics," which I originated in 1962 without giving it any specific name. A third is the study of metallic fractures and the like. If these developments "dismember" the fractal synthesis, the resulting fragments would all be harmed.

Neither is it helpful to replace the term "fractal" by "scaling." That replacement is sometimes formally correct but is invariably misleading because scaling has multiple meanings — related but not identical. Scaling occurs in probability theory since Cauchy (1853) and P. Lévy (the 1920s). It occurs in turbulence since Richardson (the 1920s) and Kolmogorov (1941). It occurs in increasingly geometric fashion in my work, since 1951 for Zipf's law, and already very explicitly in 1956.

Finally, scaling occurs in different parts of "core physics," especially in the physics of criticality since K. Wilson in 1972. Criticality had the largest number of practitioners and tempts other investigations to use its terminology. However, criticality is a very specific situation. The study of critical shapes like clusters have been greatly helped by fractal tools but there was no significant influence in the opposite direction. Not only criticality played no role in originating the chapters of fractal geometry mentioned early in this subsection, but it evolved no tools to help their study. For example, it had no use for Lévy stable distributions. Therefore, thinking in terms of criticality did not and does not bring any benefit.

Added to other reasons, the preceding comments make it useful to ponder the broader issue of the place of fractals within physics. I think of fractality as related to the emergence of a new stream of thinking sketched in Section 1.1. Being concerned specifically with roughness in all its forms, it can be viewed as providing a generalized physics. The dream of generalizing physics in this fashion is an ancient one but had long been thwarted as long as overly specific features of existing physics were preserved too faithfully.

1.7 *The role of fractal geometry in pure mathematics: renewed key role played by the "material" world and the examination of fully-fledged pictures*

Another invidious claim one hears is that fractal geometry has solved no existing mathematical problem. This claim has no merit, either, but it is true that I provided few difficult proofs but many separate conjectures of all kinds. Each turned out to be difficult and opened a new field that continues and prospers long after I move to other concerns. Notable examples will be mentioned in Section 7 devoted to the Mandelbrot set, Section 2 devoted to the dimension 4/3, and several subsections throughout devoted to multifractality. Other conjectures are scattered elsewhere in this text.

The perceived importance of those contributions to pure mathematics varies but a common feature is that they did not arise from earlier mathematics but in the course of practical investigations into diverse sciences of nature or of culture, some of them old and well-established, others newly revived, and a few altogether new. Some branches of mathematics agree that physics, numerical experimentation and geometric intuition are very beneficial but other branches proclaim physics as irrelevant, computation as powerless, and intuition as misleading — especially when it is strongly visual. A well-known irony is that history consistently proves that, as branches of mathematics develop, they suddenly either lose or acquire deep but unforeseen connections with the sciences — old and new.

As to numerical experimentation — which Gauss had found invaluable, but whose practice was long interrupted — it has seen its power multiplied thanks to computer calculations, and later, to computer graphics. This allows my practice to be dominated, in mathematics as in the sciences, by the role played by fully-fledged pictures that are as detailed as possible and go well beyond mere sketches and diagrams.

This feature destroyed a belief that was near-universal among pure mathematicians around 1980, that a picture can only lead to another, and never to fresh mathematics. Hence, my work bears on an issue of great consequence. Does pure (or purified) mathematics exist as an autonomous discipline, one that can — and ideally should — adhere to a Platonic ideal and develop in total isolation from both "sensations" and the "real" world? I believe, to the contrary, that the existence of totally pure mathematics is a myth — a useful one on occasion, but not on the long run.

My 1982 book *The Fractal Geometry of Nature*, *FGN*, was meant above all to be a "manifesto" in praise of the trained eye. I believe that computer graphics has changed the iconoclastic (anti-pictorial) dogma that prevailed in mathematics and physics into a serious liability. In search of always fresh evidence for this belief, I looked for new facts that the standard pictures leave hidden. The pictures' original goal was modest: to gain acceptance for ideas and theories that I had managed to develop without pictures and whose acceptance was reluctant and slow because of cultural gaps. To begin with, the pictures did indeed lead to acceptance, but then they went on to help me and many others generate new ideas and theories. The input of mundane questions gradually grew and became far more ambitious than originally intended or recognized.

Norbert Wiener once described his key contribution to science as bringing together — starting from widely opposite horizons — the fine mathematical points of Lebesgue integration and the physics of Gibbs and Perrin. Similarly unlike "parents" characterize the theory of fractals, which is arguably a multiple second flowering of Wiener's Brownian motion. Also (like Poincaré) Wiener was very committed (and successful) in making frontier science known to a wide public.

1.8 *The unexpectedly long history of fractals began well before nineteenth century mathematics; fractals have now been traced back to art since time immemorial*

Anticipating the difficult conjectures mentioned in Section 1.7, the early pictures I drew of old standbys like the Koch or Peano curves and the Cantor set were precise, and as a result they became inspiring. They sufficed to thoroughly disprove the previously held belief that those sets are "monsters." Quite to the contrary, they were turned around into unavoidable "cartoons" of reality. For example, I "demoted" Peano "curves" from being counter-intuitive monsters to being nothing but motions that follow plane-filling networks of rivers.

More profoundly, giving concrete uses to mathematics allowed it to be compared on more equal terms with other human activities and allowed fractals' history to slowly reveal itself as having been long and varied.

In art and decoration, they have been known since time immemorial, all over the world. I noted a few examples in *FGN* but new examples reveal themselves continually.

Far better known is the already mentioned second broad stage in history: a century ago, fractals entered the purest of mathematical esoterica and a "Polish school" of mathematics viewed itself as devoted exclusively to *Fundamenta*, added mightily to the list of monster shapes. It greatly contributed to the deep and long — but inevitably of finite duration — estrangement of mathematics from physics.

Specifically ironical, therefore, is that in a third stage my work, that of my colleagues, and now that of many scholars, made those monster shapes, and new shapes that are even more "pathological," into everyday tools of the sciences of nature and culture.

This subsection must end by a call for balance. I always agreed with John von Neumann that "a large part of mathematics which became useful developed with absolutely no desire to be useful... This is true for all science. Successes were largely due to... relying solely on... intellectual elegance. It was by following this rule that one actually got ahead in the long run, much better than any strictly utilitarian course would have permitted... The principle of laissez-faire has led to strange and wonderful results."

1.9 *The beauty of fractals*

Fractal pictures have become ubiquitous. Many strike everyone as being of exceptional and totally unexpected beauty. Some have the beauty of the mountains and clouds they are meant to represent; others are abstract and seem wild and unexpected at first, but after brief inspection appear totally familiar. In front of our eyes, the visual geometric intuition built on the practice of Euclid and of calculus is being retrained with the help of new technology.

Hence a different philosophical issue arises. Is there any relation between the beauty of these mathematical pictures and the beauty that a mathematician rooted in the twentieth century mainstream sees in his trade after long and strenuous practice? My lectures often underline these questions, by showing in full colors what certain mathematical shapes really look like.

1.10 General references

Due to space restrictions, this survey is extremely sketchy and centers around my own contributions. As the field grew, early versions appeared in 1984, 1999, 2000, and 2001. Each in turn was made obsolete by the continuing development of the field.

On fractals overall, the basic reference remains my 1982 book *The Fractal Geometry of Nature*, already referenced as *FGN*. As explained at the end of the paper, suitable other initials in italics will reference other books, some printed and others only available (now or shortly) on my web: www.math.yale.edu/mandelbrot. More specific references are made part of the text.

Alternative surveys include a) a text I wrote with M.L. Frame for *The Encyclopedia of Physical Science and Technology in Fifteen Volumes* (San Diego CA: Academic), third edition (2001): **6**, 185-208, b) the Overview chapter of *SH*, and c) several chapters of book *MF*. A useful commentary on the mathematics is provided by the Foreword Peter W. Jones contributed to *SC*.

2 Complex Brownian bridge; Brownian cluster and the dimension 4/3 of its boundary; the self-avoiding plane Brownian motion

The sequence of examples in this paper follows little order. As mathematics goes, the iconic Mandelbrot set is only mentioned in Section 7. The present Section 2 is concerned with an example that is far less widely known but is easy to understand and of greatest current interest. It provided mathematicians with difficult conjectures and a unifying theme. It provided physics with a new cluster having special virtues discussed in Section 2.5.

2.1 A historically incorrect and continuing misleading "streamlined" story

The story of the "4/3 conjecture" was exemplary by the standards of my work and this paper but very atypical by the customary standards of mathematics. Therefore it is often replaced by the following grossly "streamlined" account.

Somehow, Mandelbrot had the idea that in the plane the boundary of Brownian motion is a curve of Hausdorff-Bescovitch dimension 4/3. The conjecture attracted wide attention but turned out to be very challenging. The proof took time and came in two stages.

A "field-theoretical" physical argument has been provided by B. Duplantier, *Phys. Rev. Lett*: 82, 1999, 880; 82, 1999, 3940; 84, 2000, 1363.

A proper proof has been provided by G. Lawler, O. Schramm & W. Werner, much of it is only available on the Web (xxx.lanl.gov/abs/math.PR/0010165) as a series of preprints totaling over 100 pages, the first of which has been accepted by *Acta Mathematica*. According to a newsweekly (*Science*, 8 December 2000, pages 1883-4) it "drew rave reviews" at an important meeting and was hailed as "one of the finest achievements in probability theory in the last 20 years."

Between 1982 and 2000, a dozen or so scattered technical conjectures in mathematical analysis had been shown to be equivalent to that "4/3." Therefore, all have now been proven as corollaries and together provide an element of unity that co

2.2 *Preliminaries to the historical sequence of events. Definitions of the Brownian cluster and of self-avoiding Brownian motion*

The actual history of the 4/3 is more interesting. The key discovery reported in 1982 — *FGN*, Plate 243 — relied on a novel processing of *Brownian motion* $B(t)$ in the plane. This very old shape is, of course, a random process whose increments $B(t+h) - B(t)$ are two-dimensional Gaussian random variables with mean 0 and variance h, and are independent over disjoint time intervals. It is well-known that $B(t)$ is statistically self-affine in the sense that

$$Pr\{B(t+h) - B(t) \leq b\} = Pr\{B(s(t+h)) - B(st) \leq \sqrt{s}b\},$$

and the same is true of joint probability distributions for all finite collections of time intervals h_j.

Assuming $B(0) = 0$, a *Brownian bridge* $B_{bridge}(t)$ was defined by N. Wiener as the periodic function of t, of period 2π, that is defined for $0 \leq t \leq 2\pi$ by

$$B_{bridge}(t) = B(t) - (t/2\pi)B(2\pi).$$

In distribution, $B_{bridge}(t)$ is identical to a sample of $B(t)$ conditioned to return to $B(0) = 0$ for $t = 2\pi$. Wiener wrote $B_{bridge}(t)$ as a trigonometric series whose nth coefficient is $G_n/\sqrt{n}$, where the G_n are independent reduced Gaussian random variables. Combining two statistically independent Brownian bridges yields the complex function $B_{bridge}(t) = B_r(t) + iB_i(t)$.

The *Brownian plane cluster* Q is defined in *FGN*, Plate 243, as the set of values of $B_{bridge}(t)$. This is the (non-traditional) map of the time axis by the complex function $B_{bridge}(t)$. The classical map of the time axis by $B(t)$ is everywhere dense in the plane, and the map of a time interval by $B(t)$ is an inhomogeneous set. In contrast, conditioning the origin Ω of the frame of reference to belong to Q makes all the probability distributions concerning Q independent of Ω. Therefore Q (see *SN*, Chapters 8, 9 and 10) I called Q a *conditionally homogeneous* set. This property is not only aesthetically attractive, but, as will be seen, proved inspiring.

The self-avoiding planar Brownian motion $\tilde{Q}$. This random object is defined in *FGN* as being the closed set of points P in Q accessible from infinity by a path that does not intersect $Q - P$. This $\tilde{Q}$ is also conditionally homogeneous.

2.3 *Steps that led to the Brownian cluster being defined*

Today, after the fact, the boundary of Brownian motion or cluster seems a "natural" notion. After all, the overall appearance of planar Brownian motion is known at least since J. Perrin, as evidenced in *FGN*, Plate 13. It inspired Norbert Wiener in the 1920s, then pictures' evocative power was exhausted. In the absence of suitable "graphic rendering," the earlier pictures of samples of $B(t)$ did not highlight a boundary. Worse, they gave no hint of anything worth studying.

This boundary came up during a "fishing expedition," an aimless search motivated by the feeling that a careful fresh look at $B(t)$ using better tools may lead to new insight. Plate 242 of *FGN* exemplifies the finite duration samples of $B(t)$ with which I began; those pictures "did not talk to me." I figured that those finite samples' non-homogeneity may overwhelm and hide interesting facts. When the eye

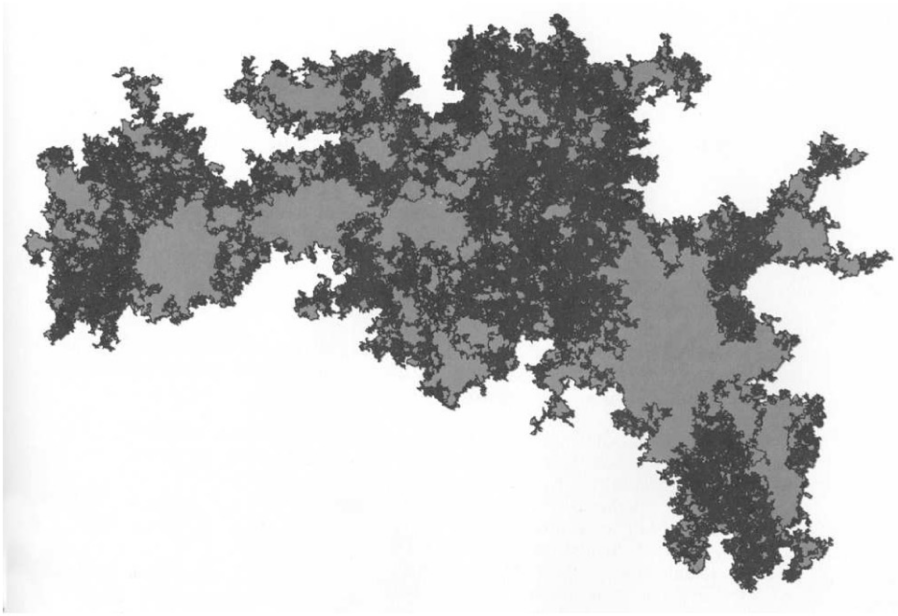

Figure 1. This is Plate 243 of *FGN*, representing the original sample of Brownian cluster.

is to be trusted, it is good practice to help it and in particular to avoid burdening it by extraneous complications — such as non-homogeneity.

To the contrary, the Brownian cluster is homogeneous by design. Therefore, I asked my assistant to produce a Brownian cluster and also to "paint" its interior in order to enhance the graphics.

The outcome became Plate 243 of *FGN*, reproduced here as Figure 1. It triggered an "eurêka" moment. With no prompting, what I saw looked to me like an island with a clearly visible and especially wiggly coastline. Hence visual intuition nourished by experience in geomorphology suggested $D \approx 4/3$. This value was confirmed by my direct numerical tests.

2.4 Comment on the relation between the dimension 4/3 *and self-avoidance*

Originally, the term "self-avoiding Brownian motion" came to my mind because $\tilde{Q}$ is a shape related to Brownian motion and does not self-intersect. The term became strengthened because I recalled the dimension 4/3 found in the plane for the self-avoiding random walk (SARW) on a lattice. The value 4/3 for SARW is unquestioned but physicists obtained it by analytic arguments that are geometrically opaque; its interpretation as a dimension implies yet another unproven conjecture, which no one doubts.

2.5 *Differences between the self-avoiding Brownian motions defined it the cluster and via the "streamlined" account*

The mathematicians who take the "shortcut" described in Section 2.1 define "self-avoiding Brownian motion" as the boundary of a finite sample of Brownian motion. The same Hausdorff Besicovich dimension of 4/3 holds for two clearly distinct fractal curves. I suspect that the cluster boundary is the more interesting topic.

This ambiguity recalls one that specialists in SARW on a lattice have observed long ago: a standard definition and a "true" one had vied for attention. This topic is interesting but space lacks to develop it.

2.6 *Brownian clusters, as compared to the clusters of statistical physics*

Section 5 will survey several major clusters in statistical physics: percolation, Ising, DLA. All belong to physics on a prescribed lattice. Contrary to fractals, their construction does not proceed by an interpolation that converges strongly to a limit, but by extrapolation.

It is the case that down-scaled versions of those physical lattice clusters, converge weakly to fractals? This is what I conjectured and precise forms of the conjectures are widely believed and studied. For DLA (Section 5.3) the issue is murky.

By contrast, Brownian clusters did not originate in physics but have a special asset they follow an explicit definition and involve no conjectural limit process.

2.7 *Squigs and a wide open issue that combines fractals and topology*

Being obtained by extrapolation, SARW is difficult to study. In the spirit of Section 2.6, *FGN* (Chapter 24) introduced recursive alternatives to SARW, called *squigs*, that create self-avoidance by interpolation. For the simplest squig my heuristic argument yielded the dimension $\log_2 2.5 \approx 1.3219...$ This value was confirmed by J. Peyrière, *C.R. Acad. Sc. Paris*: 286A 1978, 937 and *Ann. Institut Fourier*: 31, 1981, 187. The discrepancy between 4/3 and $\log_2 2.5$ clearly follows from the fact that only the squigs — not the clusters — involve a discrete and recursive subdivision of the plane into triangles, squares, or other indefinitely interpolable tessellations. Viewing this discrepancy as of secondary importance, I suspect that self-avoidance is linked in a profound and intrinsic way to the dimension 4/3. The nature of this link is a mystery and a challenge.

3 Explosive multiplication of new fractal constructions, dimensions (including negative ones), and Hölder exponents

Until fractal geometry became organized, the numbers of distinct fractal constructions and of distinct definition of fractal dimension were both very small. Moreover, the values of distinct dimensions used to coincide, except for contrived "counterexamples." As fractals became common tools in the sciences and favorites in computer graphics new constructions multiplied. Moreover, differences between the values of distinct dimensions ceased to be exceptional; in many contexts they became the rule with every variant contributing its share to an overall description. Fractional

Brownian motion and multifractal measures led to a rich mathematical literature that is exemplified in *SH* and *SN*, respectively. Other new constructions are less well known. Section 3.1 describes one example. The remainder of the chapter tackles the multiplicity of dimensions.

3.1 A promising but little-explored novelty: embedding the stable processes and fractional Brownian motions in a the broader class of functions: the fractal sums of pulses

Brownian motion was generalized in two deeply different ways by the introductions of Lévy stable processes (LSM) and fractional Brownian motions (FBM). The LSM depend on a parameter α, with $0 < \alpha \leq 2$ and $\alpha = 2$ yielding the Brownian as a limiting case. They are investigated, among many other places, in *SE*. The FBM depend on a parameter H, with $0 < H < 1$ and $H = 1/2$ yielding the Brownian as a critical case. They are investigated, among many other places, in *SH*. By the definition of $B_H(t)$, the increment $B_H(t) - B_H(t')$ is a Gaussian random variable of expectation 0 and standard deviation $|t - t'|^H$.

Numerous formal analogies exist between the respective studies of LSM and FBM. Those analogies changed from surprising to very natural when I imbedded both families in a far broader family, the "fractal sums of pulses" (FSP). The FSP also allow a variety of additional behaviors that are useful in science and may be of mathematical interest. The latest reference is my contribution to *Long-Range Dependent Processes* (eds. G. Rangarajan and Ming Ding) Springer 2003, pp. 118-135.

3.2 Multiplicity of alternative definitions of dimension

Linearly self-similar sets are iconic but exceptional. For them, the many definitions of fractal dimension yield identical values. A set S is self-similar if it is constructed recursively and its generator consists of N copies of itself, the *i*th copy S_i being obtained from S by a similarity with contraction factor r_i. The calculation of the fractal dimension D is relatively simple. Under a mild condition (the "open set" condition), D is the solution of the Moran generating equation

$$\sum r_i^\delta = 1,$$

where i ranges from 1 to N.

The original Hausdorff-Besicovitch dimension invoked in Section 2 remains essential in mathematics despite the fact that its value is often hard to obtain. But in the sciences, D_{BH} is impossible to measure because its definition contains the operation "inf." (In the case of self-similar or self-affine shapes, the operation "limit" poses no problem.) Far more important is the fact that self-similar sets are a special case. Purely mathematical needs demanded concepts of dimension distinct from D_{HB} and contrived "counter-examples" showed that, in the absence of self-similarity, those dimensions can take distinct values. More recently, concrete needs forced fractal geometry to alternative definitions that led to values other than D_{HB}. Often, considering those values together helps describe an object's geometry.

3.3 Self-similar multifractal measures

The random multiplicative singular measures that I began to construct around 1970 are described in papers from 1968, 1972, 1974 and 1976 collected in *SN*. They are now called multifractal. They were not intended to become a new kind of esoterica but a model in turbulence and (near immediately after) in finance. The conjectures I put forward created an active and prosperous subbranch of mathematics and — today — the main branch of statistical modelling of the variation of financial prices.

The topic is too rich to be dwelt upon here, but it is useful to note that a multifractal measure is, above all, described by a function $f(\alpha)$ of the parameter α. My original 1974 paper dealt with multiplicative multifractals (see Section 3.5) and deduced a function equivalent to $f(\alpha)$ from the Cramer theory of large deviations. Since they involve a function $f(\alpha)$, multifractal measures involve an infinite number of parameters.

3.4 Negative dimensions as measure of the newly introduced notion of quantitative measure of emptiness

The value of $f(\alpha)$ can be either ≥ 0 or ≤ 0, hence a fundamental distinction enters inevitably. When it is positive, f is a suitable set's fractal dimension, for example in the sense of Hausdorff Besicovitch. When it is negative, f takes an altogether different new role, as a measure of "degree of emptiness." (Mandelbrot, *J. Fourier Analysis and Applications* (Kahane issue), 1995, 409-432; *J. Stat. Physics*, 110, 2003, 739-777). Negative dimensions amply deserve closer study.

3.5 Multiplicative multifractals: microcanical, canonical, and products of pulses or other functions

Multifractals' self-affinity can be approximate or exact. Numerous approaches, some heuristic and some mathematically rigorous, apply under quite general conditions but, as unavoidable counterpart, they are not very specific. Beginning in my pioneer papers, I have taken a different tack and deliberately focussed on multifractals that — in a statistical sense — are exactly self-affine. They are less general but perspicuous and continue to yield very specific and varied results one can "tune" by changing the process.

Step by step, the constraints were made less and less strong and immensely richer structures arose. In 1974, I moved the construction from microcanonical to canonical products (*J. Fluid Mechanics* 62, 1974, 331-358 and CR (Paris) 278A; 1974, 289-292 & 355-358). Recently, the construction moved further to products of pulses and of other kinds of functions (Barral and Mandelbrot *Proba. Th. and Related Fields* 124, 2002, 409-430, *J. Math. Pures et Appl.* 82, 2003, 1555-1589 and contributions to the book *Fractals* (ed. M. Lapidus) Am. Math. Soc., 2004.)

3.6 Self-affine sets

When the transformation of S into S_i is an affinity, the evaluation of D_{HB} was successful in a surprisingly small number of cases. Contributors include McMullen, Bedford, Falconer, Peres, Kenyon, Lalley, and Gatzouras.

Furthermore, the many alternative definitions of fractal dimension yield values that differ from D_{HB} and from one another. In particular, my contribution to *Fractals in Physics* (E. Pietronero & E. Tosatti, eds.) 1986 (reprinted in *SH* as Chapters H22, H23 and H24) introduced the concepts of *local* and *global* dimension. They coincide in the self-similar case but greatly differ in the case of self-affinity. The global notions of dimension pose many open mathematical issues.

All these computations suggest that, while the notion of fractal dimension can be defined under wide conditions, its "natural domain" of practical relevance centers around self similarity.

3.7 The many forms of the Hölder (and Hurst) exponent

In the case of the graph of a self-affine function, the most "natural" quantitative description of roughness is not provided by a dimension, but by diverse forms of an exponent introduced in the 1970s by the mathematicians by Hölder and Lipschitz and in the 1950s by the hydrologist H. E. Hurst. The variable α in the multifractal function $f(\alpha)$ is a Hölder exponent. Chapter E6 of *SE* and Chapter N1 of *SN* show that the original definitions have, in response to concrete needs, branched in diverse directions.

3.8 The exponent yielded by a generalized Moran equation

As discussed in *SE* and mentioned in Section 10.3, I put forward the fractional Brownian motions of multifractal "trading time" as models of price variation. Instead of a Hölder-Hurst exponent, they involve "H" exponents of particularly great variety.

Denoting the ΔP_i the increments of such a function over arbitrarily chosen time increments Δt_i, the sum $|\Delta P_i|$ has no upper bound, hence $P(t)$ is called a function of unbounded variation. More generally, define the qth variation by starting from the formula for the ordinary variation and replacing $|dP|$ by $|dP|^q$. If the qth variation is infinite for $q < 1/H$ and vanishes for $q > 1/H$, the value $q = 1/H$ is "critical" and defines the tau dimension D_τ. (The tau dimension is independent of the trading time and concern a projection along the time axis of a complex-valued "completion" of the function $P(t)$.) The inverse $1/D_\tau$ is yet another form of Hölder's exponent. It generalizes to all processes and in many cases the equation yielding D_τ is a generalization of Moran's equation of Section 3.2.

This is, for example, the case for the "cartoons" that I described in *Quantitative Finance*, 1, 2001, 427-440.

The properties of D_τ and of the "non-Hölderian" $1/D_\tau$ deserve careful mathematical study beyond what is already known.

4 Tools of fractal analysis other than the dimensions: ramification and lacunarity

Careful analysis brings in many fractal tools, some new, other old but obscure, that are neither dimension-like nor Hölder like exponents.

4.1 *Sierpinski curves and Urysohn-Menger ramification*

As seen in *FGN*, Sierpinski's investigations in the 1900s built on two ancient decorative designs: one became known as the "carpet," and the second I called the "gasket." The Sierpinski carpet shows that a plane curve can be "topologically universal," that is, contain a (homeomorphic) transform of every other plane curve. The construction starts with a square, divides it into nine equal subsquares and erases the middle one, which I call a "trema" ($\tau\rho\eta\mu\alpha$ is the Greek term for "hole"). One proceeds in the same fashion with each remaining subsquare, and so on ad infinitum. The Sierpinski gasket is a curve with branching points everywhere. The construction starts with an equilateral triangle, divides it into four equal subtriangles and erases the middle one as trema. One proceeds in the same fashion with each remaining subtriangle, and so on ad infinitum.

During the 1920s, the distinction between the carpet and the gasket became essential to the theory of curves. Piotr Urysohn and Karl Menger took them as prime examples of curves having, respectively, an infinite and a finite "order of ramification."

FGN quotes influential mathematicians for whom the "gasket" gave prime evidence that geometric intuition is powerless, because it can only conceive of branch points as being isolated, not everywhere dense. In fact, Gustave Eiffel himself wrote (as I interpret him) that he would have made his Tower lighter, with no loss of strength, had the cost of finer materials allowed him to increase the density of double points. From the Eiffel Tower to the Sierpinski gasket is an intellectual step that one's intuition is easily trained to take.

The theory of curves that studies carpets, gaskets and the order of ramification became a stagnant corner of mathematics. Where can one find the latest facts about these notions? The surprising answer is that, after I introduced them in the statistical physics of condensed matter, physicists came to view these notions as "unavoidable." Once ridden of the cobwebs of abstraction, they prove to be very practical and enlightening geometric tools to work with. Physicists make them the object of scores of articles, and invent scores of generalizations that mathematicians did not need in 1915.

4.2 *Ramification's key role in diffusion on fractals*

Early on in the study of fractals in physics (in the wake of Gefen et al *Phys. Rev. Lett.*: 45, 1980, 855) we had to investigate random walks on lattices that approximate fractals. We found that a key role is played by those fractals' order of ramification. The theory was easy for $R < \infty$ (for example for the Sierpinski gasket). But for $R = \infty$ (for example, for the Sierpinski carpet), exact theory is impossibly difficult and we had to resort to possibly dubious approximations.

The theory of diffusion on fractals has grown into an active field of mathematics. For $R < \infty$, our heuristic arguments have been given a sound basis but the case $R = \inf$ continues to be very problematic.

4.3 *A non-dimensional and non-topological fractal tool that begs to be studied further: lacunarity*

The well-known standard construction of a Cantor dust proceeds recursively as follows. The "initiator" is the interval $0, 1$. Its first stage ends with a generator made of N subintervals, each of length r. In the second stage, each generator interval is replaced by N^2 intervals of length r^2, etc. The resulting limit set arose in the study of trigonometric series, but first attracted wider interest because of its topological and measure-theoretical properties. From those viewpoints, all Cantor dusts are equivalent. Hausdorff's and every other definition of dimension yield $D = \log N / \log(1/r)$. The value of dimension splits the topological Cantor dusts into finer classes of equivalence parameterized by D.

Fractal geometry showed those classes of equivalence to be of great concrete significance. In due time, the needs of science, rather than mathematics, required an even finer subdivision. To pose a problem, consider the Cantor-like constructions stacked in Figure 2. In the middle line, $N = 2$ and $r = 4^{-1}$; k steps below the middle line, $N = 2^k$, $r = 4^{-k}$ and the generator intervals are uniformly spaced; k steps above the middle line, $N = 2^k$, $r = 4^{-k}$, again, but the generator intervals are crowded close to the endpoints of $0, 1$. The Cantor dusts in this stack share the common values $D = 1/2$, but look totally different. The Latin word for hole being *lacuna*, motion down the stack (or up) is said to correspond to decreasing (increasing) *lacunarity*.

Challenge. As $k \to \infty$, the bottom line becomes "increasingly dense" in $0, 1$, and the top line "increasingly close to two dots." Provide a mathematical characterization of this "singular" passage to the limit.

Second challenge. *FGN*, Chapters 33 to 35, and my contribution to *Fractal Geometry and Stochastics* (ed. C. Bandt et al) Birkhäuser 199, 12-38 describe and illustrate several constructions that allow a control of lacunarity. However, for the needs of both mathematics and science, the differences between the resulting constructs must be quantified. The existing studies of this quantification show that it is not easy and also not unique. Identical reduction ratios, like in Figure 2, create special complications.

Of the alternative methods investigated in the literature, one is based on the prefactor relation $M(R) = FR^D$ that yields the mass $M(R)$ contained in a ball of radius R.

Another method is based on the prefactor in the Minkowski content.

A third method has the advantage that defines a neutral level of lacunarity that separates positive and negative levels.

On the line, this level is achieved by any randomized Cantor dust S with the following property. Granted that any choice of origin Ω in S divides the line into a right and a left half line, lacunarity is said to be neutral when the intersections of S by those half lines are statistically independent. Increasingly positive (resp. negative) correlations are used to express and measure increasingly low (resp. high) levels of lacunarity. These notions will be used in the sections that follow and in Section 6.3.

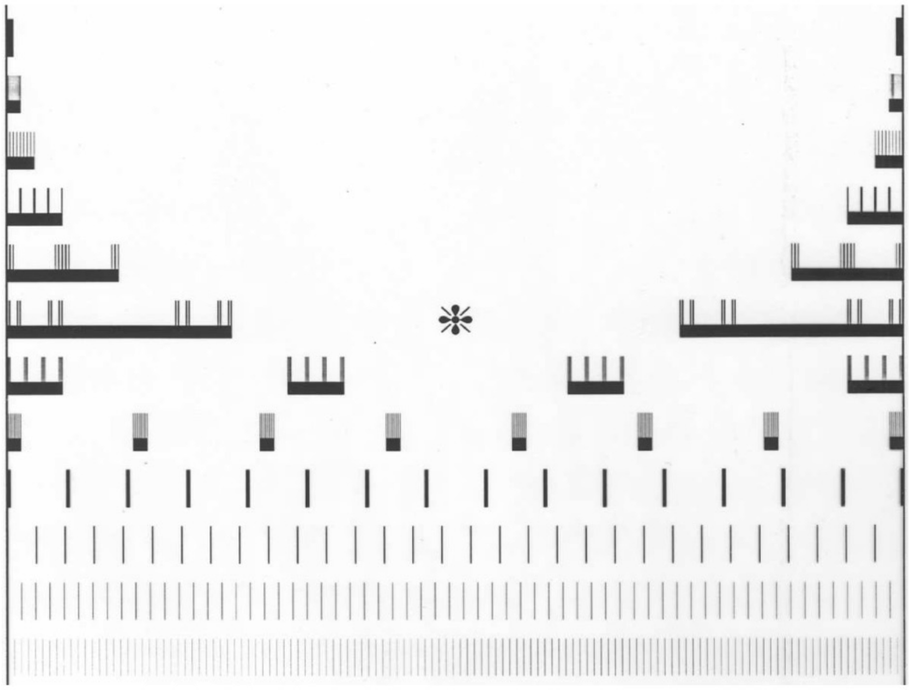

Figure 2. A stack of Cantor sets of equal dimension $D = 1/2$, whose lacunarity changes from very low at the bottom to very high at the top of the stack.

4.4 Actual geometric implementation of the formal fractional-dimensional spaces that are useful in statistical physics

The physics of criticality is very successful with spaces whose properties are obtained from those of Euclidean spaces by interpolation to "noninteger Euclidean dimensions." The dimension may be $4 - \epsilon$ or $1 + \epsilon$, where ϵ is in principle infinitesimal. Formal calculations are carried out, including expansions in ϵ. Then the final stage sets the "infinitesimal" ϵ to $\epsilon = 1$. Mathematically, these spaces remain unspecified, yet the procedure turns out to be extremely useful.

Mathematical challenge: Show that the properties postulated for those spaces are mutually compatible, show that they do (or do not) have a unique implementation; describe their implementation constructively.

Very partial solution: A very special example of such space has been implemented as a limit (*FGN*, second printing, p. 462; Gefen et al, *Phys. Rev. Lett.* 50, 1983, 145). We showed that the postulated properties of certain physical problems in this space are identical to the *limits* of the properties of corresponding problems in a Sierpinski carpet whose "lacunarity" is made to converge to 0, in the sense that it tends to 0 as one moves down the stack on Figure 2.

5 Fractality of the major fractal clusters in statistical physics

While Brownian motion is fundamental in physics as well as in mathematics, the Brownian clusters of section 2 are recent, perhaps only a mathematical curiosity. However, their property of fractality is shared by all the major real clusters (turbulence, galaxies, percolation, Ising, Potts) and all the major real interfaces (turbulent jets and wakes; metal and glass fractures discussed in Section 1.4; diffusion fronts). Each of these categories raises numerous open mathematical questions, of which a few will be commented upon.

5.1 Percolation clusters at criticality

Take an extremely large lattice of copper or vinyl tiles. Each tile is chosen at random: with the probability p, it is made of vinyl and with the probability $1-p$, of copper. Allow electric current to flow between two tiles if they have a side in common. A "cluster" can then be defined as a collection of copper tiles such that electricity can flow between any two of these tiles. The basic reference is D. Stauffer & A. Aharony. *Introduction to Percolation Theory.* Second edition. London: Taylor & Francis, 1992.

For an alternative, but equivalent, construction, define at the center of every tile a random "relief function" $R(p)$ whose values are independent random variables uniformly distributed from 0 to 1. If this relief is flooded up to level p, each cluster stands out as a connected "island." Physicists conjectured, and mathematicians eventually proved, that there exists a "critical probability" denoted by p_C, such that a connected infinite island, or connected infinite conducting cluster, almost surely exists for $p < p_C$, but not for $p > p_C$.

The geometric complexity of percolation clusters at criticality is extreme, and many of the basic new conjectures did not arise from pure thought, but from careful examination of computer-generated clusters of unusually large size.

Open conjecture A. Take an increasingly large lattice and resize it to be a square of unit side. At p_C, the infinite cluster converges weakly to a "limit cluster" that is a fractal curve.

Conjecture B. The fractal dimension of this limit cluster is $91/48$. This value was first obtained numerically, then confirmed by den Nijs, from a partly heuristic "field theoretical" argument that yields characteristic exponents, finally made rigorous by S. Smirnov.

Conjecture C. Figure 3 shows that, depending on the definition of the boundary of a percolation cluster, its fractal dimension is either $4/3$ or $7/4$. These conjectures began with experiments (Grossman and Voss, respectively) and rigorous mathematical proof have been provided by S. Smirnov.

It may be worth mentioning that proofs concerning fractal dimensions have attracted wider interest among mathematicians than the rigorous proofs of previously known facts about percolation.

Open conjecture D. Linear cross-sections of the limit cluster are Lévy dusts, as defined in *FGN*. Experimental evidence is found in Mandelbrot & Stauffer, *J. Physics*: A 28, 1995, L 213 and Hovi et. al. *Phys. Rev. Lett.*: 77, 1996, 877.

Open conjecture E. The limit cluster is a finitely ramified curve in the sense of

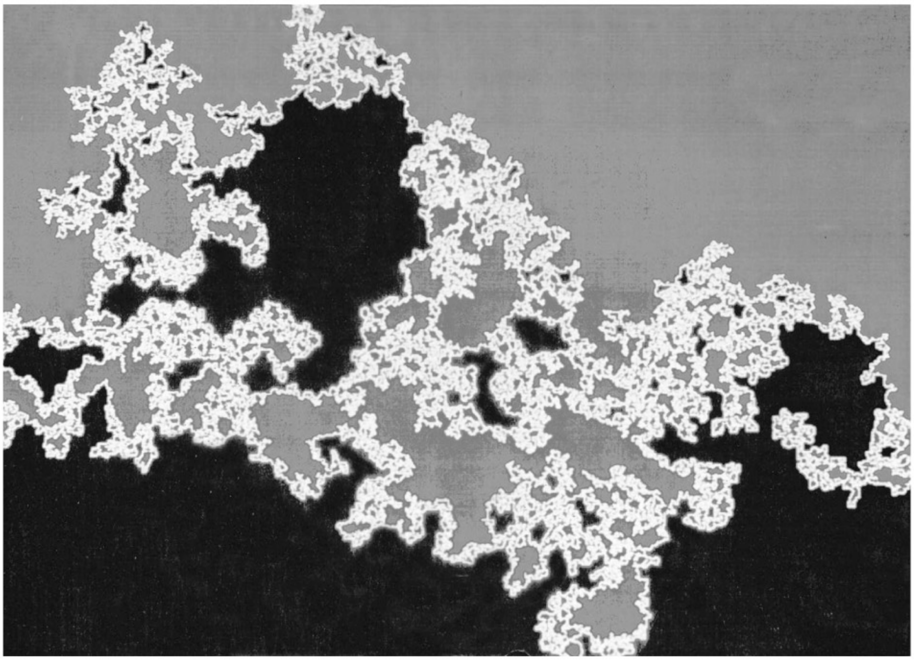

Figure 3. This figure (drawn by Bernard Sapoval for a different goal) helps explain why the critical percolation clusters have two sharply distinct boundaries. One is the curve drawn in white. It is the common boundary of the black and grey areas that it separates. It is very convoluted but without self-contact and its fractal discussion is 7/4. But there are many points where it nearly self-contacts so that it creates "pores," and plugging the pores one defines a "boundary of boundaries" of dimension drastically reduced to 4/3.

Urysohn-Menger.

5.2 The Ising model of magnets at the critical temperature

At each node of a regular lattice, the Ising model places a spin that can face up or down. The spins interact via forces between neighbors left to themselves, these forces create an equilibrium (minimum potential) situation in which all the spins are either up or down. However, a second input is added: the system is in contact with a heat reservoir, and heat tends to invert the spins. When the temperature T exceeds a critical value T_C, heat overwhelms the interaction between neighbors. For $T < T_C$, local interactions between neighbors overwhelm heat and create global structures of greatest interest.

My work touched upon several issues in the shape of the up (or down) clusters at criticality.

Long open implicit question: Beginning with Onsager, it is known that in Euclidean space R^E the necessary and sufficient condition for magnets to exist is that $E > 1$. There are innumerable mathematical differences between the R_E for $E = 1$ and $E > 1$. Identify differences that matter for the existence of magnets.

Partial answer: The specific examples of the Sierpinski curves and of related fractal lattices suggest that magnets can exist when and only when the order of ramification is infinite. *FGN*, p. 139; Gefen et al, *Phys. Rev. Lett.*: 45, 1980, 855).

Conjecture: The above answer is of general validity.

Unanswered challenge. Rephrase the criterion of existence of magnets from the present and highly computational form, to a direct form that would give a chance of proving or disproving the preceding conjecture.

5.3 The ever-mysterious clusters of diffusion-limited aggregation (DLA)

A DLA cluster is generated by allowing an "atom" to perform Brownian motion starting far away until it hits an initial "seed." In Figure 4, the seed is the (opened up) bottom of a half cylinder. When the atom and the seed hit, they are "fused," and a fresh Brownian atom is launched against the enlarged target.

Overwhelming evidence from computer simulations shows that the arrival of many atoms transforms the seed into a cluster that shows about the same high degree of complexity at all scales of observation. Hence any mathematical definition of the concept of fractal must be constrained to include DLA.

The simplicity of the growth rules the DLA and its basic role in understanding many physical phenomena have motivated extensive quantitative studies. However, a full theory even a more informal understanding of the resulting complex structure are lacking. Over many orders of magnitude, the circle of radius R centered on the original and contains a mass $M(R) \approx R^D$ with $D = 1.715$. But there are definite divergences from strict self-similarity — as seen for example in my paper in *Physica* A 191, 1992, 95-107 and my paper with Kol and Aharony in *Phys. Rev. Lett.* 88, 2002, 055501-1-4.

At an early stage, those deviations were thought to be no worse than those relative to critical phenomena. The latter has a well-developed theory, and it was hoped that a theory of DLA could be achieved in the absence of a careful and complete description. This optimistic view is no longer widely held, and a careful description cannot be neglected.

6 Interrelations between fractality and smooth variability: some cases may have a common origin in the usual partial differential equations

6.1 An apparent quandary: are smoothness and fractality doomed to coexist with no interaction?

To establish the presence of fractals in nature and culture was a daunting task to which a large portion of *FGN* is devoted. New and often important examples keep being discovered, but the hardest present challenge is to discover the *cause,* or more probably, the *causes* of fractality.

Some cases are reasonably clear. Thus, in the case of the percolation and Ising clusters in Section 5, fractality is the geometric counterpart of scaling and renormalization, that is, of the fact that the analytic properties of those objects follow a wealth of analytic "power-law relations." Many mathematical issues, some of

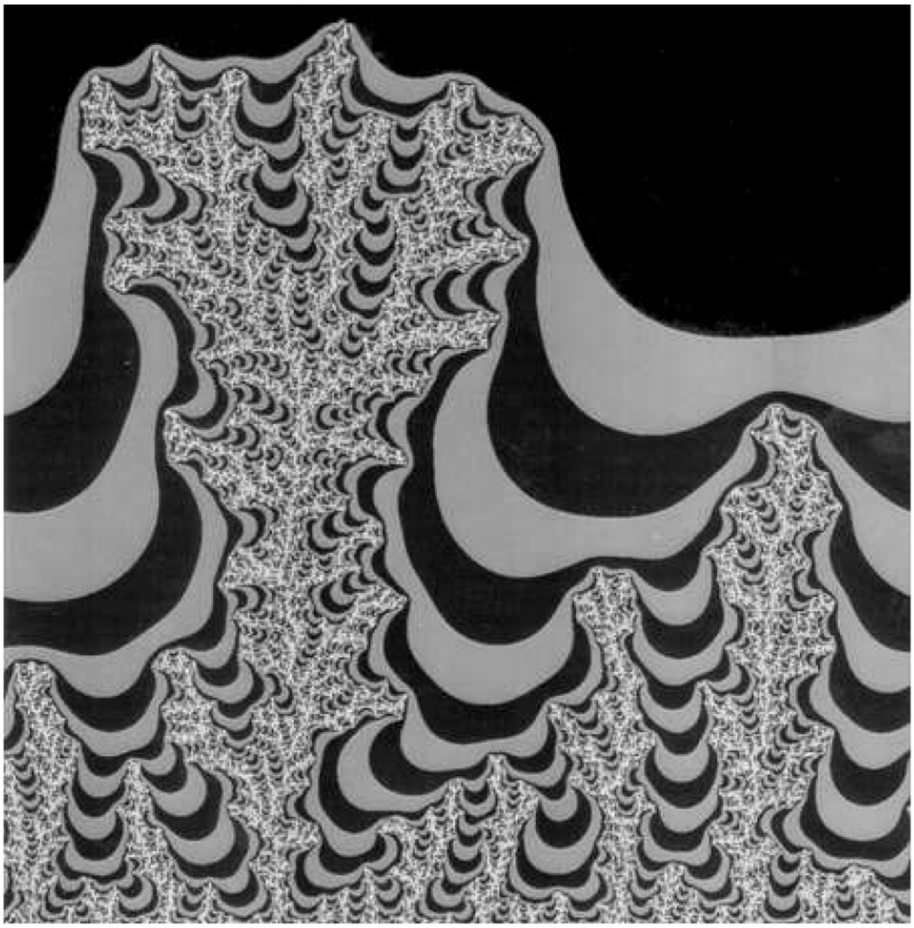

Figure 4. Reproduction of Figure C19-2 of *SC*. A smallish sample of plane DLA, called "cylindrical" because it is grown from the bottom of a half-cylinder (opened up). This DLA is small enough to compute the Laplacian potential and draw its isolines. The latter are a graphic device but also much more: an essential tool of study. A curious visual resemblance is thereby created between DLA and the Mandelbrot set. Of the two, DLA has proven the more resistant to analysis.

them already mentioned, remain open, but the overall renormalization framework is firmly rooted.

Renormalization and the resulting fractality also occur in arguments that involve the attractors and repellers of dynamical systems in a phase space. Best understood is renormalization for quadratic maps. Feigenbaum and others considered the real case. For the complex case, renormalization establishes that the Mandelbrot set (see Section 7) contains infinitely many small copies of itself.

Unfortunately, the usual renormalization fails — even in principle — to account for the diffusion-limited aggregates (DLA) and additional examples of fractality.

Yet another class important occurrences of fractality, to which we now proceed is linked to partial differential equations in real space. It is universally granted that physics is ruled by diverse partial differential equations, PDEs. Those of Laplace, Poisson, and Navier-Stokes will be referred to as "basic." All differential equations imply a great degree of local smoothness, even though closer examination shows isolated singularities or "catastrophes." To the contrary, fractality implies everywhere dense (or at least widespread) roughness and/or fragmentation. This is one of the several reasons why fractal models in diverse fields were initially perceived as being "anomalies" that stand in direct contradiction with one of the firmest foundations of science.

6.2 A conjecture stated and defended in FGN: the solutions of PDEs can be fractal

This is no longer a conjecture, insofar as many specialized PDEs have been solved and found to create fractality. To eliminate the appearance of contradiction between smoothness and fractality, *FGN* conjectured that the same is true of the "basic" equation. This implies that fractals arise unavoidably in the long time behavior of the solution of very familiar and "innocuous"-looking equations. In particular, many concrete situations where fractals are observed involve equations having free and moving boundaries, and/or interfaces, and/or singularities.

As a suggestive "principle," *FGN* (Chapter 11) described the possibility that, under broad conditions that largely remain to be specified, these free boundaries, interfaces and singularities converge to suitable fractals. Among equations examined from this viewpoint, this paper will limit itself to two examples of critical importance. In the case of DLA (Section 5.3), this argument supports self similarity, hence is disappointing, thus far.

6.3 The large scale distribution of galaxies: Newton's law as a possible sufficient generator of fractality

Background. The near universally held view is that the distribution of galaxies is homogeneous, except for local deviations.

In the past, however *FGN*, Chapter 9, Y. Baryshev & P. Teerikorpi, *Discovery of Cosmic Fractals* Singapore: World Scientific 2002) a number of philosophers or science fiction writers have played with the notion that stars (galaxies were not known) follow a hierarchical distribution patterned — long in advance! — along a spatial Cantor set. Those models are excessively regular and necessarily imply that the Universe has a center assuming hierarchies leads to no prediction, that is, implies no property that was not put in beforehand, and raises no new question. For them and other good reasons, hierarchies were dismissed as unrealistic and largely forgotten.

Conjecture that the distribution of galaxies is properly fractal. FGN, Chaps. 9, 33, 34, and 35.) Granted that the distribution of galaxies certainly deviates from homogeneity, existing improvements took two broad approaches.

One consists in correcting for local inhomogeneity by using local "patches."

My next simplest approach acknowledges that one must exclude strict hierarchies as being both physically unrealistic and in conflict with widely held principles.

But I also contend that the specific details of the hierarchical arguments are unimportant. What matters is the underlying fractality, which must be recognized as being of central importance and broad scope. To dismiss fractality with the hierarchies amounted to throwing the baby with the water.

To buttress this belief, I performed detailed mathematical and visual investigations of sample sites generated by two concrete constructions of random fractal sets. The details are given in *FGN.*

The first construction is *The Seeded Universe* that I based on a Lévy flight. Its Hausdorff-dimensional properties were well-known. I observed that its correlation properties (Mandelbrot, *C. R. Acad. Sc. Paris*: 280 A, 1975, 1075) are nearly identical to those of actual galaxy maps. The second construction is *The Parted Universe*, which is obtained by subtracting from space a random collection of overlapping sets, tremas.

In a statistical model, the self-similarity ratio is not restricted to powers of a prescribed r_0. That is, a hierarchical structure is not a deliberate and largely arbitrary input. Quite to the contrary, either of the above constructions yields sets that are highly irregular and involve no special center, yet exhibit a clear-cut clustering that was not a deliberate input. They also exhibit "filaments" and "walls," which could not possibly have been imputed, because I did not know that they had been observed.

Conjecture: could it be that the observed "clusters," "filaments" and "walls," need not be explained separately, but necessarily follow from "scale free" fractality? This would mean that all those structures do not result from unidentified features of specific models but are unavoidable consequences of random fractality — as interpreted by a human brain.

The preceding paragraph is deficient insofar as the word "conjecture" cannot be given a strict mathematical meaning, unless a mathematical meaning is advanced for the remaining terms.

Lacunarity. A problem arose when careful examination of the simulations revealed a clearly incorrect prediction. The original *Seeded Universe* proved to be visually far more lacunar than the real world, in the sense mentioned in Section 4.3. This means that the holes are larger in the simulations than in reality. The *Parted Universe* model fared better, since its lacunarity can be adjusted at will and fitted to the actual distribution.

A lowered lacunarity is expressed by a positive correlation between masses in antipodal directions. Testing this specific conjecture is a challenge for those who analyze the data.

Conjectured mathematical explanation of why one should expect the distribution of galaxies to be fractal. In a cubic box in which opposite sides are identified to form a three-dimensional torus, consider a large array of point masses subjected to Newtonian attraction. The evolution of this array obeys the Laplace equation, with an essential novelty: the singularities of the solution — which are the positions of the points — are movable. The numerous simulations I know of (beginning with those performed at IBM around 1960) all suggest the following. Even when the pattern of the singularities begins by being uniform or Poisson, it gradually creates clusters and a semblance of hierarchy, and appears to tend toward fractality. It is against the preceding background that I conjectured that the limit distribution of galaxies is fractal, and that the origin of fractality lies in Newton's equations.

6.4 The Navier-Stokes and Euler equations of fluid motion and the conjectured fractality of their singularities

Background. The first concrete use of a Cantor dust in real spaces is found in a 1963 paper on noise records by Berger & Mandelbrot (reprinted in *SN*), a work near simultaneous with Kolmogorov's work on the intermittence of turbulence. After numerous experimental tests, designed to create an intuitive feeling for this phenomenon (e.g., listening to turbulent velocity records that were made audible), I extended the fractal viewpoint to turbulence, and was led circa 1964 to the following conjecture.

Conjecture concerning the geometric nature of "turbulently dissipative" parts of spaces. Dissipation should be viewed as occurring, not in domains in a fluid with significant interior points, but in fractal sets. In a first approximation, those sets' intersection with a straight line is a Cantor-like fractal dust having a dimension in the range from 0.5 to 0.6. The corresponding full sets in space should therefore be expected to be fractals with Hausdorff dimension in the range from 2.5 to 2.6.

Actually, Cantor dust and Hausdorff dimension are not the proper notions in the context of viscous fluids, because viscosity necessarily erases the fine detail that is essential to Cantor fractals. Hence the following weaker conjecture.

Conjecture: *FGN*, Chapter 11 and Mandelbrot, *C. R. Acad. Sc. Paris*: 282A, 1976, 119, translated as Chapter N19 of *SN*). The dissipation in a viscous fluid occurs in the neighborhood of a singularity of a nonviscous approximation following Euler's equations, and the motion of a nonviscous fluid acquires singularities that are sets of dimension about 2.5 to 2.6.

Open mathematical problem: To prove or disprove this conjecture, under suitable conditions.

Comment A. Several numerical tests agree with this conjecture (e.g., Chorin, *Commun. Pure and Applied Math.*: 34, 1981, 853).

Comment B. I also conjectured that the Navier-Stokes equations have fractal singularities of much smaller dimension. This conjecture has led to extensive work by V. Scheffer, R. Teman and C. Foias, and many others, but is not exhausted.

Comment C. As is well-known to students of chaos, a few years after my work, fractals in phase space entered the transition from laminar to turbulent flow, through the work of Ruelle and Takens and their followers. The task of unifying the roles of fractals in real and phase spaces is not completed.

7 Iterates of the complex map $z^2 + c$. Julia and Mandelbrot sets

The study of iterates of rational functions of a complex variable is an old topic of pure mathematics that reached a sharp peak circa 1918 with Fatou and Julia. Those authors succeeded so well that — apart from the proof of the existence of Siegel discs — their theory remained largely unchanged for sixty years. A more recent sharp break began in 1980 and has become iconic since most "ordinary" people seem to have heard of the Mandelbrot set: it is arguably the only tangible proof known to them that mathematics is alive and well. The beginnings are now fully documented in *SC* therefore a bare sketch will suffice here.

7.1 The J-set or Julia set

The Julia set is defined as the repeller of rational iteration. For the quadratic map $z \to z^2 + c$, a more direct definition is available: the filled-in Julia set of a given c is the set of points that the map does not iterate to the point as infinity, and the Julia set is the boundary of the filled-in Julia set. With few exceptions, it is fractal: a nonanalytic curve or a "Cantor-like" dust. Julia called these sets "very irregular and complicated." The computer — which I was the first to use systematically — led to beautiful wildly colorful displays that must now be familiar to every reader. To associate forever the name of Fatou and Julia, the complement of the Julia set is best called the Fatou set and its maximal open components, Fatou domains.

Starting with the quadratic map, I explored numerically and graphically how the value of c affects the dynamics and the shape of the Julia set.

7.2 The set M_0 and the Mandelbrot set

The M_0 set. Of greatest interest from the viewpoint of dynamics, hence of physics, is the set M_0 of those values of c for which the map $z \to z^2 + c$ has a finite stable limit cycle. This set having proved to be hard to investigate directly, I moved on to the computer-assisted investigation of a set that was easier to study and seemed closely related.

The M set. The set of those parameter values c in the complex plane, for which the Julia set is connected, was called the μ-map in *FGN* (Chap. 19), but Douady and Hubbard called it *the Mandelbrot set.*

The Mandelbrot set proved to be a most worthy object of study, first for "experimental mathematics" and then for mathematics, and it also gave birth to a new form of art! It is so well and so widely known, that no further reference is needed. But it is good to mention that the M set is a universal object. Curry, Garnett, and Sullivan (*Commun. Math. Phys.*: 91, 1983, 267) discovered that M arises also in Newton's method for cubic polynomials, a dynamical system significantly different from $z \to z^2 + c$. Following this, Douady and Hubbard (*Ann. Sc. Ec. Norm. Sup. (Paris)*: 18, 1985, 287) developed the theory of quadratic-like maps and showed the M set arises for a wide variety of functions, and in this sense is a universal object.

Also, the study of $z \to z^2 + c$ naturally suggested the study of similar questions for other polynomials. But even the generic cubic, $z \to z^3 + az + b$, has proved soberingly difficult. Intense study by extremely powerful mathematicians still leaves many questions unanswered.

7.3 Relations between M_0 and M; the incredibly stubborn conjecture that M is the closure of M_0; "MLC"

Computer graphics approximates M_0 by a smaller set and M by a larger set. Early on, extending the duration of the computation seemed to make the two representations converge to each other. Furthermore, when c is an interior point of M, not too close to the boundary, it was easily checked that a finite limit cycle exists. Those observations led me to conjecture that M is identical to M_0 together with its limits points.

In terms of its being simple and understandable without any special preparation, this conjecture comes close to the "dimension 4/3" conjecture about Brownian motion, discussed in Section 2. Again, I could think of no proof, even of a heuristic one. More significantly, the conjecture remains unanswered.

The MLC conjecture. Many equivalent statements were identified, the best known being that the Mandelbrot set is locally connected. This statement was given a "nickname," MLC. It has the great advantage of being local and was proven for a very large subset of the boundary of M — earning J. C. Yoccoz a Fields medal. But, compared to the original form, MLC has the great drawback of being far from intuitive. (For the generic cubic map, the corresponding local connectivity conjecture was proved to be false.)

8 Limit sets of Kleinian groups

A collection of Möbius transformations of the form $z \to (az+b)/(cz+d)$ defines a group that Poincaré chose to call Kleinian. With few exceptions, their limit sets S are fractal. For the closely related groups based on geometric inversions in a collection C_1, C_2, ..., C_n of circles, there is a well-known algorithm that yields S in the limit. But it converges with excruciating slowness as seen in the top panel of Figure 5. For a century, the challenge to obtain a fast algorithm remained unanswered, but I met it in many cases as seen in the middle panel of Figure 5. For details, see Chapter 18 of *FGN* and *Mathematical Intelligencer*: 5(2), 1983, 9, both reproduced in *SC*.

An interesting contrast. By leading to the 4/3 conjecture, fractal geometry opened a brand new mathematical problem and gave it a very active constituency; but it failed to contribute to solving it. With inversion groups, fractal geometry dealt with *a very old* problem long viewed as so difficult that it had long to have an active constituency. Not only was the problem solved to a significant degree, but it was made, in a literal sense, childishly easy: it is a nice example used in the high school classes examined in this paper's Section 11.

The fast algorithm first described in FGN and illustrated in Figure 6. The limit set of the group of transformations generated by inversions covers the complement of S by a denumerable collection of circles that "osculate" S. The circles' radii decrease rapidly, therefore their union outlines S very efficiently.

When S is a Jordan curve (as on Plate 177 of *FGN*), two collections of osculating circles outline S, respectively, from the inside and the outside. They are closely reminiscent of the collection of osculating triangles that outline Koch's snowflake curve from both sides in a construction that is described in Plate 43 of *FGN* and dates to the 1900s. Because of this analogy, the osculating construction appears, after the fact, to be entirely "natural." But this appearance is thoroughly misleading, as proven by the gap of roughly hundred years that elapsed before it was discovered. It was not obvious at all because of the mood of mathematics: even

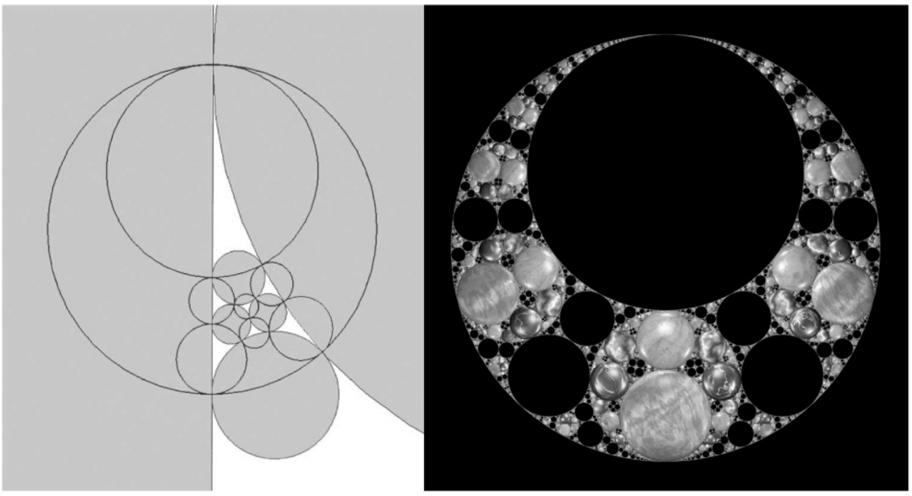

Figure 5. This is Figure C16-2 of *SC*. It elaborates upon Plate 199 of *FGN* and page 129 of *SN*. The "generator" part of the diagram consists in six circles filled in gray. The inversions with respect to those circles, when combined with prescribed probabilities, define a "decomposable dynamical system" also called IFS. The limit set is a self-inverse fractal for which I discovered a new algorithm using the diagram's remaining eight bold circles. The decorative "Pharaoh Breastplate" represents four of those circles and their successive inverses represent by four kinds of "semi-precious stones."

after computer graphics had become available, it continued to scorn pictures. The algorithm did not start to be viewed as natural until it literally burst out after respectful examination of pictures of many special examples.

A particularly striking example is seen in Figure 5, called "Pharaoh's breastplate," a black-and-white rendering of Plate 199 of *FGN*, of the cover of *SN* and of a figure in *SC*. A more elaborate version of this picture appears on the cover of *SN*. This is the limit set of a group generated by inversion in the 6 circles drawn as thin lines on the small accompanying diagram. Here, the basic osculating circles actually belong to the limit set and do not intersect (each is the limit set of a Fuchsian subgroup based on three circles). The other osculating circles follow by all sequences of inversions in the 6 generators, meaning that each osculator generates a "clan" with its own color.

By inspection, it is easy to see this group also has three additional Fuchsian subgroups, each made of four generators and contributing full circles to the limit set.

Pictures such as Figure 5 are not only aesthetically pleasing, but they helped breathe new life into the study of Kleinian groups, recently exemplified by the book by Mumford, Series, & Wright: *Indra's Pearls* (Cambridge University Press, 2002) Thurston's work on hyperbolic geometry and 3-manifolds opens up the possibility for limit sets of Kleinian group actions to play a role in the attempts to classify 3-manifolds. The Hausdorff dimension of these limit sets has been studied by Sullivan Canary and others.

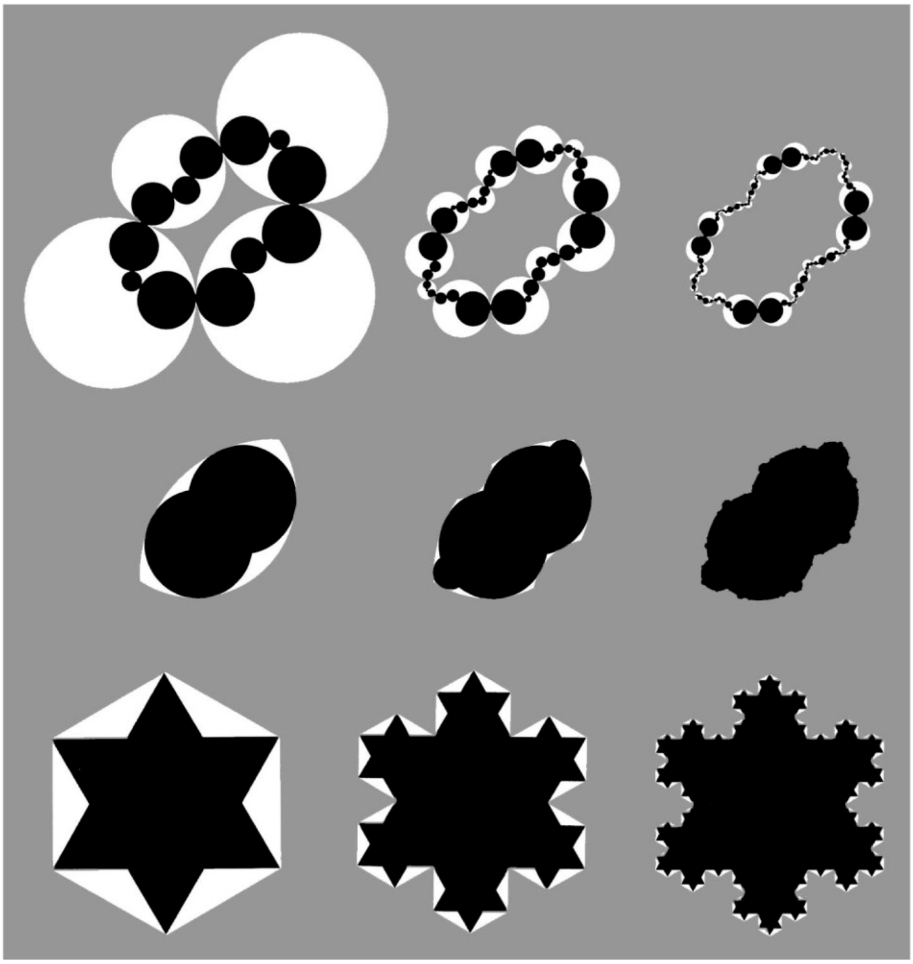

Figure 6. This is Figure C16-1 of *SC* and a composite of page 173 and Plates 177 and 43 of *FGN*. The two top panels represent two constructions of the limit set of a group based on inversions. The top panel shows the slowly converging classical construction (Poincaré). The middle panel shows my fast-converging proposed alternative. The latter recalls the Cesaro construction of the Koch snowflake that is shown in the lower panel.

Challenge. Incorporate lacunarity and multifractal measures into the study of 3-manifolds through these limit sets.

9 The study of power law probability distributions and the notion that variability and randomness can fall into distinct "states," ranging from "mild" to "slow" and "wild"

9.1 The evolution of power-law probability distributions, from a neglected periphery of statistics to a central position in fractal geometry and the topic of active interest on its own

The most widely known analytic tool of fractal geometry consists in power-law relations and power-law probability distributions. They are ancient since Ohmori discovered such a law for earthquake aftershocks in 1894, predating even the Pareto law for the distribution of personal income which was discovered in 1896. Around 1950, however, power laws were viewed by statisticians and scientists alike as scattered anomalous. They were often arbitrarily replaced by the lognormal distribution, or otherwise questioned and played down. When I explained and demystified the Zipf law of word frequencies (*CR (Paris)* 232, 1951, 1638-1640 and 2003-2005), the situation changed completely. To bring power laws together credibly I devoted papers too numerous to be listed. References, reprints, and expositions are found in my *Selecta* books. In many sciences, those papers moved power laws to the forefront, interpreting them as evidence of the broad geometric scaling property of invariance that led to the concept of fractal.

9.2 A basic distinction between "mild" and "wild" "states of variability:" practical aspects

My early investigations of turbulence and price variation arose in the 1960s and used closely related procedures, thus confirming the saying that the Stock Market is as unpredictable and irregular as the weather. The analogy has gone much farther than one may have expected.

It led to general considerations about randomness that converged in Chapter 5 of *SE* to a distinction that may seem philosophical but is in fact very practical. In *principle*, Kolmogorov unified probability theory by providing unquestioned foundations. But in *practice* it is best to consider a function as belonging to one of several distinct "states of variability and/or randomness." Among random variables, iconic examples are the Gaussian distribution for the *mild* state, the power-law distribution for the *wild* state, and the lognormal distribution for a *slow* state in between.

The key underlying fact contradicts a widespread but unfounded belief. The law of large numbers and the central limit theorem are *not* universal truths that one can blindly rely upon in model making. They are special properties that characterize cases of exceptional simplicity that define mild or slow randomness. The contrary is true of all the stochastic processes I used in investigating turbulence, finance and other fractal phenomena. All are examples of "wild" randomness.

This distinction deserves further discussion from the mathematical viewpoint and is bounded to play an active role in physics as well. The power-law long tails and/or dependence that rule all fractal phenomena are clear-cut symptoms of wild variability.

In many cases, the fractal or multifractal models that I put forward have been subjected to counter-claims. Alternative models put forward satisfy all the usual central limit theorems and appear to avoid both the formal mathematical difficulties and the "uncomfortable" consequences of wild randomness. Some of those models do not try hard and are content truncating the fractal models. Central limit behavior is thereby saved but only in an asymptotic sense that is useless because it is not reached in practice. Others proceed more indirectly but amount to the same thing.

9.3 A small purely mathematical aspect of the mild-slow distinction. The boundary between these states provides the classical moment problem of classical analysis with a new wrinkle that originated (of all things!) in finance

This subsection brings us back from finance to the purest mathematical analysis that flowered from stieltjes in the 1890s to the 1930s.

The boundary between the wild and slow states involves the classical central limit theorem, a key idea of probability theory. To the contrary, the boundary between the mild and the slow states is not at all traditional but marked by what I call the *criterion of short-run inequality.* Let $P(x)$ be the tail probability of X and $P_N(x)$, the tail probability of the sum of N independent variables having the same distribution as X. Then, for fixed N and x tending to infinity, *FGN*, Chapter 5 showed the importance of the criterion that $P_N(x)$ behaves like $NP(x)$.

That very simple criterion entered my work in 1960 for very practical reasons. But it turns out to run close to several complicated criteria that occur in the "moments problem" and the theory of quasi-analytic functions. This opened up a very interesting issue: could it be that time has come to study again those old topics once classical but lately very much out of fashion?

10 The variation of financial prices

Historically, my investigation of roughness was comparatively late in turning to physics and mathematics. It began in the early 1960s with investigations in economics that amounted to characterizing the roughness of financial charts. In the 1990s, this work became the foundation of "econophysics." No other application illustrates more vividly the potency of the notion of fractal geometry as the beginning of a science of roughness.

In 1800, Louis Bachelier invented Brownian motion as a model of the variation of financial prices. Even before this model became widely accepted in academia, mine was the first voice to warn against its pitfalls. I pointed out that its two key features are thoroughly unrealistic, hence unacceptable. Having discovered that each involves an empirical power law distribution, I modelled both, first separately (Sections 10.1 and 10.2) and then jointly (Section 10.3), on the basis of the emerging concept of fractality. Under the term "scaling in finance," this concept is the topic of Chapter 38 of *FGN*. Scaling became important in finance before it became important in the physics of criticality.

10.1 The essential importance, even in a first approximation, of large sudden price discontinuities

I was the first to argue that the neglect of discontinuities in the Brownian model is unjustified. They are not "outliers" one can safely disregard or study separately. To the contrary, I argued in 1963 (see *SE*) that their distribution is much more important than that of the "background noise" constituted by near Brownian small changes.

I followed this critique by showing that the big discontinuities and the small "noise" fall on a single power-law distribution and represented them by a scenario based on Levy stable distributions. Howard Taylor and I introduced in 1967 the new notion of intrinsic "trading time." The originality of this work had been recognized all along. In 1964 P.H. Cootner called it "revolutionary." Cootner also raised many questions that have all been answered. Forty years later, the "revolution" is bearing fruit in many diverse ways. Fractal trading time and my 1963 model have gained wide acceptance.

10.2 The fact that the "background noise" of small price changes is of variable "volatility"

That the so-called "price volatility" is itself "volatile" could not be denied but was ordinarily viewed as a symptom of non-stationality that must be studied separately. To the contrary (see *SE*), I interpreted this variability in 1965 as indicating that price changes differ from being statistically independent. In fact, for all practical purposes, their interdependence should be viewed as extending to an infinitely long term. Indeed, it too follows a power-law dependence. In particular, it is not limited to the short term that is studied by Markov processes and more recently ARCH or GARCH and its variants. I followed this critique and illustrated long dependence by introducing a process called fractional Brownian motion which has become very widely used.

10.3 Multifractal models of price variation

I introduced multifractility (minus the term) in 1968 in the context of turbulence (see *SN*). But I immediately observed and pointed out in 1972 (see *SE* and *SN*) that — because it combines long power-law tails and long power-law dependence — multifractility also apply to finance. I also introduced "cartoons" that realize long tails and long dependence and a very simple process understandable to experts and beginners alike.

Fractional Brownian motion in multifractal time, and its use in financial modelling. One half of *SE* is made of reprints of previously published works of mine, but Chapter 6 consists in material never previously published. It advances a new model of variation of prices that is further explored in many publications of mine, in particular, in *Quantitative Finance*: 1,2001, 113-123, 124-130, 427-440, 641-649, and 558-559.

This model represents price as a fractional Brownian motion B_H, that, instead of the clock time, t, is followed in a "trading time," θ. Those two times are related by a multifractal function $\theta(t)$ that is the integral of a random multifractal measure. That is, $P(t) = B_H[\theta(t)]$. At this early stage of the theory, I assumed the functions B_H and θ to be statistically independent. This process is specified by the properties of B_H, primarily its exponent H (a Hölder exponent) and the properties of $\theta(t)$, beginning with its $f(\alpha)$ spectrum. This process was found to fit diverse financial data very well. From most other viewpoints, it is wide open for exploration.

11 The directly useful fractal

Early on, I used to point out a striking contrast: in raw nature smooth shapes are rare exceptions but in manufactured goods they were the near-universal rule. Tables are meant to be horizontal planes with near-linear or circular edges. Walls are meant to be vertical planes.

My early standard of fractality, the Eiffel Tower, was not accepted as counterexample: it remains a masterpiece of engineering but one never meant to be useful. Engineering seemed to be a systematic reaction against the roughness of raw nature.

An invidious claim added to those voiced in Sections 1.4 and 1.6 was that fractals have not contributed to any existing engineering problems.

All this initially led to a question "Did I expect fractals to become practically useful, and, if so, how soon?" I used to recommend patience, recalling the fate of astronomy: while every stage in its development had immediate users who helped support it, all those users were astrologers.

In due time (and with no direct help from me) fractals have indeed become widely useful. Too bad that each real use hits only a specific group of users, so that hardly anyone notices. The following list, very schematic and incomplete, can only touch fields that allow open publication. This excludes finance where what is published may never reflect what is actually used.

Traffic on the internet. Early efforts to squeeze the traffic's extreme variability into the familiar Poisson process soon failed. The multifractal model is now generally acknowledged as being the best and it is the topic of intense study.

Road traffic. The data are less abundant but one often needs multifractality.

Antennas. Stick antennas' properties are easy to analyze mathematically but inadequate and for antennas made of even a few sticks the mathematical analysis rapidly becomes very complicated. The properties of fractal antennas are both far better and easily calculated.

Capacitors. To achieve one farad, flat capacitators need a very large area and a little folding makes the mathematical analysis very complicated. Fractally folded one-farad capacitors are easy to calculate and fit on a pinhead.

Sound-absorbing road barriers. Houses close to roads want to be protected from traffic noise. Early on, flat protection panels simply reflected noise. Incoming panels with a fractal pattern are noise-absorbent.

Chemical engineering. When two gases are meant to react, it is best to control the surface of reaction. This is achieved by bringing one reactant in the midst of the other with the help of a spatial tree. The reaction is faster and cleaner with fewer impurities.

12 Fractals in the college and school classroom

Several examples in this paper share a very nice feature that is also very unusual. Among fields of research, fractal geometry may well exhibit some of the shortest distances and the greatest contrasts between a straightforward core and multiple new frontiers. The latter are filled with major difficulties of every kind, including conjectures that everybody can understand but no one can prove.

Starting with *FGN*, the core has by now become widely known, even to children and adult amateurs. This has opened a wonderful new opportunity that deserves brief mention all by itself.

At issue is the abyss between mathematics and a wider community. Its story is old but in the 1960s and 1970s the "new math" fad made it deeper. I think that no one benefits from this abyss, yet some continue to welcome it, and many more can think of no suitable bridge and view the abyss as inevitable. Therefore, it persists. M.L. Frame and I have convinced the Mathematics Department at Yale University that, in fact, a strong bridge can be based on fractal geometry.

The upshot: for the last several years, Yale has been offering an undergraduate course and associated summer workshops that teach fractals to several groups of non-mathematicians. Their attractiveness to students depends heavily on three assets.

One is the already mentioned unusually short distance from the simple to the complex and even the impossibly difficult. To the contrary, from the viewpoint of mathematics education, one of the worst features of most topics is that prerequisites are interminable. They are unavoidable but respond to needs that do not become compelling until the ends of long paths that allow many opportunities to drop out.

A second asset is that the history of fractals reaches back for several millennia, proving that fractality is "natural" to the culture of our species.

A third asset is, of course, that the ubiquity of roughness translates into a large number and variety of current applications of every kind in the works of Nature and Man.

Fourth asset: as a very valuable by-product, our course teaches the meaning of rigor by the most efficient method: when a program is buggy, the computer immediately screams Error! at the programmer.

The book *FM* explains and motivates our course and reproduces stories from several colleagues who work along the same lines. It also refers to two items on the web: an extensive set *course-notes* and a *Panorama* that collects innumerable examples of fractality. Everyone is invited to add to this collection!

Acknowledgements

Conversations with Professor Michael Frame of Yale University helped greatly in the preparation of this text. Professor Kenneth Monks of the University of Scranton drew Figure 3 using the algorithm described in *FGN*.

Thinking in Patterns: Fractals and Related Phenomena in Nature.
(Fractal 2004, Vancouver CN).
Edited by Miroslav Novak.
Singapore: World Scientific, 2004, 1-33.

SELECTED TOPICS IN MATHEMATICS, PHYSICS, AND FINANCE ORIGINATING IN FRACTAL GEOMETRY

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The bulk of this text consists in nonsystematic sketches of the current status of diverse very difficult questions in various mathematical sciences. All were triggered by actual fractal pictures generated by computer. In physics some of those questions outline a nascent "rational rugometry," involving quantitative measures of roughness. Other questions concern diverse clusters and turbulence. In mathematics, some of those questions have been settled — one of them, the 4/3 conjecture, in 2000. Other questions, however, including a basic property of the Mandelbrot set, resist repeated efforts to answer them. In finance, Mandelbrot's models starting in 1963 became the foundation of "econophysics." In all cases, many questions on the research frontier — solved or not — can be understood by a good secondary-school student, which is why fractal geometry is increasingly affecting high school teaching. All those questions involve in essential fashion some shapes long called "monsters" and guaranteed to belong to esoteric mathematics lacking any contact with the real world. Fractal geometry reveals them as being extremely "natural" and also as having been familiar to artists since time immemorial.

1 Introductory comments of various kinds

1.1 Presentation

Fractal geometry ranges over many parts of the mathematical sciences but the questions sketched in this text mostly belong to either pure mathematics or the interfaces between mathematics and physics. Specific sections or subsections are free-standing and do not require acquaintance with one another or with fractal geometry as a whole.

The paper may also interest those already familiar with fractal geometry because it includes recent developments and/or because many of my opinions have either evolved or become more focussed. Hence — even though the overall tone is by no means introductory — it is appropriate to begin with several separate introductory remarks concerned, first, with science, then with mathematics.

1.2 Dilation invariance and a reinterpretation of fractal geometry, as the first step towards a "quantitative rational rugometry"

A basic issue must be touched first: what is fractal geometry today? Largely after the fact, it is best characterized as being the first systematic and quantitative approach to the study of roughness — in both in pure mathematics and in mathematical sciences of the "real world." The latter includes nature (turbulence, clusters of statistical physics, broken solids, noises, galaxy distributions, geomorphology) and "culture," that is, the works of Man (finance, spoken discourse, the internet, and even art).

Roughness is, of course, ubiquitous in the real world and has long been counted among the basic "sensation" of Man. However, its study lagged; even finding a