

FRACTAL ASPECTS OF THE ITERATION OF  
 $z \rightarrow \lambda z(1 - z)$  FOR COMPLEX  $\lambda$  AND  $z$

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INTRODUCTION

Given a mapping  $z \rightarrow f(z, \lambda)$ , where  $f$  is a rational function of both  $z$  and  $\lambda$ , consider the iterated maps  $z_n = f(f(\dots f(z_0) \dots))$  of the starting point  $z_0$ . To achieve a global understanding of these iterates' behavior, it is necessary to allow  $\lambda$  and  $z_0$ , hence  $z_n$  also, to be complex variables. Contrarily, the extensive recent studies of the mapping  $z \rightarrow \lambda z(1 - z)$ , for example, those found in Reference 1 and in the present volume, are largely restricted to  $\lambda$  real  $\in [1, 4]$  and  $z$  real  $\in [0, 1]$ . Hence, they are powerful but local and incomplete. The global study for unrestricted complex  $\lambda$  and  $z$  throws fresh light upon the results of these restricted studies, and reveals important new facts. In this light, an immediate change of emphasis from the restricted studies to even more general mappings  $\{x \rightarrow f(x, y, \lambda); y \rightarrow g(x, y, \lambda)\}$  appears to be premature.

The present paper stresses the role played in the unrestricted study of rational mappings by diverse fractal sets, including  $\lambda$ -fractals (sets in the  $\lambda$  plane), and  $z$ -fractals (sets in the  $z$  plane). Some are fractal curves (of topological dimension 1), and others are fractal "dusts" (of topological dimension 0). The  $z$ -fractals are of special interest, since they can be interpreted as the fractal attractors of appropriately defined (generalized) discrete dynamical systems, based upon inverse mappings. This role is foreshadowed in the work of P. Fatou<sup>2,3</sup> and of G. Julia<sup>4</sup> (and even that of H. Poincaré, in the related context of Kleinian groups), but the topic was never pursued. Indeed, an explicit and systematic concern with fractals only came with my book,<sup>5</sup> in which, for the first time, the notion itself was defined and given a name: A fractal set is one for which the fractal (Hausdorff-Besicovitch) dimension strictly exceeds the topological dimension. This paper's illustrations are fresh (and better than ever) examples of what this definition implies intuitively. The text is a summarized excerpt from Reference 6. A related excerpt concerning the fractal attractors of Kleinian groups is Reference 7. The final section comments on "strange" attractors.

THE  $\lambda$ -FRACTAL  $Q$

We denote by  $Q$  the set of values of  $\lambda$  with the property that the initial points  $z_0$  for which l.u.b.  $|z_n| < \infty$  include a closed domain (that is, a set having interior points). It is well known that it suffices that l.u.b.  $|z_n| < \infty$  hold when the initial point is the "critical" point  $z_0 = 0.5$ . The portion of  $Q$  for which  $\text{Re}(\lambda) > 1$  is illustrated in FIGURE

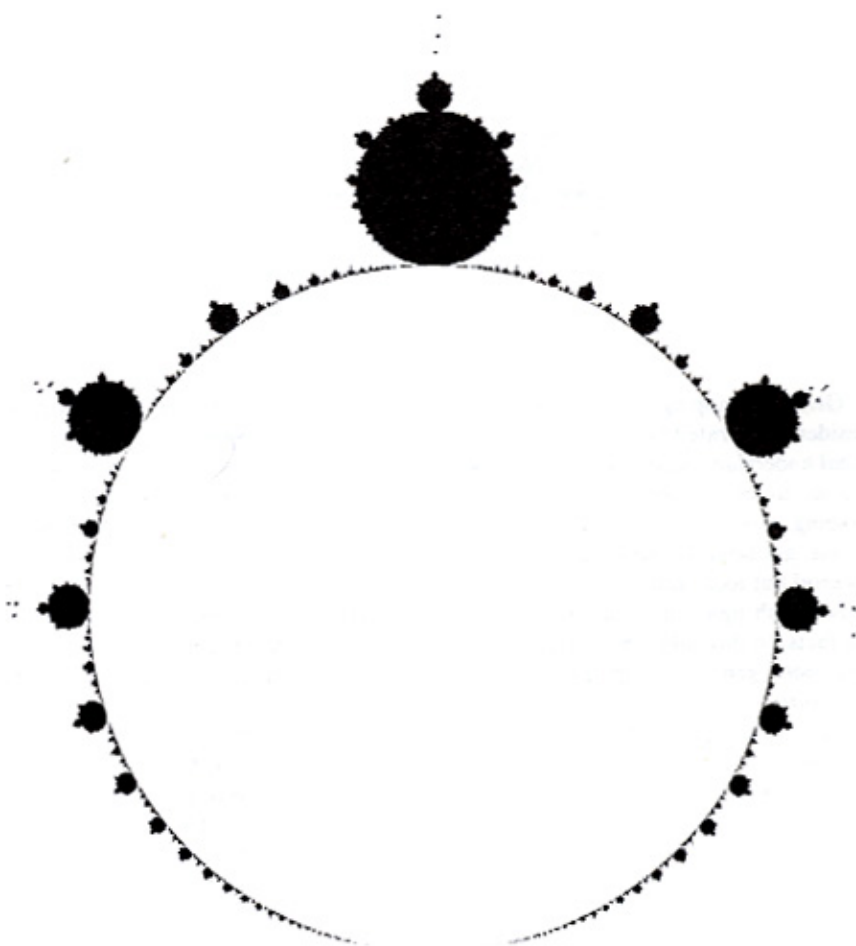


FIGURE 1. Complex plane map of the  $\lambda$ -domain  $Q$ . The real axis of the  $\lambda$ -plane points up from  $\lambda = 1$ . The center of the circle is  $\lambda = 2$  and the tip of the whole is  $\lambda = 4$ .

1, the remainder of  $Q$  being symmetric to this figure with respect to the line  $\text{Re}(\lambda) = 1$ .

A striking fact, which I think is new, becomes apparent here: FIGURE 1 is made of several disconnected portions, as follows.

*The Domain of Confluence  $\mathcal{L}$ , and Its Fractal Boundary*

The most visible feature of FIGURE 1 is the large connected domain  $\mathcal{L}$  surrounding  $\lambda = 2$ . This  $\mathcal{L}$  splits into a sequence of subdomains one can introduce in successive stages.

The first stage subdomain,  $\mathcal{L}_0$ , is constituted by the point  $\lambda = 1$  plus the open disc  $|\lambda - 2| < 1$  (left blank on FIGURE 1 to clarify the remainder's structure). If  $\lambda \in \mathcal{L}_0$ , there is a finite stable fixed point. (Proof: When  $\operatorname{Re} \lambda > 1$ , the stable limit point, if it exists, is  $1 - 1/\lambda$ ; the condition  $|f'(1 - 1/\lambda)| < 1$  boils down to  $|\lambda - 2| < 1$ .)

The remaining, and truly interesting, portion of  $\mathcal{L}$  is shown in black on FIGURE 1. The fact that this black area is "small" means that the mapping  $z \rightarrow \lambda z(1 - z)$  is mostly not bizarre. However, many interesting and bizarre behaviors (some of them unknown so far, and others thought to be associated with much more complex transformations) are obtained here in small but nonvanishing domains of  $\lambda$ .

Each of the second stage subdomains of  $\mathcal{L}$  is indexed by one or several rational numbers  $\alpha/\beta$ . The subdomain  $\mathcal{L}(\alpha/\beta)$  is open, except that we include in it the limit point where it attaches "sprout"-like to  $\mathcal{L}_0$ ; this is the point  $\lambda - 2 = -e^{-i\theta} = -\exp[-2\pi i(\alpha/\beta)]$ . When  $\lambda \in \mathcal{L}(\alpha/\beta)$ , the sequence  $z_n$  has a stable limit cycle of period  $\beta$ . This cycle can be obtained through a single  $\beta$ -fold bifurcation by a continuous change of  $\lambda$  that starts with any stable fixed point, for example, with the stable fixed point  $z_0 = 0.5$  corresponding to  $\lambda_0 = 2$ .

Each of the third stage subdomains of  $\mathcal{L}$  is indexed by two rational numbers:  $\mathcal{L}(\alpha_1/\beta_1, \alpha_2/\beta_2)$ ; it is open, save for the point where it attaches, again sprout-like, to  $\mathcal{L}(\alpha_1/\beta_1)$ . When  $\lambda \in \mathcal{L}(\alpha_1/\beta_1, \alpha_2/\beta_2)$ , the sequence  $z_n$  has a stable limit cycle of period  $\beta_1\beta_2$  resulting from two successive bifurcations, respectively  $\beta_1$ -fold and  $\beta_2$ -fold, which started with a stable fixed point in  $\mathcal{L}_0$ .

Further series of subdomains are similarly indexed by increasingly many rational numbers  $\alpha_1\beta_1 \dots \alpha_r/\beta_r$ .  $\mathcal{L}$  combines all the values of  $\lambda$  that lead either to stable limit points of  $z_n$  or to stable limit cycles that can be reduced to stable limit points by the inverse of the bifurcation process. I propose for this process the term *confluence*, which is why I call  $\mathcal{L}$  the domain of confluence.

The domains  $\mathcal{L}(\alpha/\beta)$  etc. are nearly disc shaped, but not precisely so. More generally, the boundary of each sprout is nearly a reduced scale version of the whole boundary of  $\mathcal{L}$ . Recalling the classic construction of the "snowflake curve,"<sup>15</sup> one can have little doubt but that the boundary of  $\mathcal{L}$  is a fractal curve.

### *The Transformed Domain $\mathcal{M}$*

Using the often invoked transformed variable  $w = (2z - 1)\lambda/2$  re-expresses the mapping  $z \rightarrow \lambda z(1 - z)$  into  $w \rightarrow \mu - w^2$ , where  $\mu = (\lambda^2/4) - (\lambda/2)$ . This leads to the replacement of the  $\lambda$ -set  $\mathcal{L}$  by a  $\mu$ -set  $\mathcal{M}$ . The counterpart to the discs  $|\lambda - 2| \leq 1$  ( $= \mathcal{L}_0$ ) and  $|\lambda| \leq 1$  (the symmetric of  $\mathcal{L}_0$  with respect to  $\operatorname{Re}(\lambda) = 1$ ) is a shape  $\mathcal{M}_0$  bounded by the fourth order curve of equation  $\mu = e^{2i\phi}/4 - e^{i\phi}/2$ . The sets  $\mathcal{M}$  and  $\mathcal{M}_0$  will be needed momentarily. Hence, the scholars' familiar hesitation between the notations involving  $\lambda$  or  $\mu$  is not resolved here: the shape  $\mathcal{L}_0$  is far simpler than  $\mathcal{M}_0$ , but  $\mathcal{M}$  is more useful than  $\mathcal{L}$ .

### *The Domains of Nonconfluent, or K-Confluent, Stable Cycles*

In addition to  $\mathcal{L}$ , the domain  $\mathcal{Q}$  is made of many smaller subdomains. Indeed, I discovered that at least some of what are, apparently, specks of dirt or ink on FIGURE 1



are indeed real: more detailed maps reveal a well-defined "island" whose shape is like that of  $\mathcal{M}$ , except for a nonlinear one-to-one deformation. Each island is, in turn, accompanied by subislands, doubtless ad infinitum.

When  $\lambda$  lies in an island's deformed counterpart to  $\mathcal{L}_0$ ,  $z_n$  has a stable limit cycle of period  $\omega > 1$ . When  $\lambda$  lies in an island's deformed counterpart to  $\mathcal{L}(\alpha_1/\beta_1, \dots, \alpha_k/\beta_k)$ ,  $z_n$  has a limit stable cycle of period  $\omega\beta_1, \dots, \beta_k$ . One would like again to be able to reduce these cycles, through successive confluences provoked by continuous changes in  $\lambda$ , to the fixed point  $\lambda_0 = 2$ . But this is impossible. None of these fixed cycles is confluent to a fixed point.

Some islands of  $\mathcal{L}$  that intersect the real axis create intervals that have been previously recognized and extensively studied. It was clear that a cycle with  $\lambda$  in such an interval is not confluent to  $\lambda = 2$  through real values of  $\lambda$ . We see that it is not confluent through complex  $\lambda$ s, either.

### *The Radial Patterns in the Distribution of the Domains of Nonconfluence*

The islands that intersect the real axis can be called "subordinate" to the value of  $\lambda = 3.569$ , which is known to mark the right-most point of  $\mathcal{L}_0$  and corresponds to an infinite sequence of successive 2-bifurcations. More generally, I observe that every island is subordinate to a  $\lambda$  corresponding to an infinite sequence of successive bifurcations. The subordination is spectacular (on a detailed  $\lambda$ -map) when the first of these bifurcations is of high order, that is, when  $\theta_1/2\pi = \alpha_1/\beta_1$  with a high value of  $\beta_1$ . But the subordination is already apparent in FIGURE 1 for the outermost point of the sprout linked to  $\mathcal{L}_0$  at  $\theta_1/2\pi = 1/3$ . Moreover, the islands are arrayed along  $\beta_1$  directions radiating from an "offshore point." In particular, if  $\lambda$  corresponds to several successive bifurcations, the other  $\beta_i$  do not affect the number of radii. For details, see Reference 6.

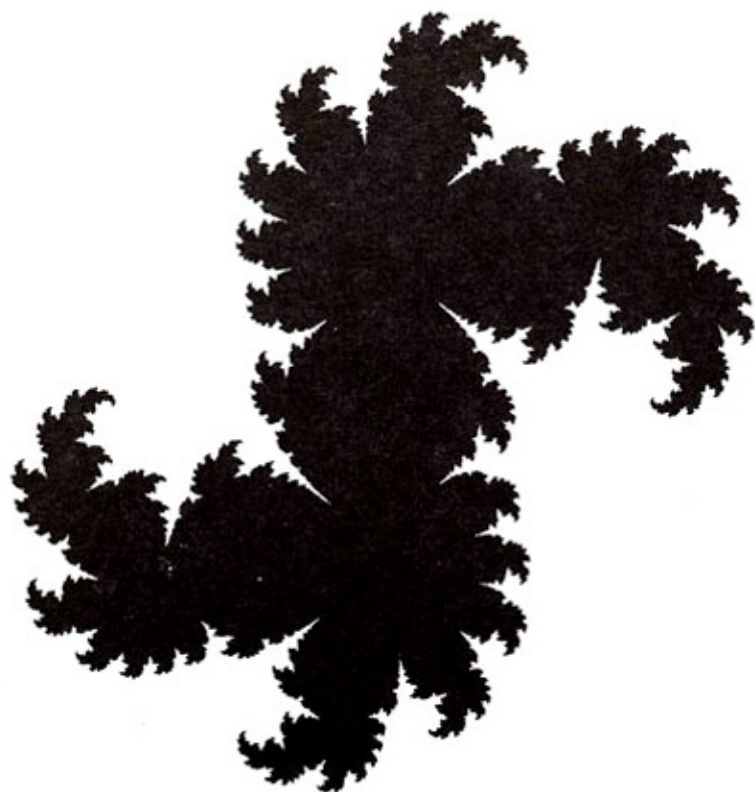
### THE $z$ -FRACTAL $\mathcal{F}(\lambda)$

We proceed now to a family of  $z$ -plane fractals associated with  $z \rightarrow \lambda z(1 - z)$ .

#### *Definition*

First, recall that  $z = \infty$  is a stable fixed point for all  $\lambda$ . (Proof: In terms of  $u = 1/z$ , the mapping is  $u \rightarrow g(u) = u^2/\lambda(u - 1)$ ; we see that  $g'(0) = 0 < 1$ .) For each  $\lambda$ , the  $z$ -fractal  $\mathcal{F}(\lambda)$  is defined as the (closed) set of points  $z_0$  such that  $\text{l.u.b. } |z_n| \neq \infty$ , that is, as the set of points whose iterates fail to converge to  $\infty$ . This set,  $\mathcal{F}(\lambda)$ , is never empty: it includes  $z_0 = 0$ , which is an unstable fixed point, all of whose iterates also satisfy  $z_n = 0$ , plus all the finite preimages of  $z_0$  and their limit points. Furthermore,  $\mathcal{F}(\lambda)$  is always bounded: it is easy to see that it is contained—with room to spare!—in the circle  $|z - 0.5| = 2.5$ . The boundary of  $\mathcal{F}(\lambda)$  is to be denoted by  $\mathcal{F}^*(\lambda)$ .

FIGURE 2 shows an example involving a 7-fold bifurcation.

FIGURE 2. Map of  $\mathcal{F}(\lambda)$  for  $\lambda$  near a 7-fold bifurcation.*Exceptional Values of  $\lambda$ , for Which  $\mathcal{F}(\lambda)$  is a Standard Shape*

The only such values of  $\lambda$  are  $\lambda = 4$ ,  $\lambda = 2$ , and  $\lambda = \infty$ .

For  $\lambda = \infty$ ,  $\mathcal{F}^*(\infty)$ , hence  $\mathcal{F}(\infty)$ , reduces to the points 0 and 1. Obviously,  $z_n = \infty$  except conceivably for  $z_0 \neq 0$  and  $z_0 = 1$ : these values yield an indeterminate expression  $z_1 = 0 \cdot \infty$ . The expression is made determinate by noting that the inverse transform leaves these points invariant. The relevance of the inverse transform will be made clear below.

For  $\lambda = 4$ ,  $\mathcal{F}^*(4)$ , hence  $\mathcal{F}(4)$ , reduces the segment  $[0, 1]$ . Indeed, introducing the new variable  $w = -(2z - 1)$  changes  $z \rightarrow z(1 - z)$  into  $w \rightarrow 2w^2 - 1$ , and the further new variable  $u = \cos^{-1} w$  yields  $u \rightarrow 2u$ , hence  $u_n = 2^n u$ . When  $\text{Im}(u_0) \neq 0$ ,  $|\text{Im}(u_n)| \rightarrow \infty$  and  $|z_n| \rightarrow \infty$ . Hence, the  $u$  coordinate representation of  $\mathcal{F}(4)$  is the real axis, implying that  $w \in [-1, 1]$  and  $z \in [0, 1]$ .

For  $\lambda = 2$ , the same variable  $w$  changes  $z \rightarrow 2z(1 - z)$  into  $w \rightarrow w^2$ , meaning that  $\mathcal{F}^*(2)$  is the circle  $|w| = 1$ , i.e.,  $|z - 0.5| = 0.5$ , and  $\mathcal{F}(2)$  is the closed disc bounded by this circle. Clearly,  $z_n \rightarrow 0$ , hence  $z_n \rightarrow 0.5$  iff  $z_0 \in \mathcal{F}(2) - \mathcal{F}^*(2)$ , and  $w_n \rightarrow \infty$ , hence

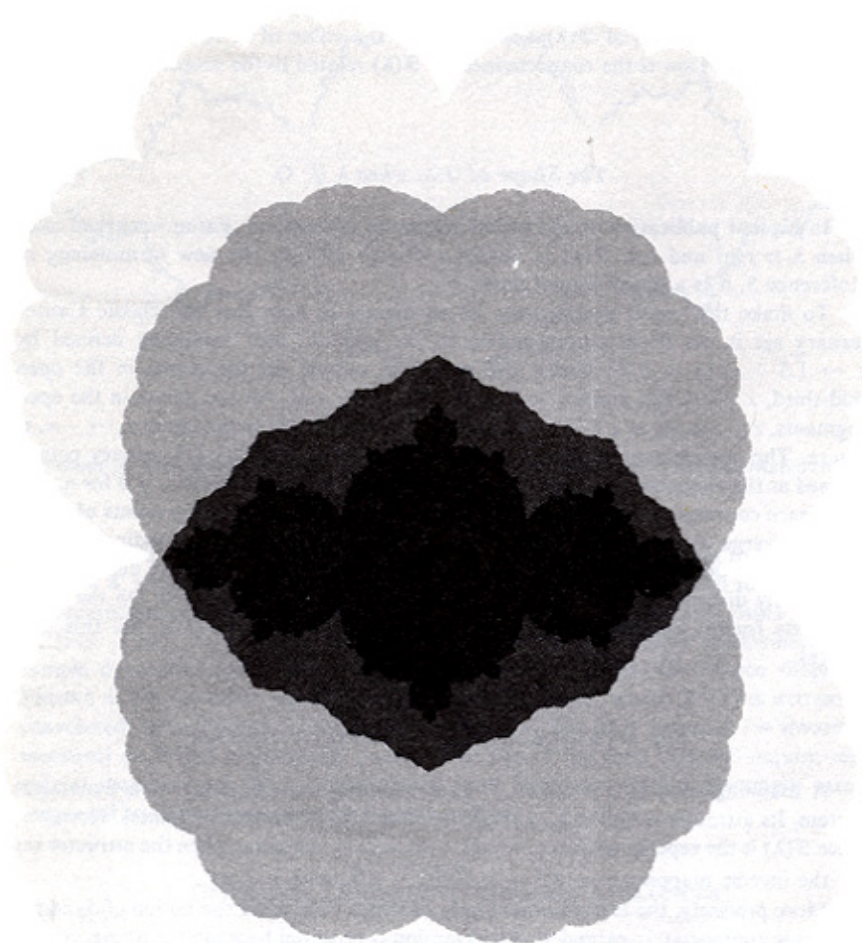


FIGURE 3. Composite  $\mathcal{F}(\lambda)$  for several real valued  $\lambda$ 's:  $\lambda = 1$  (whole picture),  $\lambda = 1.5$  (from mid-grey to black),  $\lambda = 2.5$  (dark grey to black)  $\lambda = 2.9$  (black only). The shape for  $\lambda = 3$  is called the San Marco shape in the text. The two points that are common to all four diagrams are  $z = 0$  and  $z = 1$ .

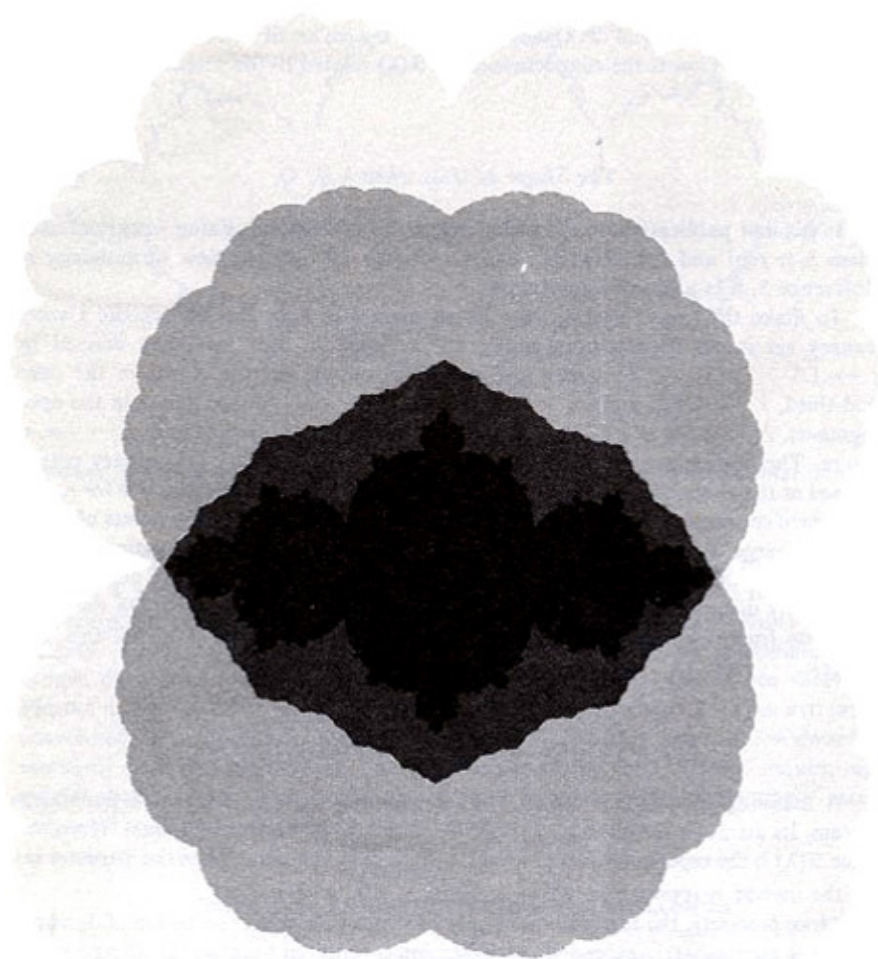


FIGURE 3. Composite  $\mathcal{J}(\lambda)$  for several real valued  $\lambda$ 's:  $\lambda = 1$  (whole picture),  $\lambda = 1.5$  (from mid-grey to black),  $\lambda = 2.5$  (dark grey to black)  $\lambda = 2.9$  (black only). The shape for  $\lambda = 3$  is called the San Marco shape in the text. The two points that are common to all four diagrams are  $z = 0$  and  $z = 1$ .



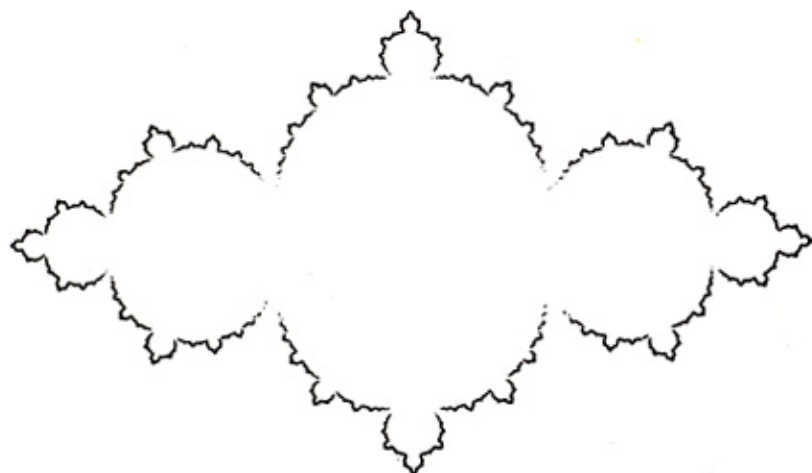


FIGURE 4. The 64 000 first positions of a dynamic system attracted to the San Marco fractal shape.

For  $\lambda = 2$ , when  $\mathcal{F}^*(\lambda)$  is a circle, the invariant measure is known to be uniform. For  $\lambda = 4$ , when  $\mathcal{F}^*(\lambda)$  is  $[0, 1]$ , the invariant measure is readily seen to be the real axis projection of a uniform measure on a circle; hence, it has the "arc-cosine" density  $\pi^{-1}[x(1 - x)]^{-1/2}$ . Both are found empirically to be very rapidly approximated by sample dynamical paths. See also the approximation of the San Marco shapes in FIGURE 4. On the other hand, the most interesting cases, where  $\mathcal{F}^*(\lambda)$  is extremely convoluted, as in FIGURE 2, involve a complication. The limit measure (= invariant measure) on  $\mathcal{F}^*(\lambda)$  is extremely uneven. The tips of the deep "fjords" require very special sequences of the  $\epsilon_n$  to be visited, and hence are visited extremely rarely compared to the regions near the figure's outline.

#### A DIGRESSION CONCERNING "STRANGE ATTRACTORS"

Lately, the term "attractor" has often been associated with the adjective "strange," and the reader may legitimately wonder whether strange and fractal attractors have anything in common. Indeed, they do.

##### *First Point*

The fractal (Hausdorff-Besicovitch) dimension  $D$  has been evaluated for many strange attractors, and found to exceed strictly their topological dimension. Hence, these attractors (and presumably other ones, perhaps even all strange attractors) are fractal sets. The  $D$  of the Smale attractor is evaluated in Reference 8. And the Saltzman-Lorenz attractor with  $v = 40$ ,  $\sigma = 16$ , and  $b = 4$  yields  $D \approx 2.06$ ; this result was obtained independently by M. G. Velarde and Ya. G. Sinai, who report it in private conversations but neither of whom has, to my knowledge, published it. (Last



minute addition: A preprint by H. Mori and H. Fujisaka confirms my value of  $D$  for the Smale attractor and the Velarde-Sinai value of  $D$  for the Saltzman-Lorenz attractor. For the Hénon mapping with  $a = 1.4$  and  $b = 0.3$ , it finds that  $D = 1.26$ .) The fact that  $D \approx 2.06$  is very close to 2, but definitely above 2, means that the Saltzman-Lorenz attractor is definitely not a standard surface, but that it is not extremely far from being one.

Since the relevance of  $D$  in this context may puzzle those who only know of fractal dimension as a measure of the irregularity of continuous curves, let me point out that in this instance,  $D$  is not a measure of irregularity but of the way smooth surfaces pile upon each other—a variant of the notion of fragmentation, which is also studied in Reference 5.

Let us also recall from Reference 5 that the Hausdorff-Besicovitch discussion was not the sole candidate for fractal dimension, but was selected because (1) it is the most thoroughly studied, (2) it has theoretical virtues, and (3) in most instances, the choice does not matter, because diverse reasonable alternative dimensions yield identical values. In an interesting further development in the same direction, a relation has recently been conjectured to exist, and verified empirically on examples, between a strange attractor's fractal dimension and its Lyapunov numbers (preprints by H. Mori and H. Fujisaka and by D. A. Russell, J. D. Hanson, and E. Ott.)

### *Second Point*

One is tempted, conversely, to ask whether the fractal attractors I study are "strange." It depends which meaning is given to this last word. Using its old-fashioned "meaning," as a milder synonym to "monstrous," "pathological," and other epithets once applied to fractals, the answer is "Yes, but why bother to revive a term whose motivation has vanished when fractals were shown, by Reference 5, to be no more strange than coastlines or mountains." Unfortunately, the term "strange" has since acquired a technical sense, one so exclusive that the Saltzman-Lorenz attractor must be called "strange-strange." In this light, many fractal attractors of my dynamic systems are not strange at all. Indeed "strangeness" reflects nonstandard topological properties, with the nonstandard fractal properties mentioned above coming along as a seemingly inevitable "overhead." In this sense (1) a topological circle (intuitively, a closed curve without double points) is not strange, however crumpled it may be; hence, (2) the fractal attractors  $\mathcal{F}^*(\lambda)$  for  $|\lambda - 2| < 1$  are surely not strange.

However, the fractal attractors associated with other rational mappings I have studied are topologically peculiar.<sup>6</sup> Thus, the answer to our question is confused. But it is not an important question: the term "strange" has, in my opinion, exhausted its usefulness, and ought to be abandoned.

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