LECTURE V

FRACTALS AND TURBULENCE: ATTRACTORS AND DISPERSION

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The renewal of interest in the mathematical aspects of turbulence has several independent and near simultaneous sources. The dynamics approach well represented in this seminar is rooted in the combined arguments of Lorenz 1963 and Ruelle & Takens 1970. A separate approach started with the combined arguments of Kolmogorov 1962, Berger & Mandelbrot 1963, and Novikov & Stewart 1964; the most recent statement is found in Mandelbrot 1976 and in the book <u>Fractals</u>, Mandelbrot 1977.

The two approaches are bound to converge, if only because both--and also those exemplified by the work of U. Frisch--make vital use of nonstandard sets I have termed <u>fractals</u>. One notion that is or should be stressed particularly in this kind of work is the Hausdorff-Besicovitch dimension D. Since this notion is classical but, so to say, somewhat obscure, it will be defined and motivated below. However, it may be useful to say immediately that (in <u>Fractals</u>) a fractal set is defined as being such that

Hausdorff Besicovitch dimension > topological dimension.

For the standard sets of Euclid, on the contrary, these dimensions coincide. The term <u>fractal</u> <u>structure</u> may be defined loosely as synonymous with structure involving D. This quantity becomes known as the fractal dimension.

The prototypical fractal is the Cantor set, and the product of a Cantor set by an interval is also a fractal. This last

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example enters in the well-known 1967 paper by Smale (see also Lecture III above) and in Ruelle & Takens 1970. Each stage of the Smale construction contracts the intercept of a torus into $N>1$ domains contained in it, with the usual illustration assuming $N=2$. In a different guise, contraction with $N>1$ also underlies the processes due to Hoyle and to Novikov & Stewart which Fractals describes under the name of "curdling". (The presently available physical motivations are sketchy in both cases.) Curdling also involves a second (weakly motivated) assumption, which has a counterpart in the the theory of contraction as restated in Smale's lecture above (but not in the original). The assumption is that each iteration replaces a set (either a curd or the meridian intercept of a domain) by N subsets that are similar to the original in a known ratio $r<1$.

The assumption concerning N is topological, but the assumption concerning r is metric in character. One metric property to which it points is the fractal dimension, which we shall see is given in this context by $D=\log N/\log(1/r)$. There are many ways of estimating the D from data (see Fractals) and their practical importance suggests that the dynamics approach ought to be developed beyond topology, to include the fractal aspects. The same remark applies to studies *à la* Lorenz 1963; there is no doubt (though the fact remains to be proved) that the corresponding "worse than strange" attractor is fractal; but its dimension is not known to me. To evaluate it would be of intrinsic interest and might help assess quantitatively rather than qualitatively to what extent natural turbulence is modeled by simplified systems of this kind (e.g., Hénon's model). (The value of D may play the role occasionally played by the exponent in the spectral density. It seems sometimes that simplified dynamic systems cease to be meant to derive the Kolmogorov $k^{-5/3}$ spectrum, the quality of a simplified system being judged on its ability to predict the $-5/3$ exponent.)

Thus, the term *strange attractor* used in Ruelle & Takens 1970 may well be a victim of the very success of the underlying approach, a more positively descriptive term becoming desirable.

One may suggest fractal attractor. (One could go so far as to argue that the first words in the title of this talk are descriptive of the whole object of this seminar; however, this is not a suggestion I want to promote.)

Two aspects of the notion of dimension: motivation. The mathematical characteristic which the Lorenz and other "strange" attractors seem to share with the sets used in Fractals is the following: It is known of the latter and suspected of the former that two alternative definitions of the notion of "dimension" yield distinct numerical values. The first is the topological dimension D_T. The second is the dimension D defined by Hausdorff and Besicovitch. Before we recall its definition, it is good to motivate D through the related concept of similarity dimension illustrated on Figure 1. (However, said illustration can be skipped; it is a variant of many in Fractals.) Figure 1 is the composite of two very-many-sided polygons one may call teragons. In Greek, teras = a wonder or a monster, and in the metric system tera = 10^{12}. One of these teragons is violently folded upon itself, being an advanced stage of the construction of a plane-filling curve. By way of contrast, the second curve can be called a wrapping. Both are constructed by a von Koch cascade, from (a) an initial polygon, and (b) a standard polygon. The first construction stage replaces each side of the initial polygon by a rescaled and displaced version of the standard polygon. Then a second stage repeats the same construction with the polygon obtained at the first stage, and so on ad infinitum.

The early construction stages are illustrated in Figure 2. The initial polygons are, respectively, a unit square and an irregular open polygon with N=17 sides. (It goes through every vertex of a certain lattice that is contained in the square.) Then each side of this 17-polygon is replaced by an image of its whole reduced in the ratio of $r=1/\sqrt{17}$. The result fills almost uniformly the shape obtained by replacing each side of the square by a certain polygon made of N=7 sides of length $r=1/\sqrt{17}$.

FIGURE 1

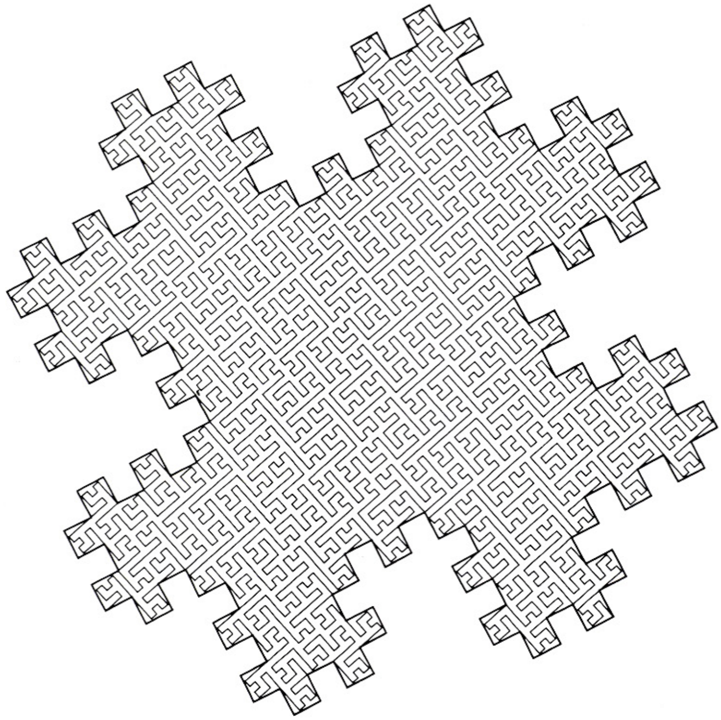
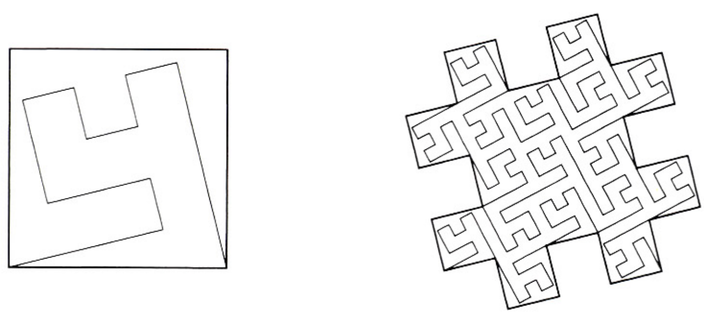


FIGURE 2



Incidentally, the familiar Peano curve and its variants drawn circa 1900 fill a square or a triangle, but recent Peano curves, like the present one, tend to involve more imaginative boundaries.

Since each construction stage multiplies length by a fixed factor $Nr>1$, both limit curves are of infinite length. But the filling tends to infinity more rapidly than its wrapping. This is expressed mathematically by the notion of similarity dimension. An intuitive explanation uses the following elementary fact: for every integer γ, the "whole" made up of a D dimensional parallelepiped may be paved by $N=\gamma^D$ "parts" which are parallelepipeds deduced by a similarity of ratio $r(N)=1/\gamma$. Hence, $D=\log N/\log(1/r)$. A dimension thus expressed as an exponent of self similarity continues to have formal meaning whenever the whole may be split up into N parts deducible from it by similarity of ratio r (followed by displacement or by symmetry). Such is the case with both limits here. For the wrapping, $N=7$ and $r=1/\surd 17$, hence $D=\log 7/\log\surd 17=1.3736$. For the filling, $N=17$ and $r=1/\surd 17$, hence $D=\log 17/\log\surd 17=2$. Thus the impression that the filling is more infinite than its wrapping is quantified by the inequality between their dimensions. The fact that the filling really fills a plane domain is confirmed by its dimension being $D=2$.

Hausdorff Besicovitch dimension and fractals. The first step in a general definition of D is to define the Hausdorff d-measure. Given a set S in a metric space and $\rho>0$, one covers S by balls with radii $\rho_m\leq\rho$, and one forms the sum $\Sigma\rho_m{}^d$; one takes the infimum of this sum over all coverings that satisfy $\rho_m\leq\rho$, then the limit of the infimum for $\rho\to 0$. The resulting $m_d(S)$ is by definition the Hausdorff d-measure of S. There exists a value of d, to be denoted by D, such that when $d>D$, $m_d(S)=0$ and when $d<D$, $m_d(S)=\infty$. This D is by definition the Hausdorff Besicovitch dimension.

Clearly, D is a metric rather than a topological property; I describe it as being a "fractal" property. More precisely, by a theorem of Szpilrajn (Hurewicz & Wallman 1941, p. 107), the topological dimension D_T and the above D are related by $D \geq D_T$. This explains the definition of fractals through $D > D_T$. The wrapping in Figure 1 is a curve of topological dimension 1; hence it is a fractal curve. For the triadic Cantor set, $D_T=0$ while $D=\log 2/\log 3$; hence it is a fractal. For the Cantor set considered by Smale in Lecture III above, $D=-\log 2/\log \varepsilon_2<1$. (However, by making N larger, one could obtain any $D<2$ in this fashion.) In the case of homogeneous Kolmogorov turbulence in the Gaussian approximation, the isosurfaces of scalars satisfy $D_T=2$ and $D=8/3$, hence they are fractal surfaces. (While this value of D is used extensively in Fractals, a complete formal proof became available too late to be included in the bibliography; the reference is Adler 1977.)

Very frequently, D coincides with the similarity dimension examined in the preceding section.

Formal relation between fractal dimension and entropy-information. By theorems of Besicovitch and Eggleston (see Billingsley 1965), the fractal dimension frequently takes the form of an entropy-information; for example, there often exist an integer C and a discrete probability p_m such that $D=-\Sigma p_m \log_C p_m$. Example: for the triadic Cantor set, $p_1=1/2$, $p_2=0$, and $p_3=1/2$, while $C=3$ (hence $C=\gamma=1/r$); thus $D=\log_3 2$. The corresponding topological entropy comes to mind: it is equal to log2 with an unspecified basis for the logarithm. The metric entropy specifies this basis as being $1/r$.

The preceding formal relation may help bring the topological dynamic aspects and the fractal aspects of turbulence together.

Dispersion of a fluid line or tube. It is not for fractals and fractal dimension to provide ready-made theories, but they often help formulate empirical observations into geometric conjectures that suggest further experiments and mathematical problems. For turbulence, consider dispersion starting with a smooth curve such as a straight segment. According to one theory, homogeneous turbulence causes the length to increase exponentially in time. It is easier to visualize the effects of a single "pinch" of turbulent energy left to decay. Within the Richardsonian view of self similar turbulence, one can argue that said effects subdivide into a sequence of stages, each of which multiplies the curve's length by some factor, either a fixed one or a random one with a fixed distribution. This picture is a variant of the Koch cascade of Figures 1 and 2. If it could be carried out ad infinitum (neglecting the viscosity cutoff and the effects of molecular diffusion), it would involve a fractal limit, and the following alternative emerges: is this limit space-filling, so that $D=3$, or such that $D<3$?

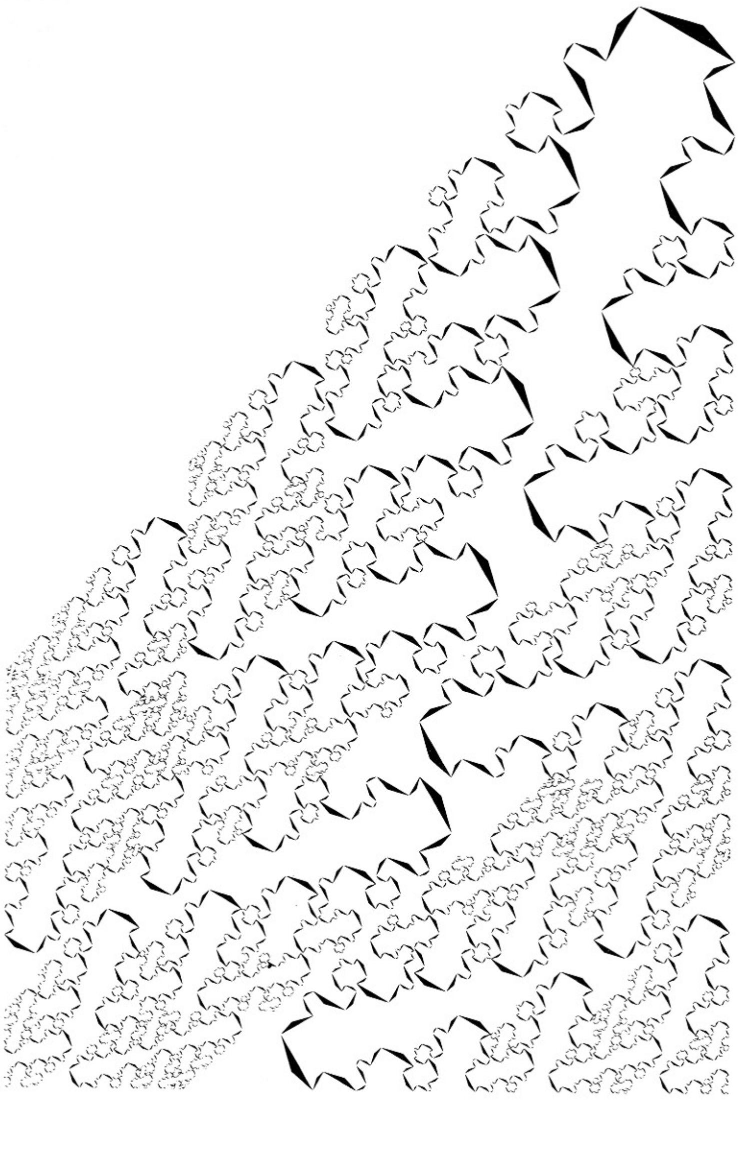
In planar reduction, $D<3$ corresponds to a curve like the wrapping in Figure 1, while $D=3$ corresponds to the filling. Let us first explore this second alternative. It views each 17-sided polygon in the Koch construction of the filling as an eddy (involving a net overall transport of matter). Observe that two intervals of the initial curve having equal lengths are mapped on two domains having equal areas. (In space: equal lengths map on equal volumes.) However, this interesting complication is avoided if the Koch cascade does not initiate on the bottom side of a square but on a domain. This domain can be taken to be the whole square and the first cascade stage can be assumed to replace this square by 17 squares collectively bounded by the first stage of the wrapping, and so on. In this fashion, our curve-to-domain application is embedded into a domain-to-domain application and the image of a curve by the first application is identical to the image of a domain by the second application. Any other initial set again yields the same image if it is included in the square and includes its bottom side. For example, one can represent a fluid tube by the bottom 1/10-th of the original square, rounded off to be shaped like one half of a sausage link. Our cascade of transformations will make it into 17 smaller links, then 17^2, etc. The limit will again be the whole interior of the wrapping on Figure 1. Each stage of the mapping is discontinuous along the lines where the preceding stage's link has been "pinched".

The (conjectural) turbulent mixing thus illustrated is grossly nonstationary. It is completely different from the usual stationary mappings such as the baker's transformation. Second difference: this "turbulent" mixing involves fixed points in exponentially increasing number and the baker's mixing has one fixed point. Third difference: there are reasons to expect the successive stages in this kind of turbulent cascade to proceed increasingly rapidly and the limit to be attained in finite time. (Denoting this time by t^*, the length will vary like $(t^*-t)^{-\alpha}$ with α a constant.)

The preceding model is readily generalized. Assuming that overall laminar motions are added, the final shape is no longer "globular", rather a long narrow strip, its overall shape being ruled both by turbulence and the laminar flow, and its detailed structure ruled only by turbulence. Furthermore, eddies can be made to be of different sizes, as for example in Figure 3. In comparison with the Koch method, the algorithm used here involves some complications, on which we shall not dwell. Broadly speaking, the initial shape ("sausage link") is a triangle. An earlier stage of the construction is visible (in reduced scale) in the eight triangles near the top of the picture, to the right.

An alternative conjectural view of turbulent dispersion involves the transformation of a smooth curve with $D=1$ into a fractal curve with $D<3$. The corresponding planar reduction is easily expressed; it would transform $D=1$ into $D<2$, as exemplified by the bottom fourth of the wrapping on Figure 1. However, the spatial form of this conjecture is hard to visualize. It is easier to imagine a spatial domain bounded by a surface of dimension $D=2$ being dispersed into a domain bounded by a surface of dimension $D \in]2,3[$. See, for example, <u>Fractals</u>, p. 52, where (somewhat weak) reasons are given to believe that $D=8/3$. Similarly, one can work with a fluid filament with a diameter smaller than the original outer scale L and greater than the viscous inner scale η. As the cascade progresses and energy splits into eddies of decreasing diameter, this filament is taken to stretch and fold on itself. When the filament and eddy diameters are

FIGURE 3



assumed proportional, we are led back to D=3. In order to achieve D<3, the eddy diameter must decrease more rapidly than the filament diameter. Such eddies will stretch the filament until it and the eddy have equal diameters. Thereafter, eddies will only affect the detail of the filament's surface; the filament's effective length will cease to change.

Of course, the preceding argument remains to be randomized, possibly along lines suggested by Robert Kraichnan. However, the feasibility of significant randomization depends strongly upon the dichotomy between D=3 and D<3. In the latter case, there is much room for it. In the former case, there is very little.

Now let us go from discrete pinches of turbulent energy on to homogeneous turbulence, approximating its effects to those of a sequence of pinches. In the case D<3, the filament's length will increase exponentially in time, as postulated by the theory to which we referred at the start. The sequence of successive transformations affecting it will be stationary.

Before attempting to model turbulent dispersion in detail, it may be advisable to analyze the evidence again, better than I could do here, to determine which of the above listed possibilities--or a farther variant--represents it properly. Once their task of helping sort alternatives is performed, the above cascade arguments should cease to be taken seriously, but the geometry they involve is likely to remain applicable.

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