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Statistics

Stochastic models for the Earth's relief, the shape and the fractal dimension of the coastlines, and the number-area rule for islands

(mathematical geomorphology/Hausdorff dimensions/Brownian random surfaces/Gaussian processes)

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ABSTRACT **The degree of irregularity in oceanic coastlines and in vertical sections of the Earth, the distribution of the numbers of islands according to area, and the commonality of global shape between continents and islands, all suggest that the Earth's surface is statistically self-similar. The preferred parameter, one which increases with the degree of irregularity, is the fractal dimension, *D*, of the coastline; it is a fraction between 1 (limit of a smooth curve) and 2 (limit of a plane-filling curve). A rough Poisson-Brown stochastic model gives a good first approximation account of the relief, by assuming it to be created by superposing very many, very small cliffs, placed along straight faults and statistically independent. However, the relative area predicted for the largest islands is too small, and the irregularity predicted for the relief is excessive for most applications; so is indeed the value of the dimension, which is $D = 1.5$. Several higher approximation self-similar models are described. Any can be matched to the empirically observed *D*, and can link all the observations together, but the required self-similarity cannot yet be fully explained.**

Geometry arose from the description of measurements of the surface of the Earth. The two disciplines split apart almost immediately, but geomorphology is bringing them together. One broad task is to sort out two elements, which are best contrasted using electrical engineering terminology. (*a*) The first is a "signal," defined as a reasonably clear-cut feature one hopes to trace to a small number of tectonic, isostatic, or erosional causes. (*b*) The second is a "noise," defined as a feature one believes is due to many distinct causes that have little chance of being explained or even disentangled. This paper is mainly directed towards this "noise" aspect. The fit between the models and the empirical relationships they aim to represent will prove good, and even surprising, since there was no objective way to tell *a priori* whether the relationships to be represented are indeed mostly noise-related. In fact, even the simplest model will generate ridges, of a kind one is tempted *a priori* to classify as signal-like. Thus, our study of noise will end up by probing the intuitive distinction between it and the signal.

The strategy and the tactic to be used are in part very familiar, having proven successful in taming the basic electric noises.* The first novel aspect of the present problem (and a major difficulty) is that it deals not with random curves, but surfaces. The second novel aspect follows as a consequence. Since an acoustic or an electric noise is not visible (save as a drawing on a cathode ray tube), geometric concepts do not enter in its study until late and in an abstract fashion. For the Earth's relief, the opposite is true. However, ordinary geometric concepts are hopelessly underpowered here; new mathematics will be needed. For our purposes, remembering that the Earth is roughly spherical overall would only bring insignificant corrections. We shall therefore assume that, overall, the Earth is flat with coordinates x and y.

GOALS

Quantified Goals. To be satisfactory, a model will be required to either explain, or at least relate to each other, the following theoretical abstractions from actual observations. (*a*) Korcak's empirical number-area rule for islands (1): the relative number of islands whose area exceeds A is given by the power-law $\Phi(A) \sim A^{-K}$. A fresh examination of the data for the whole Earth yields $K \sim 0.65$. More local (and less reliable) estimates using restricted regions range from 0.5 for Africa (one enormous island and others whose sizes decrease rapidly) up to 0.75 for Indonesia and North America (less overwhelming predominance of the largest islands). (*b*) The concept that, even though coastlines are curves, their wiggliness is so extreme as to be practically infinite. For example, it is not useful to assume that they have either well-defined tangents (2), or a well-defined finite length (references are given in ref. 3). Specific measures of the length depend upon the method of measurement and have no intrinsic meaning. For example, as the step length G of a pair of dividers one "walks" along the coast is decreased, the number $N(G)$ of steps necessary to cover it increases faster than $1/G$. Hence, the total distance covered, $L(G) = GN(G)$, increases without bound. (*c*) Richardson's empirical power law $N(G) \sim G^{-D}$. This D is definitely above 1 and below 2; it varies from coast to coast, a typical value being $D \sim 1.3$ (references in ref. 3). (*d*) Mandelbrot's proposal (3) that it is useful in practice to split the concept of the dimension of a coastline into several distinct aspects. Being a curve, it has the topological dimension 1, but the behavior of $L(G)$ suggests that from a metric viewpoint it also has a "fractal dimension," equal to Richardson's D. Curves of fractional (Hausdorff) dimension have been known for over half a century, as an esoteric concept in pure mathematics, until Mandelbrot (3) injected them into geomorphology. The notion of fractal dimension has also acquired applications in several other empirical sciences (see refs. 4 and 5). Implicit in ref. 3 was the further concept that the surface of the Earth has the dimension $D + 1$, contained between 2 and 3. (*e*) Vertical sections of the Earth have been studied less thoroughly than the coastlines, but a model of the Earth should embody whatever is known on their behalf (6).

* An opinion, that may be shared by other readers, was expressed by a referee who criticized my approach for "a narrow focus on purely tectonic processes of the simplest kind and a belief that (*i*) it is both good and important to make stochastic models whose realizations agree with the largest scale behavior, and (*ii*) if this can be done, it is right and wise to think of these largest scale phenomena as in fact stochastic." Lacking space to reply in detail, I shall simply say that, while the validity of models of the kind I shall describe must eventually be discussed with great care and skepticism, I see only benefits in first developing them in some detail.

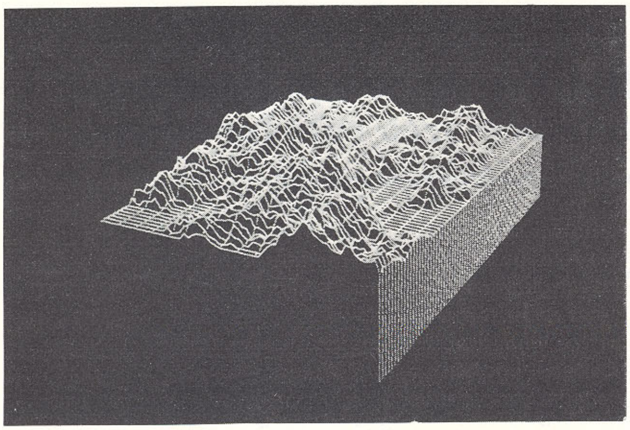

FIG. 1. Perspective view of a sample of a Brownian surface of Paul Lévy, with a sea level included to help enhance detail.

Subjective Goals: Resemblance in External Appearances. Still another property to be explained is obvious but not easy to quantify. Roughly speaking, it is hard to tell small and big islands apart, unless one either recognizes them or can read the scales. One possible germ of an explanation would consist in arguing that the determinants of overall shape are *scaleless,* hence are not signals but noises. (In fact, many islands look very much like distorted forms of whole continents, which is perhaps too good to be believed: why should self-similarity extend to the plates?)

A final goal for the model maker has recently arisen from advances in computer simulation and graphics. The validity of a stochastic model need no longer be tested solely through the quality of fit between predicted and observed values of a few exponents. It is my belief (perhaps a controversial one) that the degree of resemblance between massive simulations and actual maps or aerial views must be treated as evidence.

STRATEGY

First Strategy: An Explicit Mechanism and Its Limit Behavior. *Self-similarity.* Our first strategy is the one used to explain thermal noise in electric conductors through the intermediacy of shot noise, which is the sum of the mutually independent effects of many individual electrons. The analogous "Poisson-Brown" primary model of the Earth has isotropic increments. It is simple, explicit, direct, and intuitive, and, in general terms, does fulfill the goals that have been listed. In particular, it predicts that coastlines are not rectifiable, that the above power laws are valid, and that D must be greater than 1. In fact, the relief that it yields is self-similar. This last concept quantifies the first of the subjective goals listed above. It expresses that the altitude $Z(x,y)$ has the property of spatial homogeneity: take $Z(0,0) = 0$, and pick an arbitrary rescaling factor $h > 0$. Then $Z(hx,hy)$ is identical in statistical properties to the product of $Z(x,y)$ by some factor $f(h)$. Unfortunately, the predicted value $D = 1.5$ is not satisfactory. No single value can represent the relief everywhere, and most observed values are well below 1.5. This discrepancy is confirmed by the excessively irregular appearance of the simulated primary relief and coastlines.

Second Strategy: Self-Similarity and Use of Limits. To do better than Poisson-Brown, as demanded by both numerical and perceptual reasons, one must unfortunately resort to an indirect strategy, more complex, less powerful, and less convincing. *First postulate:* Without attempting to describe any specific mechanism, we preserve the assumption that the noise element in the relief is the sum of many independent contributions. It follows, for example, that the increment $Z(P') - Z(P'')$ between the two points P' and P'', must belong to a very restricted family of random variables. Besides the Gaussian, it includes other members (so far, ill-known in the applications; see below). *Alternative postulate:* We adopt the self-similarity of the relief as an excellent quantification of our first subjective goal, and hence as a summary of many quantified goals. Even after self-similarity is assumed, the identification of an appropriate $Z(x,y)$ continues to pose a challenge. We shall examine several possibilities in order of increasing complication, and show that they make it possible, while staying within the restricted domain defined by the first postulate, to fulfill our various goals in a way that relates them to each other.

THE POISSON AND BROWN SURFACES

A Spatial Construction on the Principle of Poisson Shot Noise. Start with an Earth of zero altitude, then break it along a succession of straight faults, and in each case displace the two sides vertically to form a cliff. The terms "fault" and "cliff" are to be understood in purely geometrical terms, with no tectonic implication. The resulting relief will be denoted by $\Pi(x,y)$. It is convenient to choose an origin (0,0), and to maintain $\Pi(0,0) = 0$, but changing the origin only adds a constant to $\Pi(x,y)$. (This model obviously neglects the basic roles of isostasy and of erosion.)

The positions of the faults and the heights of the cliffs are assumed random and mutually independent, the former being isotropic with a high average density, and the latter having zero mean and finite variance (implying that large values are very rare). A computer simulation is exhibited in Fig. 1, showing a perspective view of the relief, and in Fig. 3B, showing a larger piece of coastline. The quantitative properties of a primary relief have already been summarized by stating it is self-similar with $D = 1.5$. Let us now examine them one by one.

Vertical Sections. The effect of a set of isotropically random and mutually independent faults is named after Poisson because of the property (which could be used as a definition) that their points of intersection with any straight line (parametrized by u) form a Poisson point process $\Pi(u)$ (while the angles of intersection are distributed uniformly between 0 and 2π). Denote the average number of points of intersection per unit length by λ. Each primary vertical section can be said to be a Poisson random walk. It differs from an ordinary random walk because the instants when it moves are not uniform but Poissonian.

Limit Vertical Sections. Divide $\Pi(u)$ by $\lambda^{1/2}$, thus rescaling the cliff heights to make them decrease as their number increases, and then let $\lambda \to \infty$. As is well-known, the distribution of the Poisson steps becomes increasingly irrelevant. By the central limit theorem, $\Pi(u)\lambda^{-1/2}$ tends to a Brownian motion $B(u)$. This limit is a continuous process, which fact expresses that even the highest contributing cliff is swamped into being negligible. The overall resemblance between Brownian and real vertical sections has been pointed out in ref. 8 (p. 435), but other authors must have noted it earlier. It is confirmed by spectral analysis (6).

Relief. $Z = \Pi(x,y)$ can be called a Poisson surface, and $Z = B(x,y)$ is called a Brownian surface. It had been defined by Paul Lévy (9) through the characteristic property that, for every two points P' and P'', $B(P') - B(P'')$ is a zero mean Gaussian random variable of variance $|P'P''|^{2H}$, with

$H = 0.5$. $B(x,y)$ is self-similar with the factor $f(h) = h^{1/2}$. Until the present application, it was known simply as a mathematical curiosity. Its use as a model could have been introduced directly and dogmatically as just another instance of the oft-successful tactic, which approaches every new statistical problem by trying to solve it by the simplest Gaussian process. However, the detour through Poisson faults improves the motivation.

Coastlines. An island is defined as a maximal connected domain of positive altitude. A coastline, being simply a horizontal section of the relief, has the same degree of irregularity as a vertical section. The coast of a Brown island has infinite length, however small its area A; for a Poisson island of area $A \gg \lambda^{-2}$, the total length is very large, of the order of $A^{1/2}\lambda^{1/2}$. In either case, when $G \gg \lambda^{-2}$, one has $N(G) \sim G^{-1.5}$.

The Number-Area Rule for Islands. For islands defined through $B(x,y) \geq 0$, $\Phi(A) = A^{-3/4}$ for all values of A. For islands defined through $\Pi(x,y) \geq 0$, $\Phi(A) = A^{-3/4}$, so long as $A \gg \lambda^{-2}$.

ANISOTROPIC STRETCHING AND ADDITION OF SPECTRAL LINES

A striking feature of sample Brown surfaces (Fig. 1) is the invariable presence of clear-cut ridges. They are merely an unexpected consequence of continuity, but their presence expresses that each sample is grossly nonisotropic. Since these ridges have no privileged direction, they are quite compatible with isotropy of the mechanism by which $B(x,y)$ is generated. If we did not know them to be expressions of noise, we might say they are signals. That is, if we did not know them to be due to the superposition of many effects, we might try to explain them by some single cause.

Nevertheless, they do not remotely have the regularity of (say) the Appalachians or the Andes, which are profoundly nonisotropic. We shall list two easy ways of accounting for them by invoking "signals" superposed upon an primary "noise." *First signal:* It consists in a controlled degree of anisotrophy introduced into either $\Pi(x,y)$ or $B(x,y)$. One may, for example, make the probability of faults greater along some direction than faults along its perpendicular. Alternatively, one can stretch the plane. Either change will make our ridges tend to become parallel to each other and form mountain ranges; by adjusting the degree of stretching, the overall fit can be improved. On the other hand, the values of the parameters K and D would remain unchanged. *Additional signal:* A different approach to the problem of nonisotropy is best explained in spectral terms. The spectrum of $B(P)$ is continuous, with a density proportional to ω^{-2}. Just as in communication technology, the signal may be assumed to take the form of a pure spectral line. It would induce sinusoidal up- and down-swells in the relief, hence a tendency towards parallel ridges.

FRACTIONAL BROWNIAN RELIEF

The most satisfactory model, among those currently available, combines either of the above signals with the following noise.

A Gaussian Secondary Model with Either $1 < D < 1.5$ or $1.5 < D < 2$. We now adopt a second strategy, and fulfill its first requirement by continuing to assume that $Z(x,y)$ is a Gaussian surface, meaning that for any set of points P_n $(0 \leq n \leq N)$, the N dimensional vector of coordinates $Z(x_n,y_n) - Z(x_0,y_0)$ is Gaussian. The combination of isotropy of the increments with self-similarity turns out to require $Z(x,y)$ to be proportional to either the above Brownian function $B(x,y)$, or a generalization, which I propose to call fractional Brownian function and to denote by $B_H(P)$. It is defined by $E\{B_H(P') - B_H(P'')\}^2 = |P'P''|^{2H}$ with $0 < H < \frac{1}{2}$ or $\frac{1}{2} < H < 1$. [If $H = \frac{1}{2}$, $B_H(P) = B(P)$.] The fractional Brownian function of time has served the author in modeling a variety of natural time series (10). The present multiparameter $B_H(P)$ has been fleetingly mentioned in the literature (references in ref. 4), but here it can be applied; selected simulations are illustrated in Figs. 2, 3A, and 3C. To satisfy our numerical goals with any desired D, it suffices to select $H = 2 - D$. If $D \sim 1.3$ and $H \sim 0.7$, the subjective goal of familiarity of appearance is fulfilled also. As to the exponent K, its theoretical value is $K = D/2$ (meaning that the distribution of the typical length $A^{1/2}$ is hyperbolic with the exponent D). The single Earth-wide estimate $K \sim 0.65$ is a compromise between different regions, and indeed it fits the worldwide compromise $D \sim 1.3$. Local estimates for Africa and Indonesia also fit the local estimates of D, and the empirical relationship between D and K seems to be monotone in-

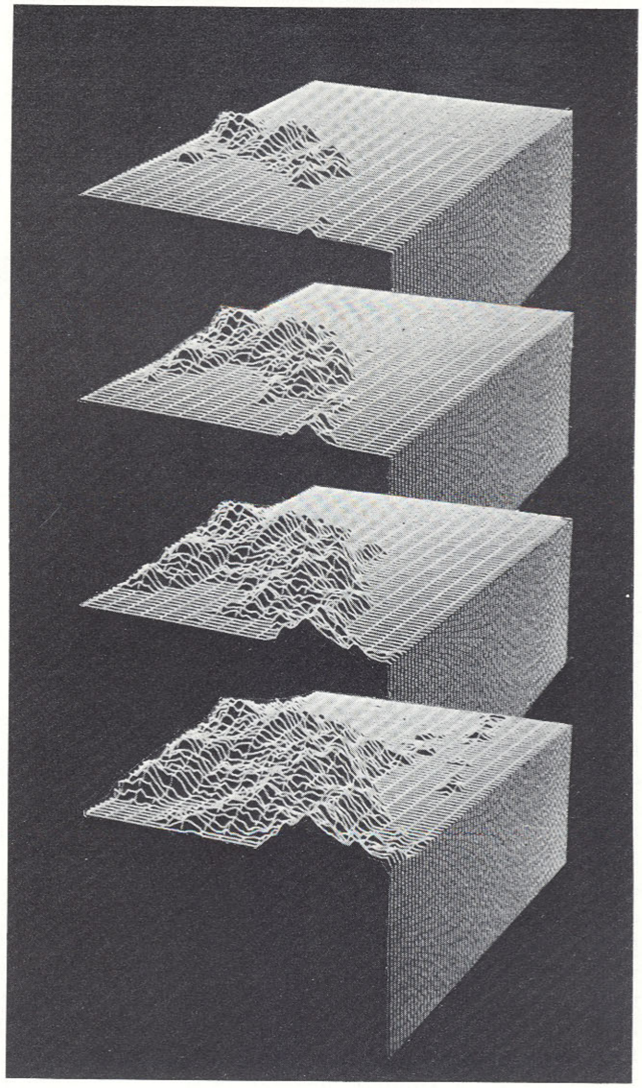

FIG. 2. Several perspective views of a sample fractional Brownian surface for $H = 0.7$ drawn using the same random generator seed as Fig. 1. Letting the sea level recede further enhances the shape of the relief. This $H = 0.7$ gives the best fit from all viewpoints.

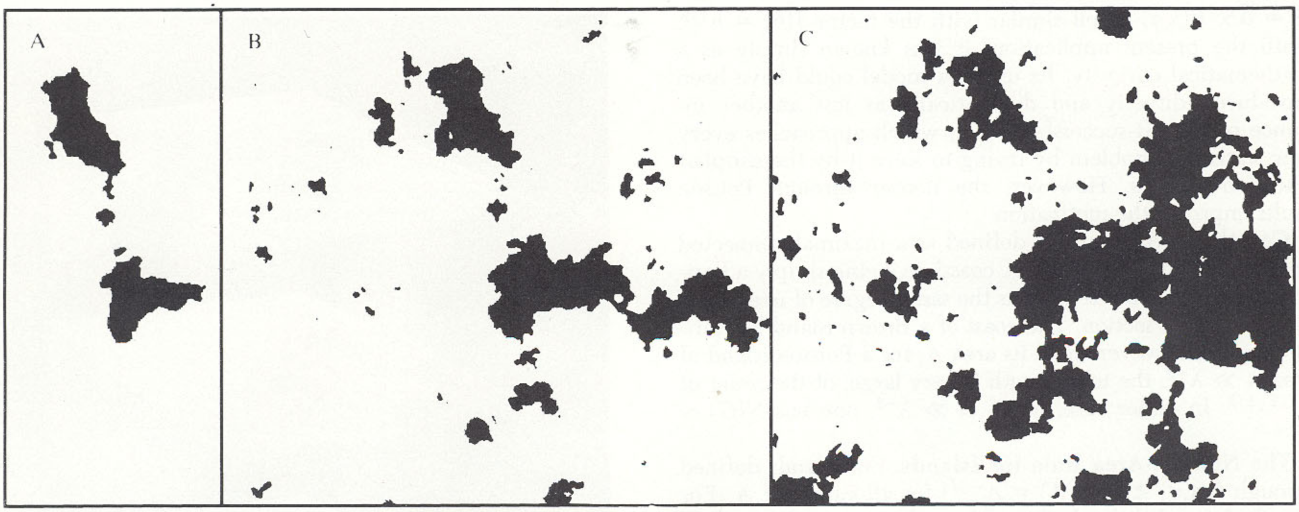

FIG. 3. Several coastlines, defined as the zero level lines of fractional Brownian surfaces corresponding to different values of H, drawn using the same random generator seed. For $H = 0.5$, one has the Brownian surface.

creasing. This feature, if confirmed (an interesting topic for further study), would provide an unexpected link between the local and the very global properties of the relief in different Earth's areas.

After the Fact Partial Rationalization of $B_H(P)$. There are at least two approaches, each of which is more reasonable in different regions of the Earth. *First method:* One notes that the spectrum of $B_H(P)$ is continuous with a density proportional to $\omega^{-(2H+1)}$. When $H > 0.5$, it differs from the ω^{-2} density of $B(P)$ by being stronger in low, and weaker in high frequencies. The replacement of $B(P)$ by $B_H(P)$ could be viewed as due to yet another signal. Its overall effect is in the direction of smoothing; its local aspects could well be associated with erosion, while its global aspects may relate to isostasy. The fact that D varies around the globe, for which we have not yet accounted at all, would result from local variability in the intensity of such erosion. However, none of the common methods of smoothing, such as local averaging, would do, because each only affects a narrow band of frequencies. The smoothing required here must involve a very broad band; and therefore, it would have to combine a whole collection of different narrow band operations. In addition, their relative importance should take a very specific form.

Alternatively, one may obtain $B_H(P)$ directly (this approach is used and described in ref. 4), by resorting to cliffs with a very special kind of profile. They must rise very gradually but forever on both sides of each fault.

A NON-GAUSSIAN SECONDARY MODEL

Though $B_H(P)$ gives a surprisingly good phenomenological description of the relief, it is continuous, which implies it will not fit some data. However, discontinuity, if required, happens to be within easy grasp. It suffices to proceed to random surfaces that follow one of the non-Gaussian distributions that may apply to sums of many independent addends, namely, a stable distribution of Paul Lévy. They can be injected into the Poisson model (see above) by making cliff heights have infinite variance and fulfill other requirements. The resulting Poisson-Lévy model cannot be described here. It suffices to say that the largest contributing cliff no longer becomes relatively negligible as the number of contributions increases, but continues to stand out. Without question, we will be tempted to interpret it as a signal.

GENERALIZATIONS

The above models not only are closely related to some recent work in turbulence (4), but have many other fairly immediate applications. They are readily translated to account for such phenomena as the distribution of minerals and of oil.

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