On the Dynamics of Iterated Maps VIII:
The Map $z \to \lambda(z+1/z)$, from Linear to Planar Chaos,
and the Measurement of Chaos

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0. Introduction

While the terms "chaos" and "order in chaos" prove extremely valuable, they elude definition and it remains important to single out instances when the progress to planar chaos can be followed in detailed and objective fashion. This paper proposes to show that an excellent such example is provided by the iterates of the map $z \to g(z) = \lambda(z+1/z)$, when z and λ are both complex. This map is touched upon in [1], but only on page 465, which was added in second printing. Therefore, the present paper is self-contained.

The map g(z) was singled out because of its valuable properties. A) Within a broad domain of λs, there are two distinct limit cycles, symmetric of each other with respect to z = 0. B) Suitable changes in λ cause both cycles to bifurcate simultaneously into n>2 times larger cycles. C) The chaos which prevails for certain λ extends over the whole z plane. These features A), B) and C) all fail to hold for the complex map $z \to f^*(z) = z^2 - \mu$ (e.g. [1], [2], [3], [4]). Indeed, for every μ, one of the limit cycles of f* reduces to the point at infinity; this point never bifurcates; and chaos, when it occurs, consists in motion over a small subset of the z-plane.

1. Summary. Relativity of the Notion of Chaos

The bulk of this paper consists in explanations for a series of figures that illustrate, for diverse λ, the shape of the Julia set $\mathscr{F}^*$, that is, of boundary of the open domains of attraction of the stable limit points and cycles. Different sequences of figures follow different "scenarios" of variation in λ, and yield maps that transform gradually from linear chaos and planar order, to either questionable or unquestioned planar chaos.

In order to put these illustrations in perspective, the paper includes comparisons with the polynomial maps. To begin with, the special map $z \to z^2 - 2$ restricted to the real interval [−2,2] is called thoroughly chaotic. However, the very same map generalized to the complex plane should be called almost completely orderly, since all z_0 that are not in the real interval [−2,2] iterate to ∞. As is well-known [5], there are many other μ's for which

the maps $z \to z^2 - \mu$ are chaotic on a suitable real interval. But the very same maps are *least* chaotic in the plane, in the sense that the domain of exceptional z_0 that fail to iterate to ∞ is smaller for a chaotic μ_0 than for any of the nonchaotic μ that can be found arbitrarily close to μ_0.

Thus, there is a clear need for an objective measure of the progress towards chaos. An obvious candidate for measuring orderliness is the fractal dimension D of the Julia set $\mathscr{F}^*$. This paper finds that D is indeed appropriate for some scenarios, but raises very interesting complications for other scenarios, when $\mathscr{F}^*$ involves more than one shape, hence more than one dimension.

The best known scenario, pioneered by J. Myrberg and very well explored in many contexts [5], proceeds from linear to planar chaos by an infinite series of finite bifurcations of arbitrary order. When this scenario is applied to g(z) (Section 5), $\mathscr{F}^*$ remains a fractal curve whose D grows from 1 in to 2, hence its codimension 2-D is indeed an acceptable measure of orderliness.

In one alternative scenario, which can be credited to C. L. Siegel, D also varies steadily from 1 to 2, but intuition tells us that the limit is very incompletely chaotic in the plane. The key of this paradox is that the corresponding $\mathscr{F}^*$ involves two different shapes, hence two distinct dimensions.

In the third scenario to be examined, planar chaos is approached without bifurcation, and D tends to 2.

2. When $\lfloor \lambda \rfloor > 1$ or λ are real, Iteration is Orderly Except on $\mathscr{F}^*$

For $\lambda = 0$, all points other than 0 and ∞ move in one step to 0, henceforth the motion is indeterminate. For $\lfloor \lambda \rfloor > 1$, there is an attractive fixed point at ∞, which contradicts our requirement A).

For real $\lambda > 0$, the map g(z) preserves the sign of Re(z), and for real &la<0, the iterated map $g_2(z)$ preserves Re(z). More generally the Julia set is the imaginary axis for all real $\lambda \neq 0$.

3. Non-real λ's that Satisfy $\lfloor \lambda \rfloor < 1$

For these λ, the Julia set $\mathscr{F}^*$ is either the whole complex plane or a fractal curve. In the latter case, $\mathscr{F}^*$ has the following properties.

$\mathscr{F}^*$ is (obviously) symmetric with respect to $z = 0$, and is self-inverse with respect to the circle $|z| = 1$.

$\mathscr{F}^*$ includes $z = 0$ and is unbounded. This is obvious when the fixed points $z = \pm\sqrt{\lambda}/\sqrt{1-\lambda}$ are stable: if z_0 iterates to one of the fixed points, $-z_0$

iterates to the other fixed point, hence the circle of radius $\mod(z_0)$ must intersect $\mathscr{F}^*$. (The origin $z = 0$ must be added because $\mathscr{F}^*$ is a closed set.)

$\mathscr{F}^*$ is asymptotically self-similar for $z\to\infty$. Indeed, if $|z| \gg 1$ and z_0 iterates into some cycle, $\lambda(z+1/z)\sim\lambda z$ iterates into the same cycle. Since $\mathscr{F}^*$ is self-inverse, it also follows that $\mathscr{F}^*$ is asymptotically self-similar for $z\to 0$. When $\mathscr{F}^*$ is topologically a line, it winds around a logarithmic spiral for $z\to\infty$ and for $z\to 0$. These spirals wind in the same direction, but *are not* the continuation of each other, because scale invariance fails near $|z| = 1$.

We wish to start with a real λ for which $\mathscr{F}^*$ is a straight line, and then to change λ and follow $\mathscr{F}^*$ as it changes from a straight line to an increasingly wiggly curve. This requires drawing the semi-open variant of the $\mathscr{M}$-set ("Mandelbrot set") as defined in MANDELBROT [2].
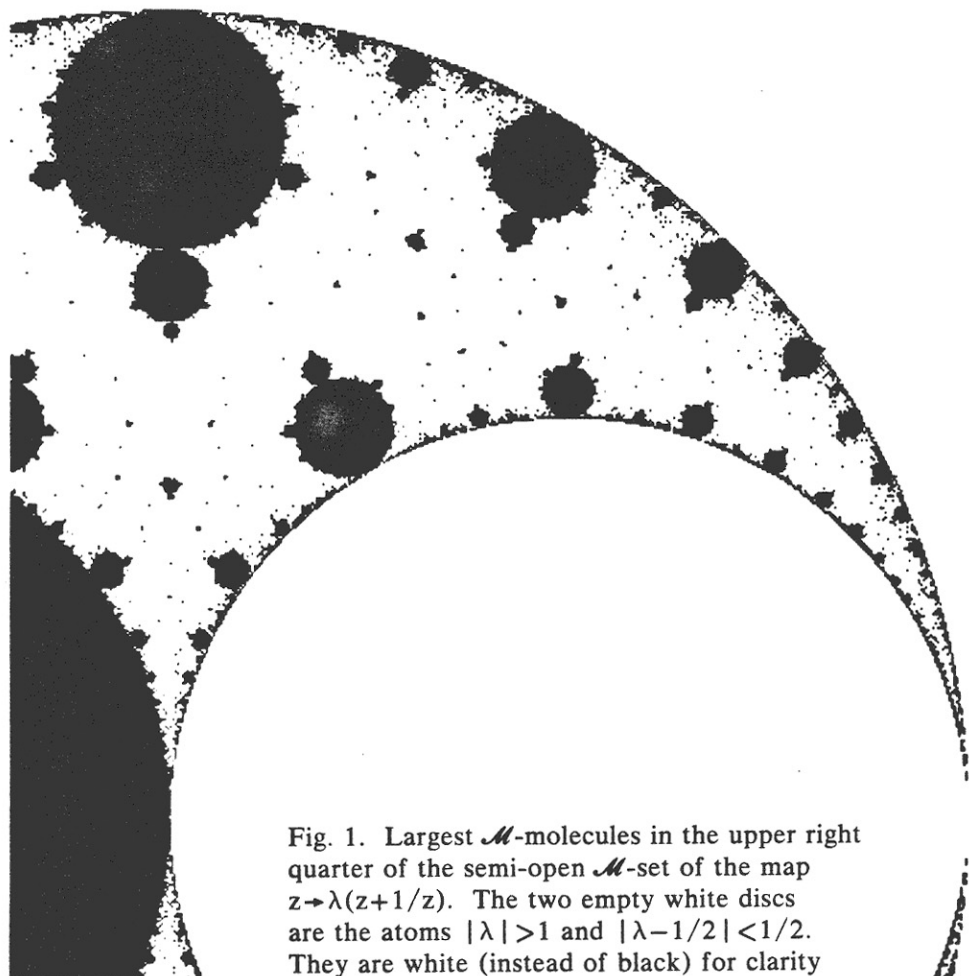

Fig. 1. Largest $\mathscr{M}$-molecules in the upper right quarter of the semi-open $\mathscr{M}$-set of the map $z\to\lambda(z+1/z)$. The two empty white discs are the atoms $|\lambda| > 1$ and $|\lambda-1/2| < 1/2$. They are white (instead of black) for clarity

The semi-open $\mathcal{M}$-set is the maximal set of λ's, such that the iteration of the map has a finite limit cycle. Its closure of is the ordinary $\mathcal{M}$-set. The semi-open $\mathcal{M}$-set of $z \to \lambda(z+1/z)$ is shown on Fig. 1 (reproducing Plate x of [1], second and later printings. Note that p. 465, which explains Fig. x of [1] omits to say it is the semi-open $\mathcal{M}$-set, and replaces z by iz.)

Inspection shows that the semi-open $\mathcal{M}$-set is made of $\mathcal{M}$-molecules, each made of $\mathcal{M}$-atoms, both shapes being the same in the case of the map $\lambda(z+1/z)$ and in the deeply studied case of the maps $z^2-\mu$ [2, 3, 4].

The present study is concerned with three different scenarios that start from the extreme order represented by λ's in the real interval $]0,1[$—hence a Julia set identified with the imaginary axis—and end in planar chaos. We focus on the $\mathcal{M}$-molecule that includes the disc-shaped $\mathcal{M}$-atom $|\lambda-1/2| < 1/2$. It is easy to see that this atom collects all λ's for which the iteration of f(z) has 2 limit points, $z = \pm\sqrt{\lambda}/\sqrt{1-\lambda}$.

4. From a Flat Sea to a Great Wave: the Computer's Homage to Katsushika Hokusai (1760-1849) (Fig. 2)

For all λ in the disc $|\lambda-1/2| < 1/2$, the Julia $\mathcal{F}^*$ is topologically a straight line that winds for $z \to 0$ or $z \to \infty$ around logarithmic spirals symmetric of each other with respect to 0. The spirals are both nicest and most educational when they are neither too loose not too tight. Let us therefore scatter a few parameter values, well within the $\mathcal{M}$-atom $|\lambda-1/2| < 1/2$, between $\lambda = 1/2$ and the neighborhood of $1/2+i/2$. To deemphasize the non-spiral complications near $|z| = 1$, the window (portion of the complex plane that is shown) is 200 units wide, and the $\mathcal{F}^*$-sets are rotated to become easier to compare. The $\mathcal{F}^*$-sets show as the boundaries between black "water" and white "air", which are the domains of attraction of two limit points. As intended, the first part of Fig. 2 evokes a completely flat black sea, hence planar order. And the figures that follow counter-clockwise evoke increasingly threatening black waves.

In parallel, the fractal dimension D of $\mathcal{F}^*$ increases. In this context, D tells how many decimals of z_0, in the counting base b, are needed to know whether z_0 is attracted to $\sqrt{\lambda}/\sqrt{1-\lambda}$, or to $-\sqrt{\lambda}/\sqrt{1-\lambda}$. To establish this fact, draw on our window a collection of boxes of relative side $r_1 = 1/b$. Roughly b^D of these boxes intersect $\mathcal{F}^*$. Write $\beta = b^{D-2}$, and choose $= x_0+iy_0$ at random in the box. With the probability $1-\beta$, the first b-decimals of x_0 and y_0 suffice to determine where z_0 is attracted. More generally, the first k b-decimals of x_0 and y_0 suffice with the probability $(1-\beta)\beta^{k-1} \propto b^{k(D-2)} \propto r_k^{2-D}$. On the average, the number of b-decimals needed to determine the limit is $1/(1-\beta)$. When 2-D is small, the expected number of base e "decimals" needed to determine the limit is $\log_e b/(1-\beta) \sim 1/(2\text{-D})$.

Fig. 2. Examined counterclockwise from
here, Julia $\mathscr{F}^*$ sets of $z \rightarrow \lambda(z+1/z)$
for several selected values of λ.
"From a Flat Sea to a Great Wave".
Homage to Katsushika Hokusai
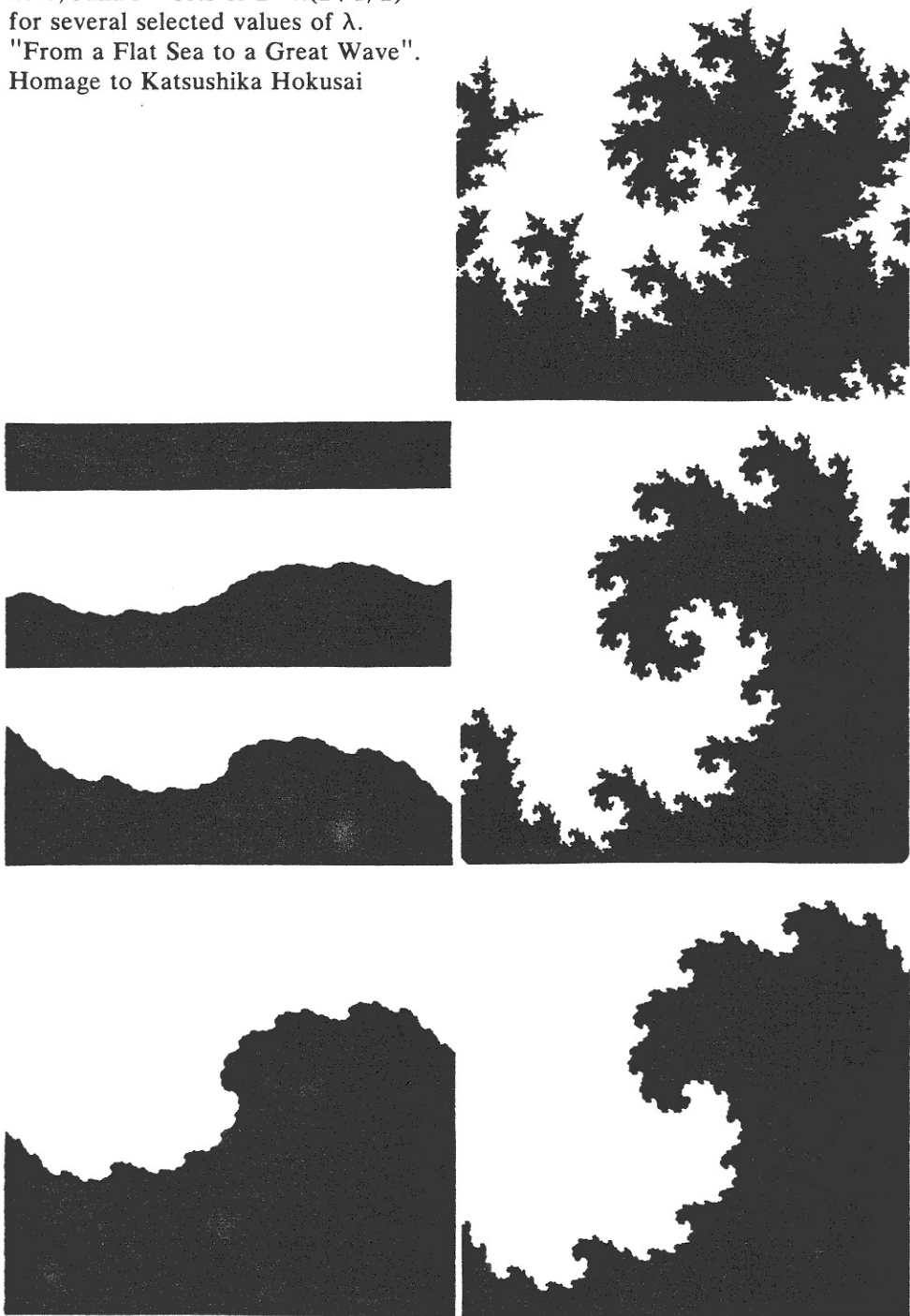
5. First Path Beyond the Great Wave. The Myrberg Scenario of Bifurcations

Figure 3 represents the $\mathscr{F}^*$-sets for two values of λ. In the top graph, λ lies past a bifurcation into 4, close to (but short of) a second bifurcation into 3. In the bottom graph, λ is reached by two successive bifurcations into 4, followed by a bifurcation into 3. Thus, the first λ lies off the center in an $\mathscr{M}$-atom off the atom $|\lambda - 1/2| < 1/2$. And the second λ lies near the nucleus in a small $\mathscr{M}$-atom off a small $\mathscr{M}$-atom off a small $\mathscr{M}$-atom attached to the $\mathscr{M}$-atom $|\lambda - 1/2| < 1/2$. (A fourth λ is seen p. viii of [1], second printing.)

The first bifurcation forms "white water" through the breakdown of connected water and connected air into larger drops, some of them quite large. The bifurcations that follow break these drops into smaller ones, without end. It is clear that one watches a gradual progression towards the ultimate replacement of separate black water and white air by something that is neither water nor air. One cannot help evoking the critical temperature of physics.

The fractal dimension D of $\mathscr{F}^*$ tends toward 2 as planar chaos is approached, and the factor β tends to 1.

## 6. Second Path Beyond the Great Wave. A Scenario of Spiraling Towards Chaos (Fig. 4)

Now select λ to be within the atom $|\lambda - 1/2| < 1/2$ but extremely close to $\lambda = 1$. The $\mathscr{F}^*$ set is illustrated by Fig. 4. It is clear that, as $\lambda \to 1$, $D \to 2$, hence $\beta \to 1$, and that *chaos is approached without bifurcation*. The facts are perhaps easier to visualize in terms of the parameter $\mu = 1/\lambda$ and the variable $u = 1/z$. This change of variable does not change $\mathscr{F}^*$. For $|\mu| < 1$, there is one limit point at $u = 0$. As μ crosses 1, this limit point bifurcates into two limit points that coexist in a chaotic situation.

7. Third Path Beyond the Great Wave. The Siegel Scenario

Figure 5 represents the $\mathscr{F}^*$-set for a value of λ within the atom $|\lambda - 1/2| < 1/2$, but very close to a point on boundary, namely $\lambda_S = 1/2 + (1/2)\exp(2\pi i\gamma)$, where γ is the irrational number (<1) whose continued fraction expansion is $(4,1,1,1,...$ ad infinitum).

This path is interesting because its limit is hard to label as chaotic or non-chaotic. One can understand the difficulty and the opportunity by viewing the boundary between colors on Fig. 5, first, as an approximate $\mathscr{F}^*$ set for $\lambda \equiv \lambda_S$ and, next, as an approximate $\mathscr{F}^*$ set for λ just short of λ_S on the way from $\lambda = 1/2$. The same figure can serve two purposes because the differences between the corresponding exact figures are erased by the inevitable limitations of actual computation. Furthermore, the roughly circular white

Fig. 3. Julia sets for two λ that yield near totally chaotic maps g(z), along the Myrberg scenario of repeated bifurcations
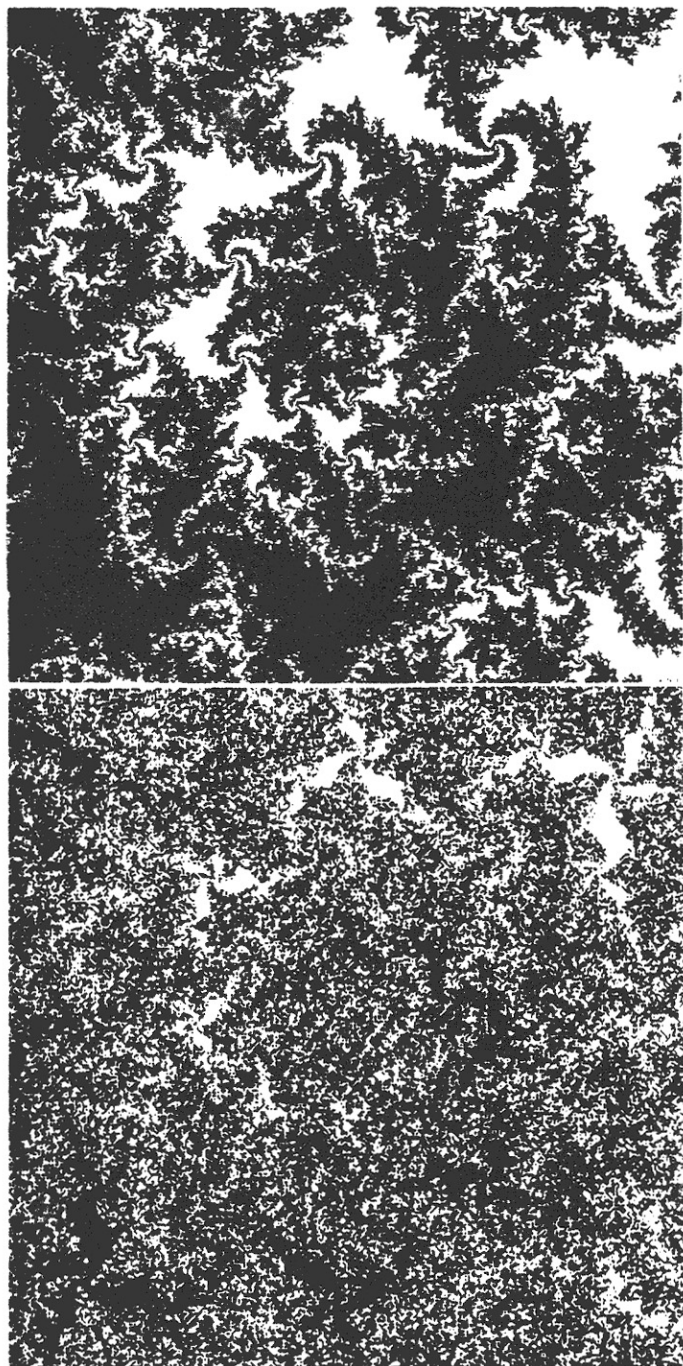
spots do not contribute to $\mathcal{F}^*$, because they too are computation artifacts: the values of z_0 that had failed to converge to either limit point after 3000.

The behavior for $\lambda \equiv \lambda_S$ is known from a theory due to SIEGEL [6]. Focus on the "cracks" that seem to separate the black wave into roughly circular black discs, and imagine that these cracks converge and join. It follows that water—and air also, by symmetry—becomes separated into discs attached to each other by single punctual bonds. Two of the discs include the points $\pm\sqrt{\lambda}/\sqrt{1-\lambda}$ and are called Siegel discs; let the remaining discs be called Siegel pre-discs. At each inter-disc bond, air and water cross each other but over most of the plane they are clearly separated by $\mathcal{F}^*$. Not unexpectedly, the fractal dimension of $\mathcal{F}^*$ takes a value of D_S that is unquestionably less than 2. One needs on the average $\sim 1/(2-D_S)$ decimals to determine whether a point z_0 is black or white. Incidentally, there is no limit point or limit cycle, but the iterates of the z_0's in the white (black) Siegel pre-discs end up in the white (black) Siegel disc. On the scale of the 200-wide window of Fig. 3, the Siegel discs are so small that the Siegel regime looks like convergence.

Next, in order to achieve an idea of how $\mathcal{F}^*$ looks for λ just short of λ_S, it is necessary to know that Siegel discs are created when a curve $\mathcal{F}^*$ that is topologically a line folds up and becomes domain- or plane-filling, as described and illustrated in [4, Paper VII]. When λ is just short of λ_S, the cracks invoked in the preceding paragraph have not converged and joined. Instead, the interior of each of the black discs is partly split by many (here, 157) very narrow "fjords", that penetrate deep into the white domains, without quite meeting, but coming close to meeting near the center of a spurious white spot. In symmetric fashion, one must visualize black fjords thrusting into the white domain.

Since the boundaries of both white and black fjords are part of $\mathcal{F}^*$, the curve $\mathcal{F}^*$ is very close to filling the whole plane. Its dimension being arbitrarily close to 2 tempts us to conclude that the corresponding map $g(z)$ is completely chaotic. But it is not. In fact, the Siegel scenario reveals an important and subtle point: We need a close look at the factor β. For very tiny values of the cell side r_k, we find $\beta \propto r_k^{2-D}$. However, as long as $r_k > \xi$, with ξ a function of 2-D, we find $\beta \equiv 1$. Thus, the smallness of 2-D expresses that *every* cell of side $> \xi$ will be intersected by $\mathcal{F}^*$. But this does not say anything about the relative proportions of black and white in the cells $> \xi$. In the present case, the fjords are so narrow that a cell $> \xi$ is mostly black or mostly white, depending on whether it is in a domain that Fig. 5 shows as solid black or solid white. The expected number of decimals of base b depends upon whether one wants to know the color of z_0 precisely, or with high probability. Absolute precision requires $1/[1-b^{D-2}]$ decimals, high probability requires only $1/[1-b^{D_S-2}]$.

Fig. 4. Julia set for a λ that yields a near totally chaotic map and is attained by yet another scenario. Topologically, this curve is a straight line
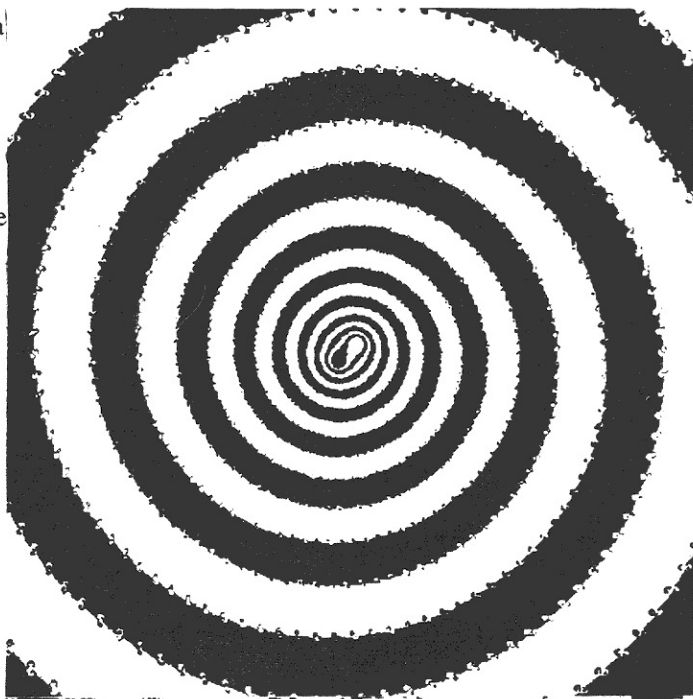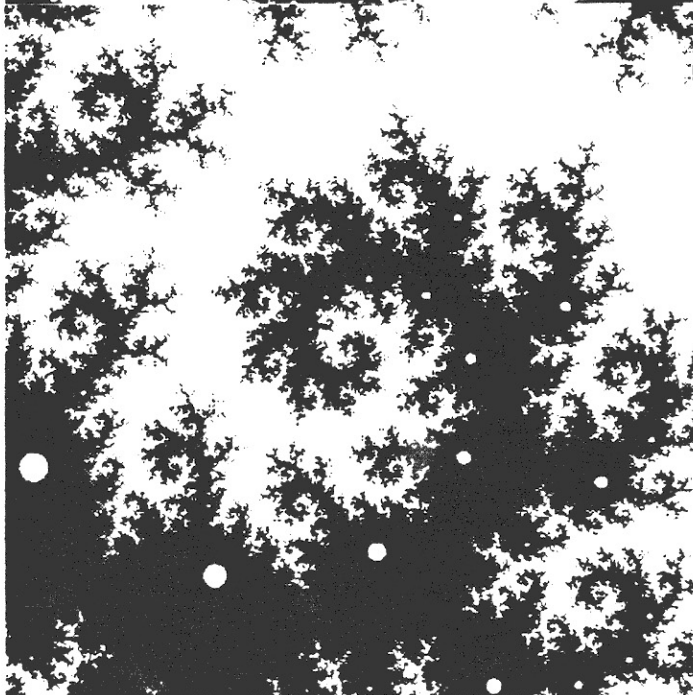

Fig. 5. Julia Set for λ that yields a questionably chaotic map g(z), and is attained by the Siegel scenario

In other words, the overall appearance that computer limitations give to Fig. 5 is not misleading at all. In fact, it helps reveal a basic truth. When λ is very near λ_S, the shape of $\mathcal{F}^*$ is ruled by *two* distinct dimensions: its own and that of the $\mathcal{F}^*$ corresponding to the nearest Siegel value of λ.

We must agree that near-complete planar chaos should require that all small cells a) intersect $\mathcal{F}^*$ and b) be about half black and half white. Under these conditions, a nearly space-filling $\mathcal{F}^*$-set of dimension nearly 2 is *not* sufficient for complete chaos. The presentation of further results on this topic must be postponed to a later occasion.

Acknowledgement

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