KINETICS OF AGGREGATION AND GELATION
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ON THE AGGREGATIVE FRACTALS CALLED "SQUIGS", WHICH INCLUDE RECURSIVE MODELS OF POLYMERS AND OF PERCOLATION CLUSTERS

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Aggregation processes yield random fractals in non-recursive fashion. Polymers and percolation clusters are also constructed non-recursively. For various purposes, it is useful to imitate and illustrate each of the resulting fractals in a manner that is both recursive and random. The present note introduces an especially convenient class of such illustrations, using the vocabulary of aggregation, and serves as extended legend to several illustrations.

1. INTRODUCTION

A very versatile recursive and random illustration of various forms of fractal aggregation is provided by a class of fractals which I introduced in 1978 and which has recently begun to attract attention (1, 2, 3, Chapter 24, 4, 5, 6). The word "squig" I later gave to these shapes refers to their "squiggly" appearance. Among the thoroughly studied examples of squigs, the simplest are called fractal "intervals", a more precise synonym of "non-branched fractals". (The idea is that a generalized interval is a shape that can be put in continuous one-to-one correspondence with an Euclidean interval.) The squig intervals may provide a model for non-branched polymers. The next simplest are fractal trees; they were originally meant to be plane-filling and to model river networks, but they can readily be modified to model non-plane-filling branched polymers. The latest thoroughly studied squig is a fractal cluster. An investment in the study of squigs (analogous to the study of the Sierpinski gasket) is likely to be of high yield. While their constructions are fully described elsewhere, it is appropriate to restate them here using the vocabulary of aggregation.

The process that generates a squig always begins with a lattice, whose cells collect in super-cells of size b^k, where b (the base) and k are integers. The lattice tiles may be bounded either by broken lines or by fractals. The construction is recursive, which allows renormalization group arguments to be carried out in mathematically rigorous fashion.

2. SQUIG TREES AND INTERVALS, AND MODELING OF POLYMERS (1, 2, 3)

When the lattice is triangular and b = 2, squigs are drawn on the "dual" hexagonal lattice of "potential bonds", obtained by linking the centers of the "neighboring cells" in the original lattice, that is, of cells that share a side. Some bonds are then deleted at random, in recursive fashion, using one of two distinct processes: "decimation" and "separation". The remaining bonds are called "activated". The construction is tractable because the processes of decimation and separation are made to be statistically independent.

The first construction stage takes 4 triangular lattice cells C(0) of linear size 1 that fit into a triangle C(1) of linear size 2, and activates all the 4 bonds between neighboring C(0)'s. The result is a small Y-shaped tree. The second construction stage takes 4 copies of C(1) that fit together in a triangle C(2) of linear size 4. The boundary between any two neighboring C(1) within one C(2) is crossed by 2 potential bonds; one of them is activated and one is decimated. The k-th stage takes 4 statistically independent replicas of C(k−1) to make a triangle C(k) of linear size 2^k. The boundary between any two neighboring C(k−1) is crossed by 2^{k-1} potential bonds; one of them is activated, and the other $2^{k-1}-1$ are decimated. One can think of the above process as a recursive aggregation of increasingly large aggregates: at each stage, only one of the many links between two aggregates is chosen at random to be "activated".

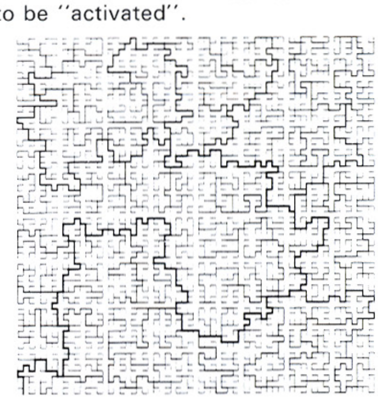

FIGURE 1: SQUIG TREE AND INTERVALS

The final outcome is a connected fractal, a "squig tree", that can be said to fill the whole plane, hence is of dimensionality $D = 2$. An example of such a tree is found in (3) page 227. To avoid duplication, Figure 1 gives a different example, which is relative to a square lattice and involves not only decimation but a form of the "separation" introduced in Section 3.

The shortest path between two points in a fractal tree is called a "fractal interval". An alternative interpolative construction is illustrated in the color Figure C1. Here, the fractal dimensionality is $D = \log_2 2.5 = 1.3219$, and other squig intervals (constructed on different lattices) invariably yield D's close to 4/3.

As this readership knows well, the fractal dimensionality $D = 4/3$ also characterizes the self-avoiding random walk and the non-branching polymers in the plane. The latter may be usefully imitated and illustrated by squig intervals.

3. SQUIG CLUSTERS AND MODELING OF PERCOLATION (4, 5, 6)

The process that generates squig clusters begins with the Sierpinski carpet of base $b = 3$. The carpet is obtained by dividing a square into 9 smaller squares then deleting the middle one, and continuing recursively. The limit's fractal dimensionality is $\log_3 8 \sim 1.8928$, a very acceptable value for percolation clusters in the plane; unfortunately, the carpet is not at all suitable as a model of percolation clusters since—unlike percolation clusters—it has no dangling bonds and is infinitely ramified (3, Chapter 14). The idea of the squig construction is to leave the carpet's dimensionality unchanged, while either bonds or sites are deleted recursively.

Again, the construction works with the carpet's "dual". After midsquares of every order have been removed, a finite approximation to the carpet is a collection of squares, each bounded by four of the usual bonds. The dual sites are centers of these squares, and the dual bonds join the centers of squares that share a side. The approximate dual carpet is the sum of 8 subcarpets, each linked to two neighbors by very many bonds (which is why the carpet is infinitely ramified). To achieve finite ramification, my squig construction "decimates" these bonds, in the sense that it deletes all but one, then continues recursively. The nondeleted bond may be either the central one (half-random version), or selected at random (random version).

Next, the squig construction creates dangling bonds via a different rule, "separation". The already decimated carpet is made of 8 subcarpets, plus 8 bonds linking neighboring subcarpets. To perform random bond separation is to delete a non-decimated bond with prescribed probability s, which is the only adjustable parameter in the model. One proceeds in the same way with each part. The case $s = 1$ yields a tree. The construction is illustrated in (5) Figures 3 and 4. Further information is provided by the color Figure C2.

By design the cluster's topology is finitely ramified and with dangling bonds, but the overall dimensionality remains $\log_3 8$. Furthermore, the dimensionalities of the parts of the squig clusters (backbone, links, etc.) were computed (5) and found to be very close to those of 2D percolation clusters.

It is easy to restate this construction in terms of aggregation of square cells. The first stage begins with 8 cells C(0) getting together into a square ring. With the probability $1-s$, any two neighboring cells C(0) are bonded together, while with the probability s, a bond arises only between 7 of the 8 cells. The k-th stage begins with 8 cells $C(k-1)$ getting together into a square ring. With the probability $1-s$, any two neighboring cells $C(k-1)$ are bonded together at only one of the numerous potential bonds, while with the probability s, such bonds only arise between 7 of the 8 cells.

ACKNOWLEDGMENTS

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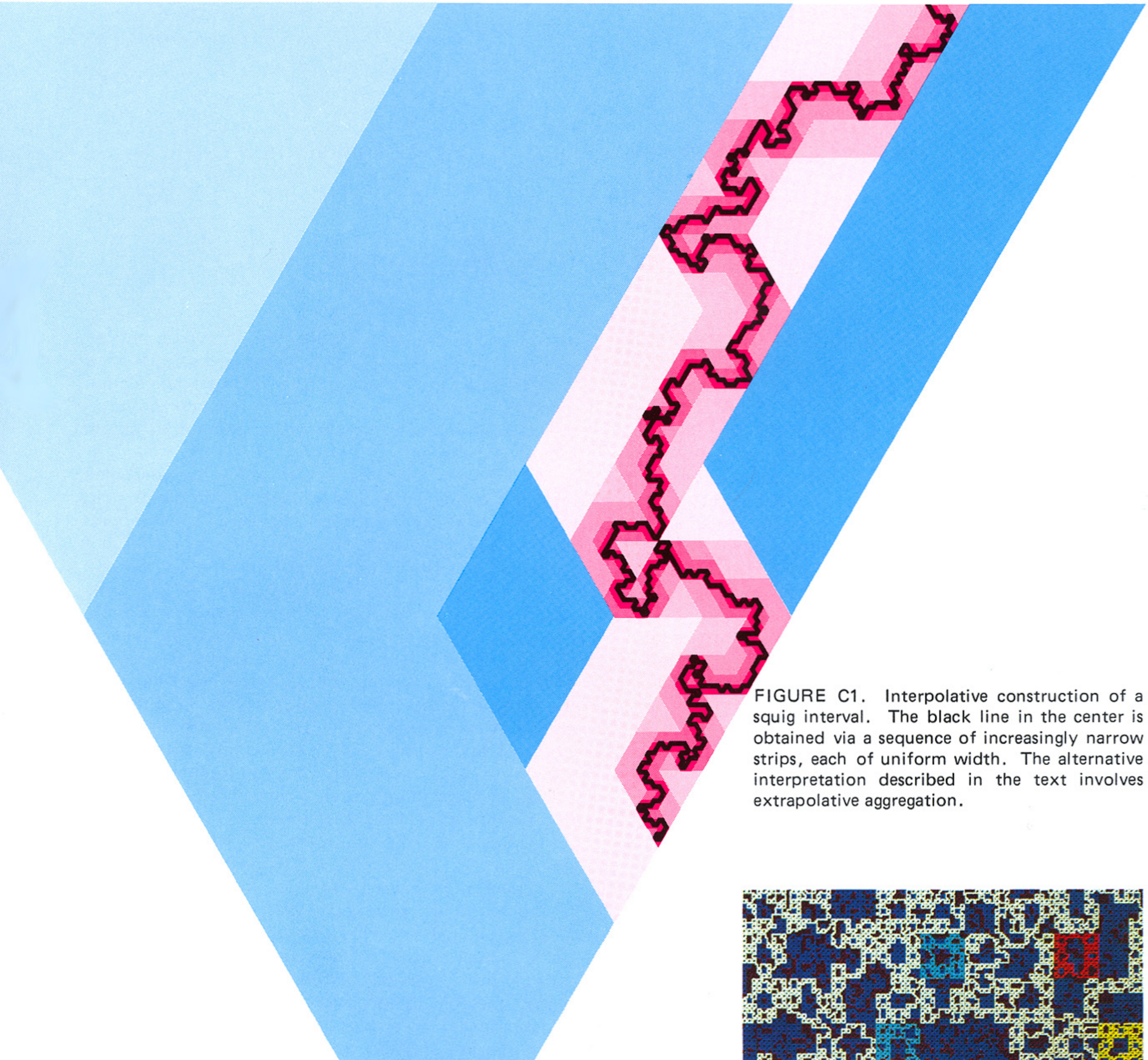

FIGURE C1. Interpolative construction of a squig interval. The black line in the center is obtained via a sequence of increasingly narrow strips, each of uniform width. The alternative interpretation described in the text involves extrapolative aggregation.

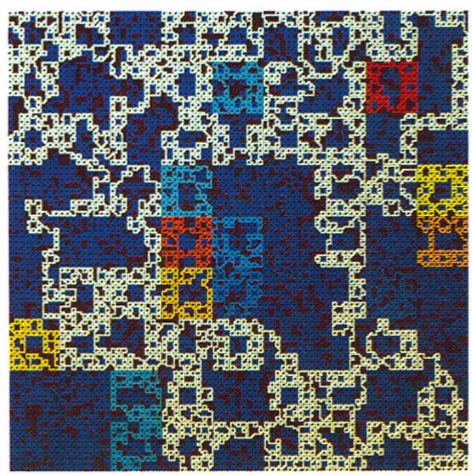

FIGURE C2. The squig clusters contained within a square include a large one, a second smaller one positioned in the large cluster's tremas, and so on. This figure shows the biconnected portion of these various clusters: the largest is shown in white, the 2nd largest in bright yellow, the 3rd in dull yellow, and successively smaller clusters are shown in shades varying from orange to red to light blue.