INTERMITTENT TURBULENCE AND FRACTAL DIMENSION: KURTOSIS AND THE SPECTRAL EXPONENT 5/3+B

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Various distinct aspects of the geometry of turbulence can be studied with the help of a wide family of "shapes", for which I have recently coined the neologism "fractals". Until now, they had been used hardly at all in concrete applications, but I have shown them to be useful in a variety of fields. In particular, they play a central role in the study of a) homogeneous turbulence, through the shape of the iso-surfaces of scalars (Mandelbrot 1975a), b) dispersion (Mandelbrot 1976a), and especially c) the intermittency of dissipation (Mandelbrot 1972 and 1974a,b). Fractals are all loosely characterized as being violently convoluted and broken up, a feature denoted in Latin by the adjective "fractus". Fractal geometry approaches the loose notion of "form" in a manner different and almost wholly separate from the approach used by topology.

The present paper will sketch a number of links between the new concern with fractal geometry and the traditional concerns with various spectra of turbulence and the kurtosis of dissipation. Some of the results sketched will lead to improvement and/or correction of results found in the literature, including further refinement of Mandelbrot 1974a,b.

One result described in Chapter IV deserves special emphasis: It confirms that intermittency requires that the classical spectral exponent $5/3$ be replaced by $5/3+\mathrm{B}$. However, the factor B turns out in general to be different from the value ordinarily accepted in the literature, for example in the treatise by Monin and Yaglom 1975. Said value, derived by Kolmogorov, Obukhov, and Yaglom, is linked to the so-called lognormal hypothesis, which is a separate *Ansatz*, and is highly questionable.

Other results in the paper are harder to state precisely in a few words. In rough terms, they underline the convenience and heuristic usefulness of the fractally homogeneous approximation to intermittency, originating in Berger & Mandelbrot 1963 and Novikov & Stewart 1964. On the other hand, they underline the awkwardness of the lognormal hypothesis. That it is only an approximation has been stressed by many writers of the Russian school, but it becomes increasingly clear that even they underestimated its propensity to generate paradoxes and to hide complexities.

Turbulence and Navier Stokes Equations (Orsay, 1975).
Edited by Roger Temam (Lecture Notes in Mathematics 565).
New York: Springer, 121-145.

My book *Les objets fractals: forme, hasard et dimension*, Mandelbrot 1975b, describes numerous other concrete applications of fractals. It can also serve as a general background reference, but its Chapter on turbulence is too skimpy to be of use here. This deficiency should soon be corrected in the English version, tentatively titled *Fractals: form, chance and dimension,* which is being specifically designed to also serve as preface to technical works such as the present one. Nevertheless, in its main points, the present text is self-contained.

I. CURDLING AND FRACTAL HOMOGENEITY. ROLE OF THE FRACTAL DIMENSION $D=2$.

The term "curdling" is proposed here to designate any of several cascades through which dissipation concentrates in a small portion of space. Absolute curdling is described by the Novikov & Stewart 1964 cascade. Its outcome was described independently, without any generating mechanism, in Berger & Mandelbrot 1963 and Mandelbrot 1965. Weighted curdling is described by the cascades of Yaglom 1966, Mandelbrot 1974b and Mandelbrot 1972. It turns out that absolute curdling is more realistic than suggested by its extreme simplicity and in addition it provides an intrinsic point of reference to all other models. Therefore it deserves continuing attention. Weighting will be examined next. It turns out that it mostly adds complications, and its apparent greater generality is in part illusory.

Absolute curdling. Before each stage, dissipation is assumed uniform over a certain number of spatial cells, and zero elsewhere. Curdling concentrates it further: each of the initial cells breaks into $C = \Gamma^3$ sub-cells and dissipation concentrates within $N \geq 2$ of these, called "curds". The quantity $\rho = 1/\Gamma$ is the ratio of similarity of sub-cells with respect to the cells.

After a finite number of stages of absolute curdling, dissipation concentrates with uniform density in a closed set, whose outer and inner scales are L and η and which constitutes an approximation to a fractal. Figure 1 represents such an approximate fractal in the plane. (We shall soon see it is nearly a plane cut through an approximate spatial fractal.)

The most important characteristic number associated with a fractal is its fractal (Hausdorff) dimension, which in the present case is most directly defined as

$$D = \log N/\log (1/\rho).$$

It is always positive (because of the condition $N \geq 2$), and it is ordinarily a fraction. On Figure 1, $\Gamma = 5 = 1/\rho$ and $N = 15$, so that $D = 1.6826$ (the values of N and ρ were chosen to make this D as close as conveniently feasible to $5/3$). One reason for calling D a dimension is elaborated upon in Figure 2.

The notion of fractal dimension also extends to shapes that are not self similar. D is never smaller than the topological dimension D_T. For the classical shapes in Euclid, $D=D_T$ and the shapes for which I had introduced the term fractals are by definition such that $D>D_T$.

When dissipation is uniform over such a fractal of dimension $D<3$, turbulence will be called *fractally homogeneous.* The modifier is of course meant to contrast it with G. I. Taylor's classical concept of homogeneous turbulence, which can henceforth be viewed as the special limit case of fractally homogeneous turbulence for $D\rightarrow 3$. The salient fact is that the generalization allows $D-3$ to be negative.

The value of D refers to just one among many mathematical structures. It follows that the same D can be encountered in sets that differ greatly from other viewpoints, for example are topologically distinct. Nevertheless, many aspects of fractally homogeneous turbulence turn out to depend solely upon D. In an approximate fractal of dimension D and scales L and η, it is clear that dissipation concentrates within $(L/\eta)^D$ out of $(L/\eta)^3$ cells of side η. This approximation's volume is $(L/\eta)^D\eta^3$. The relative occupancy ratio of the region of dissipation (measured by the relative number of curds of side η within a cell of side L) is $(\eta/L)^{D-3}$. Therefore the uniform density of dissipation in a curd must be equal to $(L/\eta)^{3-D}$ times the overall density of dissipation.

The difference $3-D$ (or, for sets in Δ-dimensional Euclidean space with $\Delta\neq 3$, the difference $\Delta-D$) will be called *codimension.* This usage is consistent with the usage prevailing in the theory of vector spaces.

The quantities evaluated in the preceding paragraph concern solely the way blobs of intermittent turbulence *spread around.* Therefore D is a so-called metric characteristic. It is conceptually distinct from the topological characteristics of the way blobs are *connected.* And indeed, as is already the case for defining inequality $D>D_T$, most of the relationships between fractal and topological structures are expressed by inequalities. Topological structures prove very difficult to investigate, but fractal structures are more easily accessible to analysis. It is fortunate therefore, as we shall show in this paper, that several structures which a casual examination would classify as topological actually turn out to be exclusively or predominantly metric, namely, fractal. One example is the degree of intermittency as measured by the kurtosis. Another is the intermittency correction to the $2/3$ and $5/3$ laws, even though intermittency may conceivably have a distinct topological facet.

Weighted curdling. This more general process proceeds as follows. Each stage starts with dissipation uniform within cells. Then the density of dissipation in each subcell of a cell is multiplied by a random factor W, with $\langle W\rangle=1$. As a concept, W is related, but not identical, to the Yaglom multiplier. The cascade underlying weighted curdling is a generalization and a conceptual tightening-up of various arguments concerning the lognormal distribution. (In the absence of viscosity cutout, weighted curdling leads asymptotically to an everywhere dense fractal that is – topologically – open rather than closed. However, this distinctive feature vanishes when high frequencies are cut out by viscosity.)

The inner scale. The value of η, of course, is determined by the dissipation and the viscosity ν. Its value in the Taylor homogeneous case is well known. In the fractally homogeneous case, it continues to be well-defined, with a value that also depends on D. In the general case, however, the notion of η involves great complications. They will be avoided in Chapters III and IV, and only faced in Chapter V.

Behavior of linear cross-sections and a deep but elementary experimental reason to believe that for turbulent dissipation $D>2$. Most conveniently, the fractal dimensions of the linear and planar cross-sections of a fractal are given by the same formulas as the Euclidean dimension of the corresponding cross-sections of an elementary geometric shape. We shall state the rule, then show that, combined with evidence, it suggests very strongly that the D of turbulent diffusion must be greater than 2.

Rule: When the fractal or Euclidean dimension D of a shape is above 2, then its cross-section by an arbitrarily chosen straight line has a positive probability of being non-empty with the dimension $D-2$. Otherwise, the cross-section is empty. For planar sections analogous results apply, except that, instead of subtracting 2, one must subtract 1 (Mattila 1975). *Finite η-approximations to fractals.* Start with $(L/\eta)^D$ curds of side η. When $D>2$, the typical line cross-section will either be near-empty, or (with a positive probability, near independent of η) will include about $(L/\eta)^{D-2}$ segments of side about η. When $D<2$, on the contrary, the probability of hitting more than a small number of curds, say two curds or more, will greatly depend on η and will tend to zero with η/L. At the limit, suppose that $D=\log 2/\log(L/\eta)$ (which, among possible values of D, is the closest to $D=0$); then everything concentrates in two curds; the probability of hitting either by an arbitrarily selected line or a plane is minute. *Illustration.* Figure 1, being of dimension $\log 15/\log 5$, has the same dimension as the typical planar section of an approximate spatial fractal with $D=1+\log 15/\log 5$.

Application. By necessity, turbulence is ordinarily studied through linear cross-sections in space-time. Under Taylor's frozen turbulence assumption, they are the same as linear cross-sections through space. Turbulence is a highly prevalent phenomenon, in the sense that the typical cross-section hits it with no effort and repeatedly. Such would not be the case if the cross-sections were almost surely empty. Hence, we have an elementary reason (and hence an especially profound one) for believing that the fractal dimension D of turbulent dissipation satisfies $D>2$.

Digression. Possible relevance of fractal geometry to the study of the Navier-Stokes and Euler equations. My approach to the geometry of turbulence is to a large extent "phenomenological", as was Kolmogorov's approach, and is geometric rather than dynamic. It cannot rely on any information drawn from the study of the Navier-Stokes and Euler equations. However, the converse could be true: any success the fractal approach may be able to achieve should assist in the notoriously difficult search for turbulent solutions. I think, indeed, that the greatest roadblock in this search has been due to the lack of an intrinsic characterization of what was being sought. One could even go as far as to argue that no one could be sure he would recognize such a solution if it were shown to him. In past studies of other equations of physics, on the contrary, the easiest procedure has frequently been to seek guidance in guesses concerning the singularities to be expected in the solution. Knowing what to look for has often made it less difficult to find it, but this approach has not yet worked for turbulence. Von Neumann 1949-1963 has noted that "its mathematical peculiarities are best described as new types of mathematical singularities", but he made no progress in identifying them.

In this vein, I propose to infer from empirical evidence that, for nonlinear partial differential equations like Euler's system (when viscosity is absent) or the Navier-Stokes system (when viscosity $\to 0$, or possibly even at a positive small viscosity), the singularities of sufficiently "mature" solutions are likely to tend towards being fractal.

The singularities of Euler solutions should be viewed as associated with curdling, as discussed above and in the body of this paper. As to the Navier-Stokes equations, the notion that the solution can possess singularities remains unproven and in fact controversial, but if singularities in the Oseen-Leray sense do in fact exist, they must be enormously "sparser" than the Eulerian ones; possibly a proper subset. Assuming they indeed exist, Scheffer 1975 has been successful in restating some of my rough hunches on this topic into precise conjectures, and has proved several of them, relating them with the work of J. Leray 1934 and opening new vistas on this ancient problem. See also Scheffer's paper in the present volume.

Given that closely related forms of intermittency are found to occur in phenomena ruled by diverse other equations, the specific characteristic of the Navier-Stokes equation, which leads to the "fractality" of the solutions, must have its counterpart in broad classes of other equations, and it may well be more useful to study them within a broader mathematical context.

Digression. A second connection between fractality and Navier-Stokes equations. This connection comes in through the shapes of coastlines. A priori, it may well be that fractality is *wholly* related to the Oseen-Leray argument that a solution with good initial data may, after a certain time, have large velocity gradients. Alternatively, we have the Batchelor & Townsend (1949) argument "that the distribution of vorticity is made 'spotty' in the early stages of the decay by some intrinsic instability and is kept 'spotty' throughout the decay by the action of the quadratic terms of the Navier-Stokes equations". However, "spottiness" may also be affected by a third factor. Indeed, the study of partial differential equations, while stressing the respective roles of the equation itself and of boundary conditions, usually fails to consider the possible effects of the shape of that boundary. More precisely, the boundary is nearly always assumed very smooth, for example is taken to be a cube. For atmospheric and ocean turbulence, this approximation may well be unrealistic. The fact is as I showed in Mandelbrot 1967 and in the book *Fractals*) that the shapes of coastlines contain features whose "typical lengths" cover a wide span, so that their fractal dimensions are greater than 1. This and the analogous statement concerning the rough surface of the Earth may well combine with intrinsic instabilities as a third contribution to the roughness of observed flow.

II. THE FUNCTION $f(h)$. THE FRACTAL DIMENSIONS ARE DETERMINED BY $f'(1)$.

All the aspects of intermittency to be studied in this paper are ruled by power laws. If one adds specific further assumptions, the various exponents are linked to each other (through the fractal dimension D of the carrier or the parameter "μ" of the Kolmogorov theory). However, in the general case they are distinct. As Novikov 1969 had observed in the case of spectra and moments, each power law is merely a symptom of self similarity. The multiplicity of different exponents shows the self similarity syndrome to be complex and multifarious.

Nevertheless, the exponents that enter in my previous papers and in the present one can all be derived from various distinct properties of the following determining function:

$$f(h) = \log_C \langle W^h \rangle;$$

recall that $C = \Gamma^3$ is the number of subcells per cell. In absolute curdling, W is a binomial random variable: it can either vanish or take one other possible value $1/p$ with the probability p, so that $\langle W \rangle = 1$. In weighted curdling, W is a more general random variable, still satisfying $\langle W \rangle = 1$. Further, the limit lognormal model of Mandelbrot 1972 also fits in the same scheme by appropriate interpretation of W. By a general theorem of probability (Feller 1971, p.155), $f(h)$ is a convex function; it obviously satisfies $f(1)=0$. Furthermore, whenever $\Pr(W>0)=1$, one also has $f(0)=0$. As a first example, the graph of $f(h)$ is a straight line if, and only if, curdling is absolute. If so, the graph passes through $f(1)=0$ but not through $f(0)=0$. As a second example, $f(h)$ is a parabola when W is lognormal. These two cases are drawn on Figure 3, which also illustrates other features of $f(h)$.

My past and present papers show that the following characteristics of $f(h)$ are of interest: $f(2/3)$, $f'(1)$, $f''(1)$, $f(2)$, $f(h)$ for h integer >2, and α_1, α_2, $\alpha_3 = \alpha$, where α_m is defined as the root other than $h=1$ of the equation $\phi_m(h) = 3f(h)-m(h-1) = 0$. The functions ϕ_m being convex, each α_m is unique, but of course one or more among them can be infinite. Since the condition $\phi_1(h)<0$ is at least as demanding as $\phi_2(h)<0$, and, a fortiori, as $\phi_3(h)<0$, we see that, if $\alpha_1>1$, one has $\alpha_1 \leq \alpha_2 \leq \alpha_3 = \alpha$.

The order in which these various characteristics have been listed in the preceding paragraph is that of increasing sensitivity to the detail of the distribution of W. The first and least sensitive – and, in my opinion, the most basic – is

$$f'(1) = \langle W \log_C W \rangle$$

This quantity was first considered in Mandelbrot 1974a,b, further results being due to Kahane 1974 and Kahane & Peyrière 1976. When $3f'(1)<3$, the carrier of intermittency is nondegenerate, and its fractal dimension is $D=3-3f'(1)$. Thus, $3f'(1) = \langle W \log_\Gamma W \rangle$ determines the codimension $3-D$. As to the planar and linear intersections, when $3f'(1)<2$, respectively when $f'(1)<1$, these intersections are nondegenerate, with fractal dimensions equal to $D_2 = 2-3f'(1)$, respectively to $D_1 = 1-3f'(1)$. As expected, $D_2 = D-1$ and $D_1=D-2$. In the case of a lognormal W, and denoting by μ the basic parameter, one has $f(h) = (h-1)h\mu/6$. Hence, $D = 3-\mu/2$ and μ is merely twice the codimension.

Next, as the present paper will show, some other characteristics of $f(h)$, more sensitive to details of W, rule the traditional concerns with second order (spectral) properties. One must distinguish an inertial and a dissipative range (these are probably misnomers). In the former, the value of $f(2/3)$ rules the corrective term B to be added to the exponent in the Kolmogorov $k^{-5/3}$ law (this will be shown in Chapter IV). Similarly, the value of $f(2)$ rules the variance, the kurtosis and the exponent of the spectrum of dissipation (this will be shown in Chapter III). The next simplest characteristic of $f(h)$ is $f''(1)$. Our last result will be that $f''(1)$ determines the width of the dissipative range. When $f''(1)=0$, an equality characteristic of the fractally homogeneous case, the dissipative range is vanishingly narrow. Otherwise, it is most significant, especially when L much exceeds the Kolmogorov inner scale.

Digression. Each property that involves a moment of higher order h is ruled by the corresponding $f(h)$. As h increases, $f(h)$ becomes increasingly sensitive to details of the distribution of W, which is why the moments computed from the lognormal assumption appear inconsistent; see Novikov 1969. This difficulty was eliminated, however, when Mandelbrot 1972,1974a,b showed that – in the absence of viscosity cutoff – the population moments above a certain order, namely α_1, α_2, or α, are in fact infinite. In my opinion, this feature explains why the experimentalists have found empirical moments of higher order to be so elusive.

III. COVARIANCE, FLATNESS AND KURTOSIS OF DISSIPATION. EXPONENTS DETERMINED BY $f(2)$.

The exponent in the covariance of the dissipation. Take two domains Ω' and Ω'' whose diameters are small compared to the smallest distance r between them, and large compared to η. We define the covariance of the dissipation density $\varepsilon(\mathbf{x})$ as the expectation of the product of the average of $\varepsilon(\mathbf{x})$ within these domains. Without entering into details, let it be stated that in the fractally homogeneous case this covariance is approximately $(r/L)^{D-3}$, and in the general case it is $(r/L)^{-3f(2)}$. The proof follows closely that of Yaglom (see Monin & Yaglom 1975, p. 614), but stops before the point where these authors approximate the product of many W's by a lognormal variable.

By the convexity of $f(h)$, $f(2) \geq f(1) + (2-1)f'(1) = f'(1)$. Thus, $3f(2) \geq 3f'(1) = 3-D$. Equality prevails if and only if $f(h)$ is rectilinear, i.e., the curdling is absolute and turbulence is fractally homogeneous. (This is, in addition, the sole case where one can make r as low as η.) In every other case, an evaluation of the codimension $3-D$ through the observed exponent $f(2)$ would lead to overestimation. For example, for the strictly lognormal W, $3f(2) = \mu$, which is the *double* of the estimate using the dimension, namely $3f'(1) = \mu/2$.

Let us now show that the same exponents play an equally central role in the study of the kurtosis of dissipation after averaging over small domains of side r. In other words, at least in intermittency generated by curdling, the covariance and kurtosis of dissipation are conceptually identical.

Kurtosis in the fractally homogeneous case. Here, we know that the dissipation vanishes, except in a region of relative size $(L/r)^{D-3}$, in which it equals $(L/r)^{3-D}$. Hence, it is readily shown that the kurtosis is simply $(L/r)^{3-D}$. It increases as r becomes smaller, and when r takes its minimum value η (as announced, we shall show that η is well-defined in the fractally homogeneous case), the kurtosis reaches its maximum value $(L/\eta)^{3-D}$. The measure of degree of intermittency depends both on the intrinsic characteristic of the fluid, as expressed by D, and on outer and inner scale constraints, as expressed by L/η, which is related to Reynolds number. Therefore, it is better to measure the degree of intermittency by D itself. The empirical value of the exponent is 0.4 (Kuo and Corrsin 1972), suggests under the assumption of fractal homogeneity that $D = 2.6$.

To explore the significance of these findings, let me begin by sketching the results of previous studies of the kurtosis by Corrsin 1962 and by

Tennekes 1967. These and other authors took it for granted that the exponent of kurtosis depends mainly upon what may be called the proto-topological shape of the carrier of intermittency, namely on whether it is a "blob", a "slab", or a "sheet". While other assumptions also entered each model, they were felt to be secondary. This impression turns out to have been unwarranted. The crucial fact is that each of these models leads to fractally homogeneous intermittency, whose dimension D is affected by *all* the assumptions made, and determines the exponent of the kurtosis.

It turns out that, in the Corrsin model, the exponent's value is $3-D=1$ (his formula 10), hence $D=2$. This fractal dimension is experimentally wrong, in fact fails to satisfy the basic requirement $D>2$. It is interesting to note that $D=2$ is the smallest fractal dimension compatible with Corrsin's featured assumption, that turbulent dissipation concentrates with uniform density within sheets of thickness η enclosing eddies of size L. In other words, Corrsin's additional assumptions cancel out: there was no surreptitious increase of D, and he worked with a classical shape rather than with a fractal.

On the other hand, the exponent in the Tennekes model can be seen to imply $D = 7/3$. This value *does* satisfy $D>2$, and is reasonably close to observations. On the other hand, it very much exceeds the minimum fractal dimension, namely $D=1$, which topology imposes on a shape including ropes. Hence, Tennekes was mistaken in featuring the assumption that dissipation occurs in vortex tubes of diameter η. The more vital assumption was that the average distance between tubes is the Taylor microscale λ. The fact that $D = 7/3$ is even higher than the Corrsin value $D = 2$ strongly underlines that a tube, if sufficiently convoluted, ends up by ceasing to be a tube from a metric-fractal viewpoint, and becomes a fractal. Finally, the experimental $D = 2.6$ satisfies $D>2$. In addition, it *does not exclude* the presence of either ropes or sheets, but *does not require* either.

Kurtosis of nonfractally homogeneous intermittency generated by weighted curdling. In this case, kurtosis is simply $\langle \varepsilon^2\rangle/\langle\varepsilon\rangle^2 = \langle\varepsilon^2\rangle$ and turns out to be equal to

$$\langle W^2\rangle^{\log_\Gamma(L/r)} = (L/r)^{\log_\Gamma\langle W^2\rangle} = (L/r)^{3f(2)}$$

We know that $3f(2) \geq 3-D$. Hence, among all forms of turbulence generated by curdling and having a given D, the fractally homogeneous case is the one where the kurtosis is smallest. Hence $3f(2) = 0.6$ only yields $D\geq 2.6$.

Digression concerning the fractally homogeneous case. The behavior of the Fourier transform. Fourier transforms do not deserve the near exclusive attention which the study of turbulence gave to them at one time (through spectra), but they are important. It may be useful, therefore, to mention that, in the fractally homogeneous case, their properties happen to involve fractal dimension. The strength of the relationship has long been central to the fine mathematical aspects of trigonometric series see Kahane & Salem 1963, but the resulting theory has thus far been little known and used beyond its original context.

It deals particularly with functions that are constant except over a fractal of dimension D. Such functions have no ordinary derivative but have generalized derivatives which are measures carried by the fractal in question. The rough result is that the Fourier coefficients of the measure in question turn out in many typical cases to decrease like k^{-D}.

In a finer approximation, however, the considerations of fractal dimensionality are logically distinct from spectra. This fact further elaborates the assertion made earlier, that the consequences of the self similarity of turbulence split into conceptually distinct aspects, dimensional, spectral and others, which are governed by different exponents of self similarity linked to each other through inequalities.

If, as I hope, the importance of fractal shapes in turbulence becomes recognized, the spectral analysis of the motion of fluids may become able at last to make some use of a considerable number of pure mathematical results relative to harmonic analysis.

IV. THE MODIFICATION IN THE 2/3 AND 5/3 LAWS. A SPECTRAL EXPONENT CHANGE DETERMINED BY f(2/3).

It has been noted in Kolmogorov 1962 and Obukhov 1962 that intermittency changes the exponents 5/3 and 2/3 by adding a positive factor to be denoted by B. A more careful examination of the problem, to which we now proceed, confirms this conclusion but yields a value of B that does not in general fit those asserted by the Russian school.

Consider two points P' and P'' separated by the distance r and denote the velocity difference $\mathbf{u}(P'') - \mathbf{u}(P')$ by $\Delta\mathbf{u}$. In Taylor homogeneous turbulence of constant dissipation denotes $\delta(\mathbf{x}) \equiv \varepsilon$, one has $\langle(\Delta\mathbf{u})^2\rangle = (\varepsilon r)^{2/3}$. To extend this result to the intermittent case, when the nonrandom ε is replaced by a random field $\varepsilon(\mathbf{x})$ with $\langle\varepsilon(\mathbf{x})\rangle = \varepsilon$, one must replace ε in $(\varepsilon r)^{2/3}$ by some quantity characteristic of said random field and also of P' and of P''. Even though this quantity may be determined in several different ways, there will be no harm in always denoting it by ε_r.

Like Yaglom, we shall first follow closely the approach of Obukhov 1962 and Kolmogorov 1962, who propose one should take as ε_r the average of $\varepsilon(\mathbf{x})$ over a sphere – we shall call it the Obukhov sphere – whose poles are P' and P''. We shall designate this domain as $\Omega(P', P'')$. In practice, in the case of curdling within cubic cells, Ω is more conveniently the smallest cell containing both P' and P. We shall find, as have Kolmogorov and Obukhov, that intermittency requires the replacement of the classical spectral density $E(k) = E_0 \varepsilon^{2/3} k^{-5/3}$ by $E(k) = E_0 \varepsilon^{2/3} k^{-5/3} (k/L)^{-B}$. On the other hand, we shall disagree with them on two basic points: a), the value of the exponent B, and b), the highest value of k for which a spectral density with the exponent $5/3+B$ is conceivable.

When expressed in terms of the dimension D, the Kolmogorov-Obukhov correction comes out as $B = (3-D)/4.5$, but this value turns out to be due to very specific and arguable features of the lognormal assumption, which is part of their model. In the case of fractally homogeneous intermittency, one finds the different and *larger* value $B = (3-D)/3$, and in general $B = -3f(2/3)$, which can lie anywhere between the bounds 0 and $(3-D)/3$. Thus the value $(3-D)/4.5$ is a kind of compromise, perfectly admissible but by no means necessary.

Secondly, the fractally homogeneous case is unique in that it allows a widely-liked approximation in which the dissipative range reduces to a point, and one can assume that $E(k)$ takes the above form all the way to the inverse of the proper inner scale, and vanishes beyond. In all other cases, this traditional approximation leads to paradox. This Chapter will evaluate B and present the paradox; Chapter V will resolve it.

The following Sections will derive the above result, then subject the approach of Obukhov-Kolmogorov to a critical analysis. Their choice for ε_r was indeed acknowledged to be to a large extent an arbitrary first trial suggested less by physics than by commodity. Other definitions of ε_r are therefore worth considering. The first alternative ε_r will be the average of $\varepsilon(\mathbf{x})$ over a domain Ω that is the interval $P'P''$. When $D>2$, as we believe is the case for turbulence, the expression for B is unchanged and the coefficient E_0, while modified, remains positive and finite. When $D<2$, on the contrary, E_0 vanishes and B becomes meaningless. To obtain the second and last alternative ε_r, we shall let Ω be determined by the distribution of intermittency, and we shall thereby bring in topology. The argument will give reasons for believing that the carrier of turbulence, not only must satisfy the metric inequality $D>2$ proven in Chapter I, but must, in some topologic sense, be "at least surface-like". However, this third choice of Ω is very tentative, and so are the conclusions drawn from it.

1. THE FRACTALLY HOMOGENEOUS CASE WHEN Ω IS THE OBUKHOV SPHERE OR AN APPROXIMATING CUBE

In this case, ε_r is the average of $\varepsilon(\mathbf{x})$ over Ω. By the theorem of conditional probabilities, one can factor $\langle \varepsilon_r^{2/3} \rangle$ as the product of a) the probability of hitting dissipation in Ω, and, b) the conditional expectation of $\varepsilon_r^{2/3}$ where "conditional" means that averaging is restricted to the cases where $\varepsilon_r > 0$.

When Ω is the Obukhov sphere or the smallest cubic eddy that contains both P' and P'', it can be shown that, as $n \to \infty$, the hitting probability becomes at least approximately equal to $p_0(r/L)^{3-D}$.

Since the product of the hitting probability by the conditional expectation of ε_r is simply the nonconditional expectation ε, the conditional expectation must be equal to $\varepsilon\ (r/L)^{D-3}$. A stronger statement, in fact, holds true. Assuming fractal homogeneity, ε_r, when positive, it can be shown to be the product of $\varepsilon r^{D-3} L^{3-D}$ by a random variable having positive and finite moments of every order. Consequently,

$$\begin{aligned}\langle (\varepsilon_r r)^{2/3} \rangle &= V_{1/3}\,(r/L)^{3-D}\,\varepsilon^{2/3}\,r^{2/3}\,(r/L)^{-(3-D)/3}\\ &= V_{1/3}\,\varepsilon^{2/3}\,L^{-(3-D)/3}\,r^{2/3+(3-D)/3}\\ &= V_{1/3}\,\varepsilon^{2/3}\,L^{-(3-D)/3}\,r^{1-(D-2)/3}.\end{aligned}$$

The corresponding spectral density is

$$\begin{aligned}E(k) &= E_0\,\varepsilon^{2/3}\,L^{-(3-D)/3}\,k^{-5/3+(D-3)/3}\\ &= E_0\,\varepsilon^{2/3}\,L^{-(3-D)/3}\,k^{-2+(D-2)/3}.\end{aligned}$$

It is important to know that the numerical coefficients E_0 and $V_{1/3}$ are positive, but their actual values will not be needed.

These expressions show that intermittency has two distinct effects: to inject L and to change the exponent of k from $5/3$ to $5/3+B$, where $B = (3-D)/3$.

Since $B \geq 0$, the exponent $5/3+B$ always exceeds the 1941 Kolmogorov value $5/3$. As expected, $B=0$ corresponds to the limit case $D=3$, when dissipation is distributed uniformly over space.

The point where $5/3+B$ goes through the value 2 occurs when $D=2$, a relationship to which we shall return in Section 4.

Even if it is confirmed that (as inferred in Chapter I) ordinary turbulence satisfies $D>2$, it is good to include the values $D<2$ for the sake of completeness. Since B satisfies $B \leq 1$, the exponent $5/3+B$ always lies below the value $8/3$. This value is seen to correspond to $D=0$, the limit case when dissipation concentrates in a small number of blobs. (In absolute curdling, we saw that D is at least $\log 2/\log(L/\eta)$ corresponding to dissipation concentrated into two curds. However, variants of curdling yield a more relaxed relationship between $D \sim 0$ and concentrates in a few blobs.) We shall see in Section 3 that, among curdling processes of given D, B is greatest in the fractally homogeneous case. Hence, the bound $B \leq 1$ is of wide generality. Sulem & Frisch 1975 were able to rederive it by an entirely different argument from the characteristic that for $D=0$ everything concentrates in a small number of blobs.

2. THE FRACTALLY HOMOGENEOUS CASE WITH OTHER PRESCRIBED DOMAINS Ω.

To discuss Obukhov's specification of Ω further, we shall find it useful (however cumbersome) to decompose this specification into parts of increasing degrees of arbitrariness: a) one should replace ε by the average ε_r of the local dissipation rate $\varepsilon(\mathbf{x})$, taken over an appropriate domain Ω; b) this domain Ω should be independent of $\varepsilon(\mathbf{x})$; c) Ω should be three-dimensional, something like the sphere whose poles are P' and P''. Without going so far as to question assumption a) above, we shall (in Section 5) question both b) and c). In the present Section we shall keep b) and question c). That is, we shall suppose that Ω is fixed but make Ω nearly one-dimensional, namely choose for it a cylinder of radius 2η and axis $P'P''$, or make it strictly one-dimensional, namely (for reasons of symmetry) the segment $P'P''$.

When Ω is $P'P''$, the results are more complex than when Ω is Obukhov's sphere, because in the limit $\eta \to 0$, the probability of $P'P''$ hitting turbulence depends on the value of D. When $D<2$, we know this probability is zero. When $D>2$, we know it to be positive because of the nondegeneracy of linear cross-sections, and it turns out that the expression familiar from Section 1 continues to be valid: the hitting probability is approximately equal to $(r/L)^{3-D}$. As a result, the dependence of $\langle(\Delta\mathbf{u})^2\rangle$ on L/r and of $E(k)$ upon Lk goes as in Section 1, except for a single change, a vital one. Here the coefficients E_0 and $V_{1/3}$ remain positive if $D>2$, but vanish if $D\leq 2$. In particular, the exponent of r is restricted to the narrower range of values between $2/3$ and 1, and the exponent of k^{-1} always lies above the Kolmogorov value $5/3$, but below the "Burgers" value 2. The latter constitutes the bound corresponding to the stage when all turbulent diffusion *within the segment* $P'P''$ reduces to a few blobs.

3. DISSIPATION GENERATED BY WEIGHTED CURDLING

As was the case for the correlation in Chapter III, the formal argument can be borrowed from Monin & Yaglom 1975, with one exception: just like in Chapter III, one *must not*, and we *shall not*, replace $\log W$ by its Gaussian approximation. The exact result, supposing that $r>>\eta$, is as follows

$$\langle \varepsilon_r r^{2/3}\rangle = V\, r^{2/3}\, [\langle W^{2/3}\rangle]^{\log_\Gamma (L/r)}$$
$$= V\, r^{2/3}\, (L/r)^{-B},$$

with $B = -3f(2/3)$, and

$$E(k) = E_0\, L^{-B}\, k^{-5/3-B}.$$

To evaluate $f(2/3)$, we shall return to the determining function $f(h)$. By convexity, the $0\leq h\leq 1$ portion of the graph of $f(h)$ lies between the h axis and the tangent to $f(h)$ at $h=1$, whose slope is equal to $f'(1) = (3-D)/3$.

As a result, given any value $D<3$, B can range from the maximum value $B = 3f'(1)/3 = (3-D)/3$ (obtained in the fractally homogeneous case) down to 0. (It is possible to show that this last value cannot be attained, but can be approached arbitrarily closely. So it is conceivable, however unlikely, that intermittency should bring no change to the $k^{-5/3}$ spectral density.)

The general inequality $B \leq (3-D)/3$ generalizes the equality $B=(3-D)/3$ valid in the fractally homogeneous case. The value corresponding to a lognormal W is (as known to Kolmogorov 1962) $\mu/9$. Written in terms of $D = 3-\mu/2$, it yields $B = (3-D)/4.5$. This value confirms that the changes in the $2/3$ exponent can be smaller than $(3-D)/3$.

4. THE "BURGERS" THRESHOLD SPECTRUM k^{-2} AND THE DIMENSION $D=2$ ARE RELATED IN ABSOLUTE BUT NOT IN WEIGHTED CURDLING. THIS LAST FACT IS PARADOXICAL.

Formally, the preceding argument is easily generalized to Burgers turbulence and more generally to turbulence with $\Delta \mathbf{u} = |P'P''|^{2H}$. The value $B = -3f(2/3)$ is simply replaced by $B = -3f(2H)$. It follows that the Burgers case $H=1/2$, and this case only, has the remarkable property that $B \equiv 0$. The value of the spectral exponent is independent, not only of D but of the random variable W. In other words, even after Burgers turbulence is made intermittent as a result of curdling, its spectral density continues to take the familiar form k^{-2}.

More generally, the "Burgers threshold" will be defined as the point where the intermittency has the intensity needed for the spectrum to become k^{-2}. It is a well-known fact (exploited in Mandelbrot 1975a) that the k^{-2} spectrum prevails when the turbulent velocity change is due to a finite number of two-dimensional shocks of finite strength. Hence it was expected that one should find that the spectrum is k^{-2} in the case of fractal homogeneity with $D=2$. This dimension marks the borderline between the cases when the segment $P'P''$ does, or does not, have a positive probability of hitting dissipation.

On the other hand, it seems that the logical correspondence between $D=2$ and $B=1/3$ fails in the case of weighted curdling. Example: for $D=2$, the lognormal approximation combined with the choice of Obukhov sphere for Ω yields $E(k) = E_0 L^{-2/9} k^{-17/9}$ with $E_0>0$. Even though (assuming it is confirmed that turbulence satisfies $D>2$) the behavior of the spectrum about $D=2$ has no practical effect, the fact that $17/9<2$ constitutes a paradox that must be resolved. We shall postpone this task to Chapter V.

5. NON-PRESCRIBED DOMAINS Ω IN FRACTALLY HOMOGENEOUS TURBULENCE AND THE ISSUE OF TOPOLOGICAL CONNECTEDNESS

Let us resume the discussion of the choice of Ω, started in Section 3. The use of any *fixed* Ω implies the belief that the mutual interaction between $\mathbf{u}(P')$ and $\mathbf{u}(P'')$ is on the average independent of the fluid flow in between the points P' and P''. It is, however, worth at least a brief consideration to envision interactions propagating along lines, say, of least resistance. In the all-or-nothing fractally homogeneous case, it may well be possible to join P' and P'' by a line Λ such that $\int_\Lambda \varepsilon(\mathbf{x})d\mathbf{x}=0$. If so, one may well argue that $\Delta\mathbf{u}$ should vanish.

One is tempted in this spirit to replace ε_r by $(1/r)$ glb $\int_\Lambda$, a short notation for the product of $(1/r)$ by the greatest lower bound of $\int_\Lambda \varepsilon(\mathbf{x})d\mathbf{x}$ along all lines Λ joining P' to P''. The principle of the new specification of Ω is radically different from $\Lambda = P'P''$, because, if accepted, it would open the door to topology. In particular, two of the shapes to be studied in the turbulence Chapter of the English version of *Fractals* (namely, the Sierpiński sponge and pastry shell) have the same $D>2$ but very different topology. For the former glb $\int_\Lambda=0$ for any P' and P'', while for the latter glb $\int_\Lambda=0$ if P' and P'' lie in the same cutout, and glb $\int_\Lambda>0$ otherwise. Since turbulence does in fact exist so that $\langle(\Delta\mathbf{u})^2\rangle>0$, the acceptance of the Λ that minimizes $\int_\Lambda$ would lead to the following tentative inference: Among all sets of two points P' and P'', selected at random under the constraint that $|P'P''|=r$, sets in which every line from P' to P'' hits the carrier of turbulence must have a positive probability. In other words, the probability of P' and P'' being separated by "sheets" of turbulence must be non-vanishing. The mathematical nature of this tentative inference is entirely distinct from the fractal inequality $D>2$; the latter was metric, while the present one combines topology with probability. A mixture of theoretical argument with computer simulations shows there exist a critical dimension D_0, such that the probability of the set generated by absolute curdling being sheet-like is zero when $D<D_0$ and positive when $D>D_0$. This D_0 is much closer to 3 than to 2.

Further, it is tempting to constrain Λ to stay in the Obukhov sphere, and designate the restricted glb by glb$'\int_\Lambda$. If so, the inference that $\langle \text{glb}'\int_\Lambda\rangle>0$ would involve a combination of topological, probabilistic and metric features; this lead can not yet be developed any further.

The preceding reference to topology is extremely tentative, by far less firmly established than the fractal inequality $D>2$. It accepts without question two results of Kolmogorov and Obukhov: the 1941 link between $\langle(\Delta\mathbf{u})^2\rangle$ and a uniform ε, and the 1962 link between $\langle(\Delta\mathbf{u})^2\rangle$ and the expectation of $\varepsilon^{2/3}$. Moreover, the all-or-nothing fractal homogeneity may well be too flimsy a model to support such extensive theorizing.

V. THE INNER SCALE AND THE DISSIPATIVE RANGE.

Thus far the existence of actual dissipation was only acknowledge indirectly, by introducing an inner scale η which, like L, was arbitrarily imposed from the outside. We shall now dig deeper into the classical result of Kolmogorov: Taylor homogeneous turbulence with the viscosity ν and a uniform rate of dissipation ε, the dissipative range is vanishingly narrow around the inverse of $\eta_3 = \nu^{3/4}\varepsilon^{-1/4}$. One reason for the notation η_3 is that the letter η was used up above; a more consequent reason will appear momentarily. One aspect of η_3 is that, if the spectrum $E(k) = E_0 \varepsilon^{2/3} k^{-5/3}$ is truncated at an appropriate numerical multiple of $k = 1/\eta_3$, the relationship $\varepsilon = \nu \int_{1/L<k<1/\eta} k^2 \; E(k)\, dk$ becomes an identity. In this Chapter we shall first examine formally the changes due to intermittency. Then we shall proceed to an actual analysis of the inner scale of curdling. In the fractally homogeneous case, the inner scale will continue to be defined as the inverse of the spectrum's truncation points. This result had already been obtained by Novikov & Stewart, but it deserves a more careful analysis. In all other cases the result is *quite different*. The analysis will show the necessity of a dissipative range that does *not* reduce to the neighborhood of any single value $1/\eta$, but has a definite width determined by the value of $f''(1)$.

1. TRUNCATION POINT FOR THE POWER-LAW SPECTRUM

From Chapter IV, the spectral density of velocity in intermittent turbulence is of the form $E(k) = E_0\, \varepsilon^{2/3} k^{-5/3} (Lk)^{-B}$. Suppose we want the relationship $\varepsilon = \nu \int_{1/L<k<1/\eta} k^2 E(k) dk$ to continue as an identity. Then, up to numerical factors, one must have $\eta = \eta'_D$, where η'_D is defined by

$$\varepsilon = \nu\, \varepsilon^{2/3}\, L^{-B}\, \eta'_D{}^{-4/3-B}$$

$$\eta'_D = [(\nu^3/\varepsilon)\, L^{-3B}]^{1/(4-3B)}$$

$$\eta'_D/L = (\eta_3/L)^{1/(1-3B/4)}.$$

Since $0 \le B \le 1$, we find that $\eta'_D << \eta_3$. For given D, η'_D is a monotone function of B. Thus, when B reaches its maximum value $B = (3-D)/3$, η'_D reaches its minimum value $L(\eta_3/L)^{4/(D+1)}$ and $1/\eta'_D$ reaches its maximum. Note that, in addition to ν and ε, the value of η'_D depends upon L.

More generally, if one stays within a sub-domain of length scale r, much smaller than L, in which the average dissipation is ε_r, one will have the new inner scale $\eta'_D(r)$ such that

$$\nu^{3/4}\, \varepsilon_r^{-1/4}/r = [\eta'_D(r)/r]^{1-3B/4}.$$

2. CRITIQUE OF AN INNER SCALE OF INTERMITTENT TURBULENCE TENTATIVELY SUGGESTED BY KOLMOGOROV

The need to reexamine the concept of inner scale had already been felt by Kolmogorov (1962). On p. 83 of this work (seventh formula) he suggested for this role the expression $\nu^{3/4}\langle\varepsilon_r\rangle^{-1/4}$ which occurs in the left-hand side of the last formula of the preceding Section. He did not explain his choice, and made no further use of it. Actually, it seems hard to retain. A first odd feature of his definition is that when $r = L$, his modified inner scale reduces to η_3. Hence, contrary to η_D, it is independent of the degree of intermittency. A second odd feature relates to $r \to 0$. To describe it, let us follow Kolmogorov in assuming $\log\varepsilon_r$ to be lognormal, with the variance $\mu \log (L/r)$ and a mean adjusted to insure that $\langle\varepsilon_r\rangle = \varepsilon$. It follows that

$$\langle\varepsilon_r^{-1/4}\rangle = \varepsilon^{-1/4} \exp[(1/2)(-1/4)(-5/4)\mu\log(L/r)]$$
$$= \varepsilon^{-1/4} (L/r)^{5\mu/32}.$$

Hence, as $r \to 0$, the modified Kolmogorov scale *increases* on the average and may exceed r. We shall not attempt to unscramble this concept.

3. INNER SCALE OF CURDLING IN THE FRACTALLY HOMOGENEOUS CASE

The truncation of $E(k)$ shows that the energy cascade must stop when reaching eddies on the order of magnitude of η'_D. But what about the curdling cascade? It too must have an end, to be followed by dissipation. We shall now identify the scale η_D at which it stops.

In the fractally homogeneous case, η_D turns out to be identical to the η_D' defined through $E(k)$. Consider, indeed, a cube of side L filled with a Taylor homogeneous turbulent fluid of viscosity ν, dissipation ε and inner scale η_3. Since we assume that the increasingly small curds created by a Novikov-Stewart cascade are themselves Taylor homogeneous; these curds are endowed with a classical Kolmogorov scale varying with the cascade stage. We assume moreover that the instability and breakdown leading to curdling are encountered if and only if the curd size exceeds the Kolmogorov scale. (This assumption can be seen to be equivalent to a little-noticed condition of Novikov & Stewart, as reported in Monin & Yaglom 1975, p. 611.)

The first curdling stage leads to curds of side L/Γ in which dissipation is equal to either 0 or $\varepsilon\,\Gamma^{3-D}$. In the empty cells, the Kolmogorov scale is infinite, and of course further curdling is impossible. In the first stage curds, the inner scale is $\eta^{(1)} = \eta_3\Gamma^{-(3-D)/4}$. In the m-th stage curds, the average dissipation is $\varepsilon\Gamma^{m(3-D)}$, the curd size is $L\Gamma^{-m}$, and the inner scale is therefore $\eta^{(m)} = \eta_3\,\Gamma^{-m(3-D)/4}$. We see that the inner scale and the curd size both decrease with $1/m$. Our postulate being that there is no further curdling after these two scales meet, we are left with the criterion $\eta_3\,\Gamma^{-m(3-D)/4} \sim \Gamma^{-m}L$, i.e., $\eta_3/L = [\Gamma^{1-(3-D)/4}]^{-m}$. The solution turns out to yield

$\Gamma^{-m} L = \eta_D$, with η_D identical to the η'_D obtained earlier in this Chapter through the truncation of $E(k)$. Hence the fractal dimension rules not only the manner in which Novikov-Stewart curdling proceeds, but the point where it stops. We find, in addition, that it is reasonable to assume that the cutoff of $E(k)$ near $1/\eta_D$ is very sharp.

Digression concerning curdling in spaces of Euclidean dimension $\Delta > 3$. The derivation of η_D has relied on the fact that, in a space of Euclidean dimension $\Delta = 3$, the decrease in η_m is less rapid than the decrease in curd size. However, this last feature is highly dependent upon $\Delta - D$, and therefore upon the value of Δ. As in many other fields of physics, a qualitative change may be observed when $\Delta \neq 3$. Indeed, our stability criterion readily yields the result that a nonvanishing inner scale *need not exist*. It exists if and only if $\Delta - D < 4$. Its value is given by the relation

$$(\eta_\Delta / L) \sim (\eta_D / L)^{1-(\Delta-D)/4}.$$

The necessary and sufficient condition $\Delta - D < 4$ for the existence of a non-vanishing inner scale is peculiar but not very demanding. One amply sufficient condition is $\Delta < 4$. (However, in order that curdling continue forever, meaning $\eta_D = 0$, the converse condition $\Delta > 4$ is only necessary, *not* sufficient.) Another amply sufficient condition for $\eta_D > 0$ is $\Delta - D < 1$, which we know expresses that linear cross-sections are *not* almost surely empty. These various conditions make it clear that a vanishing inner scale can at most be observed for phenomena that are very much sparser than the turbulent dissipation presently under study. Much sparser even than the Leray-Scheffer conjectural singularities of the Navier-Stokes equations.

Nevertheless, odd as the result may be, our criterion does indicate that, when $\Delta - D > 4$, *a curdling cascade will continue forever, without any physical cutoff, even when the viscosity is positive.* I do not know what this result means, and what its implications concerning dissipation are. It seems to be trying to tell us something about the singularities in the ultimate solution of the equations of motion of some physical system, but I cannot guess of which one.

4. INNER SCALE OF CURDLING WHEN INTERMITTENCY IS GENERATED BY WEIGHTED CURDLING FIRST ROLE OF $f''(1)$.

In the case of weighted curdling, as we shall now proceed to show, the inner scale is best studied in two approximations. The first one yields a single typical value. The interesting fact is that this value turns out to be much smaller than the quantity η'_D obtained through the truncation of $E(k)$. The second approximation shows that said typical value is not very significant and that one must deal with a whole statistical distribution. Strictly speaking, the same situation had already prevailed in all-or-nothing curdling leading to fractally homogeneous intermittency, but in that case $\eta^{(m)}$ was simply binomial, equal to either $\eta_3\Gamma^{-m(3-D)/4}$ or infinity and the latter value could be neglected. The same cannot be done in the case of weighted curdling.

Recall that, if the curdling cascade could continue forever, the dissipation density $\varepsilon(\mathbf{x})$ at the point $\mathbf{x}$ would be a product of weights W, one per cascade stage. It can be written in the form $W_{i_1} W_{i_1 i_2} W_{i_1 i_2 i_3}$ ...., where the real number $0, i_1 i_2$ ...... designates $\mathbf{x}$ in the counting base C, and the W's are an infinite sequence of independent random weights. Similarly, $\eta^{(m)}$ will be written simply in the form $\eta_3[\Pi_{1\leq n\leq m} W_n]^{-1/4}$, Curdling will stop when this random $\eta^{(m)}$ first overtakes the nonrandom $\Gamma^{-m}L$. Taking logarithms, we find that m is the first integer where

$$\Sigma_{1\leq n\leq m}[(-1/4)\log W_n + \log\Gamma] = \log[L/\eta_3].$$

After we select a probability distribution for W, the left-hand side of the above expression will define a random walk with nonrandom drift equal to $^z\log\Gamma-\log W/4^x$ and an absorbing barrier. The above-defined value of m is therefore merely an instant of absorption or, alternatively, of ruin. Absorption will occur almost surely because the drift turns out to be positive (digression: this is so as long as $\Delta-D<4$).

First approximation. When $L/\eta_3>>1$, the drift tends to overwhelm the randomness, and one can approximate m by the value $m*$ obtained by the rough approximation which consists in replacing the random walk by its expectation. The proper choice of weights in the above expectation is not obvious, but there is room only to state the result without a full justification. Moreover, in order to avoid irrelevant notational complication, we add the assumption that the values w_g of W are discrete with probabilities p_g. Then the proper intrinsic probability of w_g is not given by p_g itself. Rather, it can be shown to take the form $p_g w_g$. Since $\langle W\rangle = 1$, $\Sigma p_g w_g = 1$; therefore the $p_g w_g$ are acceptable as probabilities. Continuing to use $\langle\,\rangle$ to designate expected values under the probabilities p_g, our criterion yields

$$\eta_3/L = [\Gamma(\exp\langle W\log W\rangle)^{-1/4}]^{-m*}$$

The result stated in the last form turns out to apply also to nondiscrete W's. Since $-\langle W\log_\Gamma W\rangle = D-3$, the definition of η_D reduces formally to that applicable in the fractionally homogeneous case.

Summary of the first approximation. In weighted curdling the order of magnitude of m is the same as in the all-or-nothing curdling having the same value of D, hence of $f'(1)$. In particular, the order of magnitude of $1/\eta_D$ is much *greater* than the $1/\eta'_D$ deduced in Section 1.

Second approximation. The actual values of m scatter around $m*$. For fixed W, the scatter increases with L/η_3. For fixed L/η_3, it is useful to define a standard scatter, to be denoted by σm. It is approximately the ratio of two factors. The first is the standard deviation of the sum of $m*$ factors of the form $-\log W/4$. The variance of W is $\langle W\log^2 W\rangle - \langle W\log W\rangle^2$, which happens to be equal to $\log C\ f''(1) = 3\log\Gamma\ f''(1)$. Hence, the first factor is equal to $[3m*\log\Gamma f''(1)]^{1/2}/4 = [3\log(L/\eta_3)/(1-(3-D)/4)]^{1/2}/4$. The second factor is the expected value of $\log\Gamma - \log W/4$, that is $\log\Gamma - f'(1)\log C/4 = \log\Gamma[1-(3-D)/4]$. Combining the two factors, we obtain the two alternative forms

$$\sigma m = (2/\log\Gamma)(1+D)^{-3/2}\,[3\log(L/\eta_3)]^{1/2}\,[F''(1)]^{1/2}$$
$$= (3/\log\Gamma)^{1/2}(1+D)^{-1}[m*]^{1/2}[f''(1)]^{1/2}$$

This is the first time that the value of $f''(1)$ enters in the present discussion. The value of $f'(1)$ enters also, through D, but the result is not very sensitive to it.

5. THE DISSIPATIVE RANGE.

The methods used in Chapters III and IV to evaluate exponents and exponent changes only apply to scales for which curdling has a small probability of having stopped, that is, roughly, from $(1/L)$ to $k\sim(1/L)\Gamma^{m*-\sigma m}$. Going towards higher wave numbers, one encounters next the range from $k\sim(1/L)\Gamma^{m*-\sigma m}$ to $k\sim(1/L)\Gamma^{m*+\sigma m}$. Here, some dissipation is likely to occur in a substantial region of our fluid.

Let us make a few more comments on this topic. By the last result of the preceding Section, the width of the dissipative range, measured in units of $\log_\Gamma k$, is proportional to $[f''(1)]^{1/2}$. When $\log W$ is lognormal, $f(h)$ is parabolic and $f''(1)$ is proportional to μ. More generally, unless the distribution of W is very bizarre, one has approximately $f(2/3)\sim f(1) - f'(1)(1/3) + f''(1)(1/3)^2/2$. That is, $(3-D)/3-B \sim f''(1)/6$. This relation holds even if B and $(3-D)/3$ do not bear to each other any numerical relationship of the kind that holds when W is lognormal and $B = (3-D)/4.5$. In other words, the width of the dissipative range – measured on the $\log_\Gamma k$ scale – is typically the square root of the defect of B with respect to the fractally homogeneous approximation.

It was to be expected that each of these quantities should be a monotone increasing function of the other. Indeed, the inequality $1/\eta_D >> 1/\eta'_D$ expresses that the spectrum $k^{-5/3-B}$ relative to the inertial range cannot be extrapolated consistently. The corresponding distribution of energy among the wave numbers decreases very much too slowly as k increases, which implies that the whole energy would be completely exhausted well before reaching $k \sim 1/\eta'_D$. The greater the difference $1/\eta - 1/\eta'_D$, the sooner must this inertial range law $k^{-5/3-B}$ cease to apply.

The expressions that apply in the dissipative range and replace coefficients such as B, will be described elsewhere.

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FIGURE 1. PLANAR FRACTAL OBTAINED BY ABSOLUTE CURDLING.

Random curdling proceeds on a square grid. We show the effect of four stages, each of which begins by dividing the cells of the previous stage into $5^2 = 25$ subcells, then "erasing" 10 of them to leave the remaining 15 as "curds".

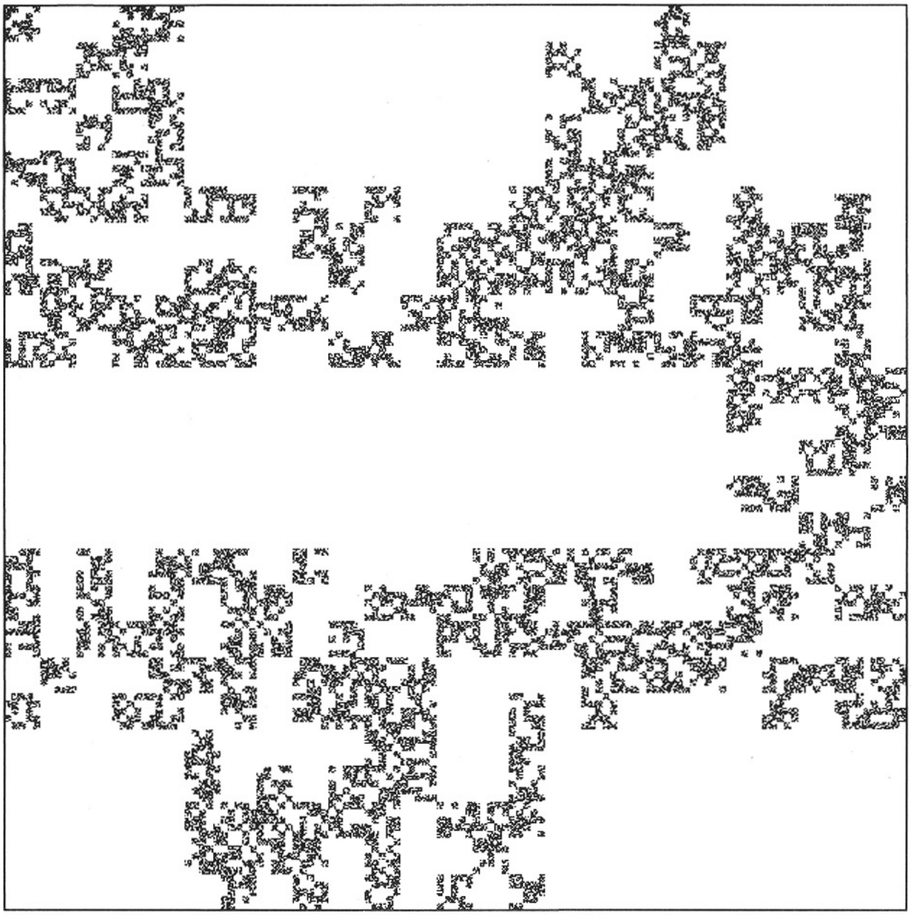

FIGURE 2. THE SELF SIMILARITY DEFINITION OF THE FRACTAL DIMENSION.

A segment of line can be paved by – and is therefore equivalent to – $N=5$ replicas of itself reduced in the ratio $r=1/5$. A square is equivalent to $N=25$ replicas of itself reduced in the ratio $r=1/5$. The same property of self similarity is obviously encountered in the pattern of Figure 1: it is equivalent to $N=15$ replicas of itself reduced in the ratio $r=1/5$. In each of the classical cases, the concept of dimension can be associated with self similarity, and one has $D=\log N/\log(1/r)$. The point of departure of fractal geometry is that this last expression a) remains well defined and b) happens to be useful for all self-similar sets such as the pattern of Figure 1, and that it is not excluded for D to be a fraction.

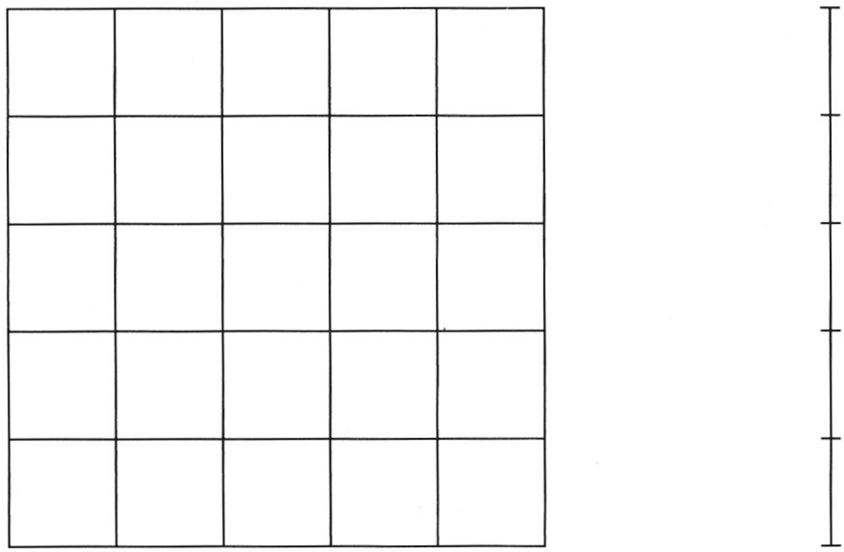

FIGURE 3. THE DETERMINING FUNCTION f(h).

The two lines represent two determining functions 3f(h) which yield the same value of $3-D=.45$. The present paper concentrates upon the roles played in the theory of intermittency by the quantities f(2/3), f(2) and f''(1). Earlier work, Mandelbrot 1974a,b, had concentrated on the role played by f'(1) and the α's (the latter are not shown here). The lower line in the present Figure is straight of equation $3f=.45(h-1)$. It corresponds to fractally homogeneous turbulence and is the lowest compatible with the given D. The upper line, which is the parabola $3f=.45h(h-1)$, corresponds to lognormal intermittency with $\mu=.9$. For other forms of curdling, the determining function can lie between the above lines or even higher than the parabola. Two examples are of interest. The fractally homogeneous case can be changed so that the value 0 is replaced by some scatter of values slightly above it, while the value of 1/p is slightly changed to keep D invariant. Alternatively, the lognormal can be truncated sharply. In either case, the resulting line f(h) will be approximately straight for abscissas to the right of $h=1$ and approximately parabolic to the left.

The concept of approximation used in the bulk of probability theory is of little value in the present context. A random variable W' may be a close approximation to W and still lead to a markedly different determining function and hence to a markedly different form of intermittency.