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H26

Limit theorems on the self-normalized bridge range

• ***Chapter foreword.*** The paper's original title was "Limit theorems on the self-normalized range for weakly and strongly dependent processes." The second half was dropped because of the changed perspective described in Chapter H5. However, theorems are unaffected by changes of perspective.

A glance shows that this paper is addressed to mathematicians. Physicists or engineers will wonder why one needs so many different notions of limit (as defined in Section 1) and so many distinct function spaces. The intuitive meaning of *weak* or *vague* convergence is discussed on page 327 of M 1982F{*FGN*.}

Footnote 3 became Section 13.5. Other footnotes were integrated into the text at the point where they were referred to. Subsection headings were added. *Bridge* is used far more freely than in the original. *Self-similar* was changed to the correct *self-affine.* The letter *d* was replaced by δ, and – for the sake of the nonprobabilists – the symbol for expectation was changed from E to $\mathscr{E}$. In the examples at the end of Section 8, three serious typographical errors were corrected. •

✦ **Abstract.** $X(t)$ being a random function of time, one can write

$$X_\Sigma(t) = \int_0^t X(s)ds;\ \ R(\delta) = (\sup - \inf)_{0 \le u \le \delta}\{X_\Sigma(u) - (u/\delta)X_\Sigma(\delta)\};$$
$$S^2(\delta) = \delta^{-1}[X_\Sigma(\delta)] - \delta^{-2}[X^2]_\Sigma(\delta);\ Q(\delta) = R(\delta)/S(\delta).$$

R is a bridge range, and Q is a self-normalized range. For certain r.f. $X(t)$, one can select the weight function $A(\delta; Q)$ so that (in a sense to be specified) the r.v. $Q(\delta)/A(\delta; Q)$, or even the r.f. $Q(e^{\Phi}\delta)/A(e^{\Phi}\delta; Q)$, has a

nondegenerate limit as $\delta \to \infty$. When such an $A(\delta; Q)$ exists, it is the key factor in a new statistical technique called *R/S analysis.* This paper describes some aspects of R/S analysis: theorems (easy to prove but unexpected) concerning weak convergence of certain r.f.. Other aspects of R/S analysis still involve conjectures based on heuristics and computer simulation.

For iid processes that either satisfy $\mathscr{E}X^2 < \infty$ or are attracted to a stable process of exponent α, it is shown that $A = \sqrt{\delta}$ independently of α. For processes that are locally dependent (e.g., Markov or autoregressive) one still has $A = \sqrt{\delta}$. Conversely, whenever $A = \sqrt{\delta}$, the r.f. $X(t)$ will be said to have a finite R/S memory.

On the other hand, suppose $X(t)$ are the finite increments of a proper fractional Brownian motion (FBM) $B_H(t)$, which is defined as the fractional integral of order $H - 0.5 \neq 0$ of ordinary Brownian motion. In that case, one has $A = \delta^H$. The r.f. $B_H(t)$ is globally dependent rather than strongly mixing and it can be said to have an infinite memory. Conversely, whenever A exists but $A \neq \sqrt{\delta}$, the r.f. $X(t)$ will be said to have an infinite R/S memory. When $A = d^H L(\delta)$, with $H \neq 0.5$ and $L(\delta)$ a slowly varying function, H will be called the R/S exponent of long-range dependence. ✦

1. INTRODUCTION TO *R/S* ANALYSIS

Let $X(t)$ be an integrable function of real time t. Functions of discrete time are interpolated as right continuous step functions of real t. For every real t and for every real $\delta \geq 0$ (to be called the lag), define the functions

$$X_\Sigma(t) = \int_0^t X(u)du;$$

$$R(\delta) = (\sup - \inf)_{0 \leq u \leq \delta}\{X_\Sigma(u) - (u/\delta)X_\Sigma(\delta)\};$$

$$S^2(\delta) = \delta^{-1}[X^2]_\Sigma(\delta) - \delta^{-2}\{X_\Sigma(\delta)\}^2;$$

$$Q(\delta) = \frac{R(\delta)}{S(\delta)}.$$

These functions are, respectively, the integral of X, the range of the bridge of X_Σ, a variance, and a self-normalized range.

1.1. Elementary properties of Q

Q takes the same value for the function $X(t)$ and for all functions of the form $\sigma[X(t)+\mu]$; thus, σ and μ need not be known to evaluate Q. Also, the first δ values of Q can be evaluated directly from the first δ values of $X(t)$. When $X(t)$ is a r.f. of time, $Q(\delta)$ is a r.f. of δ.

Q is well-determined, except when $S=0$, which is the case if and only if the $X(u)$ are identical for $0<u\leq\delta$; in this exceptional case, $R(\delta)=0$ and Q takes the indeterminate form $Q=0/0$. An indeterminate form $R/S=0/0$ is always encountered when time is discrete and $\delta=1$. When $X(t)$ is random and is not a.s. constant, the probability of finding a string of δ identical values of $X(u)$ tends to 0 with $1/\delta$. When time is continuous and Q is determined, $0\leq Q\leq\delta/2$. When time is discrete and Q is determined, the lower limit of Q is 1 when δ is even and is $\delta/(\delta-1)$ when δ is odd. Its upper limit is $\delta/2$ when δ is even and is $\sqrt{\delta^2-1/2}$ when δ is odd. The upper and lower limits are attainable and coincide for $\delta=2$, where $Q\equiv 1$. Roughly, $1\leq Q\leq\delta/2$, meaning that log Q, as a function of log δ, lies in an eighth of a plane with apex at $d=2$, $Q=1$.

1.2. Historical background

To the best of my knowledge, the statistic Q was first used in Hurst 1951, 1955 and Hurst et al 1965. Harold Edwin Hurst was an English physicist working in Cairo, who originated the idea of the Aswan High Dam and, as possibly the greatest Nilologist of all time, was nicknamed Abu Nil, the Father of the Nile. The numerator R was suggested by an old method of preliminary design of water reservoirs due to Rippl, and Hurst injected the denominator S as a natural normalizing factor, with no indication that its special virtues – to be described – had been noticed.

Steiger 1964 also defined R/S – independently of Hurst – but did not pursue the matter. There is a coincidental similarity between this self-normalization and one used in Logan et al 1973 in a work entirely independent of the present one. Logan et al attacks questions relative to independent $X(t)$; by hard analysis, he achieves strong results whose counterpart lies beyond my present technical capability. But we shall see that this counterpart is not indispensable to the practical application.

My interest in Hurst's work first led to M 1965h{H9}. Since 1968, the $\delta\rightarrow\infty$ behavior of Q has been the object of intensive study and has opened up the new field of R/S analysis; see M 1970e{H30}, 1972c{H30}, M & McCamy 1970{H28}, M & Wallis 1968{H10}, 1969a{H12,13,14}, 1969b{H27}, 1969c{H25}, Taqqu 1975.

1.3. The Hurst phenomenon: a "paradox" or a "puzzle" concerning nature

In the practice of R/S analysis, one wishes to utilize as fully as possible the information available in a sample. Therefore, one defines $R(t, \delta)$ and $S^2(t, \delta)$ for each "starting point" t, applying the above formulas to the translates $X(t+\delta)$ of the original $X(t)$, viewed as a r.f. of δ. When $X(t)$ is a stationary r.f., then, for each fixed value of the lag, $Q(t, \delta)$ is another stationary r.f. of t and is dependent on δ in distribution.

In his first application of R/S, Hurst was working with the Nile and then with other rivers and other empirical records; he estimated $\mathscr{E}Q(\delta)$ by averaging the values of $R(t, \delta)/S(t, \delta)$ for several t's and found that, roughly speaking, $\mathscr{E}Q$ "fluctuates around" δ^H, with H "typically" about 0.74. The property of Nature that is embodied in this loose statement has come to be known as the "Hurst phenomenon;" or, for reasons that will transpire momentarily, "paradox" or "puzzle."

Let us give a sharper restatement that is more readily open to precise analysis. There often exists an exponent $H \neq 0.5$ such that sample values of $Q(t, \delta)/\delta^H$ lie close to each other over the whole range where they can be estimated. One is therefore tempted to conclude that, if one could extend this sample indefinitely, Q/δ^H would neither tend to 0 nor to ∞ but, rather, to a nondegenerate limit r.v. This restatement motivates us to study the behavior of $Q(\delta)/\delta^H$ for $\delta \to \infty$. More generally, it motivates a search for other r.f.'s $X(t)$ such that $Q(\delta)/A(\delta; Q)$ has a nondegenerate limit when $A \sim \delta^H$.

A further sharpening of Hurst's finding reexamines the empirical evidence, tracing the sample $\log Q(t, \delta)$ as a function of $\log \delta$ for several values of t. One finds that it is possible to choose H so that, for sufficiently large δ, $\log Q - H \log \delta$ looks increasingly like a stationary r.f. of $\log \delta$. This finding suggests that, if $Q(\mu\delta)/(\mu\delta)^H$ is viewed as a function of μ, it may perhaps have some nondegenerate $\delta \to \infty$ limit that is a stationary r.f. of $\log \mu$. Writing $\mu = e^\phi$, we are led to conjecture that the r.f. $Q(e^\phi\delta)/(e^\phi\delta)^H$ has a nondegenerate limit that is a stationary r.f. of ϕ. For this reason, this paper is devoted to studying the r.f. $Q(e^\phi\delta)/A(e^\phi\delta; Q)$.

Hurst immediately perceived that $H \neq 0.5$ was unexpected. Indeed, a rough argument convinced him that for the simplest r.f., namely the iid process with $\mathscr{E}X^2 < \infty$, the theory predicts that H should be equal to 0.5. This was confirmed soon afterwards in Feller 1951 (whose treatment is more rigorous but not completely so, since it assumes a weak convergence lemma.) Hurst recognized that the phenomenon he discovered may express some deep characteristic of the underlying records and may yield the long-sought conceptual device that would allow cyclic but nonperiodic records to be handled properly. Several possible explanations were soon presented, but none were conclusive. Moran 1964, 1968 claims to have proved that $H \neq 0.5$ if the r.f. $X(t)$ is sufficiently far from being normal; however, many of Hurst's records are near-normal. In addition, Moran had misread the evidence, believing it to be relative to the $\delta \to \infty$ behavior of R itself, rather than of R/S; this point, as we shall see, is a major one.

Thus, the Hurst phenomenon remained a puzzle (and a spur to hydrological model making) until a r.f. for which $H \neq 0.5$ was first exhibited in my first paper on this topic. M 1965h{H9} linked the Hurst phenomenon to very long-run dependence in the r.f. $X(t)$. Later, M & Wallis 1969c, used computer simulation to demonstrate that, thanks to the normalizing denominator S, Hurst's ratio has independent interests in statistics and in probability theory. Indeed, the behavior of $A(\delta; Q)$ for large δ constitutes a powerful method for distinguishing between short-term (local) and long-term (global) dependence in r.f.'s; it makes no assumption about $\mathscr{E}X^2$ and, in fact, is essentially insensitive to the distribution of X. Thus, the main virtue of this method lies in its robustness. The statistical background is further developed in M 1972c, which is excerpted in Chapter H30.

The purpose of the present paper is to describe the current mathematical foundation of this technique, and to solicit proof or disproof of various conjectures that it uses.

We shall first state in a stronger form part of the result in Feller 1951.

1.4. Need for several notions of limit; notation

The notation *w*-lim refers to weak convergence in function space. *d*-lim refers to weak convergence of a real valued r.v.. *f*-lim refers to convergence of all finite dimensional distributions for almost all values of the argument. *as*-lim is an almost sure limit. *l*-lim is the limit of order *l*.

The number given to a theorem indicates the section in which it appears (thus, there is no Theorem 3).

1.5. A prototype theorem

The following Proposition shows that the variety of possible A's is limited.

Proposition 1. If f-$\lim_{\delta\to\infty} Q(e^{\phi}\delta)/A(\delta;Q)$ exists and is nondegenerate, $A(\delta;Q)$ is of the form $\delta^H L(\delta)$, where $L(\delta)$ is a slowly varying function.

Proof. Proposition 1 follows from Theorem 2 in Lamperti 1962.

Definition. The parameter H, when defined, summarizes as much of the information about $X(t)$ as is reflected in Q. Therefore, H will be called the "exponent of R/S dependence" of $X(t)$.

Prototype Theorem 1. If $X(t)$ is iid in discrete time with $\mathscr{E}X^2 < \infty$, then w-$\lim_{\delta\to\infty} Q(e^{\phi}\delta)/\sqrt{e^{\phi}\delta}$ in the space $C]-\infty,\infty[$ is a nondegenerate stationary r.f. $\Psi(\phi)$. In particular, d-$\lim_{\delta\to\infty} Q(\delta)/\sqrt{\delta}$ is a nondegenerate r.v. $\Psi(0)$.

Theorem 1 is nearly obvious but will be proven in Section 3. Then, having examined the continuity of certain transformations, we shall derive counterparts to theorem 1 concerning the behavior of appropriately normalized ratios $Q(e^{\phi}\delta)/A(e^{\phi}\delta;Q)$ and $Q(\delta)/A(\delta;Q)$ within increasingly broad classes of r.f.; this will culminate in Theorem 5. The crucial finding is that different classes of r.f. involve different weights $A(\delta;Q)$.

The first main result is that $H=1/2$ and that $A(\delta;Q)$ holds independently of the distribution of X, as long as X is either iid or stationary with local (short) dependence. This behavior differs greatly from that of $R(\delta)$ itself, for which the proper weight $A(\delta;R)$ is $\sqrt{\delta}$ when $\mathscr{E}X^2 < \infty$ but otherwise $A(\delta;R)$ depends on the distribution of X. The second result is that sufficiently global (long) dependence can lead to $A(\delta;Q)$ of the form $d^H L(\delta)$ with $H \neq 0.5$. Again, the exponent H is not characteristic of the distribution of X but of the intensity of long-run dependence in $X(t)$. This is the justification for normalizing R through division by S.

2. ON THE USE OF Q IN PRACTICAL STATISTICS

How well can one estimate $\mathscr{E}Q$ from a sample of $Q(t,\delta)$? When $X(t)$ is iid, the r.v. $Q(t+e,\delta)$ and $Q(t,\delta)$ are independent when $e \geq \delta$. This fact defines the r.f.$Q(t,\delta)$ as δ-dependent, and proves that reliable estimation is possible. The following statements, if true, would show estimation to be reliable under wide assumptions.

Conjecture 2. When $Q(t,\delta)$ is viewed for fixed δ as a r.f. of t, Q is "typically" (i.e., under weak conditions that remain to be specified) ergodic and short R/S dependent. The latter term means that when the function Q is computed for $Y(t)=Q(t,\delta)$, the function $A(\delta,Q)$ is $\sqrt{\delta}$.

***Conjecture* 2′.** When $w\text{-}\lim_{\delta\to\infty} Q(e^{\phi}\delta)/A(e^{\phi}\delta; Q)$ exists and is a stationary r.f. of ϕ, this limit is also "typically" ergodic and short dependent.

3. PROOF OF THE PROTOTYPE THEOREM 1 OF SECTION 1.5

We assume (without loss of generality) that $\mathscr{E}X = 0$. We note that $X(t)$, as defined, belongs to the space $D[0, \infty[$ of right-continuous real-valued functions on $[0, \infty[$ with limits on the left everywhere on $]0, \infty[$; this space is endowed with the Skorohod topology and, in general, is just like the space $D[O, 1]$. The matter has been settled in Gnedenko & Kolmogorov 1954. Also $X_\Sigma(t)$ is a continuous function in $C[0, \infty[$, and the mapping from X to X_Σ is continuous. From the assumptions, it follows from a theorem by Donsker that – when f is a fraction between 0 and 1 – $w\text{-}\lim_{\delta\to\infty} X_\Sigma(\overline{df}) / \sqrt{\delta}$ is the Brownian motion r.f. $B(f)$ with $B(0) = 0$ multiplied by $\sqrt{\mathscr{E}X^2}$. By the ergodic theorem on X^2, it follows that $as\text{-}\lim_{\delta\to\infty} S(\delta) = \sqrt{\mathscr{E}S^2} = \sqrt{\mathscr{E}X^2}$. In other words, $Q(\delta)$ only differs from R by a numerical factor. The fact that we are interested in $Q(f\delta)/\sqrt{f\delta}$ rather than in $Q(f\delta)/\sqrt{\delta}$ means that it is important to work with $D]0, \infty[$ rather than with $D[0, \infty[$. Classically, applying the continuous mapping theorem in $D]0, \infty[$, we find that $Q(\delta)/\sqrt{\delta}$ and $Q(e^{\phi}\delta)/\sqrt{e^{\phi}\delta}$ are continuous functions of $X_\Sigma(f\delta)/\sqrt{\delta}$. Hence $d\text{-}\lim_{\delta\to\infty} Q(\delta)/\sqrt{\delta}$ and $w\text{-}\lim_{\delta\to\infty} Q(e^{\phi}\delta)/\sqrt{e^{\phi}\delta}$ exist. The former is the unadjusted range of the Brownian bridge; its distribution has been derived in Feller 1951. For the stationarity part of the theorem, we use a representation given in Doob 1942 (see also Cox & Miller 1965, p. 229). This representation is $B(f) = \sqrt{f}\, J(\log f)$, where J is a stationary r.f. (namely, the Gauss-Markov r.f.). Hence one has

$$w\text{-}\lim_{\delta\to\infty} Q(e^{\phi}\delta)/\sqrt{e^{\phi}\delta} = (\max - \min)_{\gamma<0}[e^{\gamma/2}J(\phi+\gamma) - e^{\gamma}J(\phi)],$$

which is independent of γ in distribution.

4. THE GENERAL CASE WHERE X_Σ LIES IN THE BROWNIAN DOMAIN OF ATTRACTION

By transforming the first conclusion of the proof of Section 3 into an assumption, we obtain the following generalization of Theorem 1.

Theorem 4. Let X^2 be ergodic and $w\text{-}\lim_{\delta\to\infty} X_\Sigma(f\delta)/\sqrt{\delta}$ be Brownian motion. Then $w\text{-}\lim_{\delta\to\infty} Q(e^\phi\delta)/\sqrt{e^\phi\delta}$ is a nondegenerate, stationary, continuous r.s. $\Psi(\phi)$. In particular, $d\text{-}\lim_{\delta\to\infty} Q(\delta)/\sqrt{\delta}$ is a nondegenerate r.v. $\Psi(0)$. A fortiori, $\lim_{\delta\to\infty} \log Q(\delta)/\log\delta = 0.5$.

In general, the ergodicity of X^2 must be assumed separately. It need not follow from the postulated limit behavior of $X_\Sigma(fd)/\sqrt{\delta}$, except in the iid case with $\mathscr{E}X^2 < \infty$.

Theorem 4 is easy to state but difficult to apply, except when it reduces to the Prototype Theorem 1. A number of other r.f.'s that satisfy Theorem 4 are described in Billingsley 1968. Assume that the r.f. X is Gaussian with the covariance $C(\delta)$. Then the necessary and sufficient condition for ergodicity of $[X^2]_\Sigma$ and weak convergence of X_Σ to Brownian motion is that $0 < C(0)/2 + \sum_{\delta=1}^{\infty} C(\delta) < \infty$. These inequalities express that the dependence between the X is local (short). Examples include the Markov or finite autoregressive r.f.. When the r.f. is non Gaussian, the central limit problem for the dependent r.f. is well-known to be complicated. Weak convergence of X_Σ is an even difficult problem, at least in principle, since it seems that in all specific cases when the central limit theorem holds for X_Σ, the conditions of Theorem 4 are satisfied.

5. CASE WHERE THE VECTOR $\{X_\Sigma, [X^2]_\Sigma\}$ LIES IN A GENERAL DOMAIN OF ATTRACTION

Definition of H. Lemma 5. Suppose one can select the nonrandom functions $A(\delta; X_\Sigma)$, and $B(\delta;[X^2]_\Sigma)$ in such a way that the vector r.f. of coordinates

$$U(f,\delta) = \frac{X_\Sigma(f\delta)}{A(\delta; X_\Sigma)} \quad \text{and} \quad V(f,\delta) = \frac{[X^2]_\Sigma(f\delta)}{B(\delta;[X^2]_\Sigma)}$$

converges weakly (in Skorohod topology) to a limit r.f.$\{U(f,\infty), V(f,\infty)\}$, not identically equal to 0 or ∞, belonging to the space D and such that $U(f,\infty) \neq fU(1,\infty)$. Then $U(f,\infty)$ and $V(f,\infty)$ are both self-affine, in the sense that there exist two constants H' and H'' such that the distributions of $U(g,\infty)/g^{H'}$ and $V(g,\infty)/g^{H''}$ are independent of g. Moreover, $A(\delta; X_\Sigma)/\delta^{H'} = L'(\delta)$ and $B(\delta;[X^2]_\Sigma)/\delta^{H''} = L''(\delta)$ are slowly varying functions, in the sense of Karamata. Thus, writing $H = 1/2 + H' - H''/2$, the function $A(\delta; Q) = A(\delta; X_\Sigma)/\sqrt{B(\delta;[X^2]_\Sigma)}$ takes the form $\delta^H L(\delta)$, with $L(\delta)$ slowly varying in the sense of Karamata. Finally, one must have $H \leq 1$.

Theorem 5. To the conditions of Lemma 5, add either that $H<1$ or that $H=1$ but $L(\delta)\to 0$. Then f-$\lim_{\delta\to\infty} Q(e^{\phi}\delta)/A(e^{\phi}\delta;Q)$ in $D]-\infty,\infty[$ is a nondegenerate stationary r.f. $\Psi(\phi)$. In particular, d-$\lim_{\delta\to\infty} Q(\delta)/A(\delta;Q)$ is a nondegenerate r.v. $\Psi(0)$. A fortiori, $\lim_{\delta\to\infty}\log Q/\log\delta = H$. Also, $H\geq 0$.

Conjecture 5. The f-lim in Theorem 5 can be replaced by a w-lim.

Proof of Lemma 5. Note that the mapping from X to S is continuous in $D[0,\infty[$. The self-similarity of $U(\delta,\infty)$ and $V(f,\infty)$ and the form of A and B were proven in Lamperti 1962, which uses, instead of our term "self-affine," the term "semi-stable." $H>1$ would – for sufficiently large δ – contradict $[X^2]_\Sigma > ([X^2]_\Sigma)/\delta$; therefore $H\leq 1$ is necessary.

Remark. All the finite-dimensional distributions of the vector r.f. $[U(f,\delta), V(f,\delta)]$ converge to those of the vector r.f. $[U(f,\infty), V(f,\infty)]$. Then the vector r.f. converges weakly if and only if each of its coordinate scalar r.f.'s converges weakly. This result is proposed in Billingsley 1968 as Exercise 6, p. 41.

When Theorem 4 holds, Lemma 5 follows as a corollary, since $U(f,\infty)$ is Brownian motion and $V(f,\infty)$ is degenerate, in the sense that $V(f,\infty)/f$ equals a constant (the functional strong law is the same as the ordinary strong law). A second instance where $V(f,\infty)/f$ equals a constant will be encountered in Theorem 11.

In other cases, both $U(f,\infty)$ and $V(f,\infty)$ must be taken in consideration.

Proof of Theorem 5. By elementary manipulation,

$$R(e^{\phi}\delta) = (\max-\min)_{0<f<\exp\phi}\{X_\Sigma(f\delta) - fe^{-\phi}X_\Sigma(e^{\phi}\delta)\}$$
$$= (\max-\min)_{0<f<\exp\phi}\{U(fe^{-\phi}, e^{\phi}\delta) - fe^{-\phi}U(e^{\phi},\delta)\}/A(e^{\phi}\delta; X_\Sigma),$$

$$S^2(e^{\phi}\delta) = [X^z]_\Sigma(e^{\phi}\delta) - e^{-\phi}\delta^{-1}(X_\Sigma)^2(e^{\phi}\delta)]\delta^{-1}e^{-\phi}$$
$$= \{V(e^{\phi},\delta) - U(e^{\phi},\delta)[A^2(e^{\phi}\delta;Q)/(e^{\phi}\delta)]\}B(e^{\phi}\delta; X_\Sigma)(e^{\phi}\delta)^{-1}.$$

Hence,

$$\frac{Q(e^{\phi}\delta)}{A(e^{\phi}\delta;Q)}=\frac{(\max-\min)\{U(fe^{-\phi},e^{\phi}\delta)-fe^{-\phi}U(e^{\phi},\delta)\}}{\sqrt{V(e^{\phi},\delta)-U(e^{\phi},\delta)[A^2(e^{\phi}\delta;Q)/(e^{\phi}\delta)]}}\,.$$

Because of the assumptions made about H and/or $L(\delta)$, the second half of the denominator is asymptotically negligible. The indeterminate form 0/0 will have a probability tending to 0 as $\delta\to\infty$ due to the assumptions that $U(f,\infty)\neq fU(1,\infty)$. Further, (R,S), which is a random element in $D[0,\infty[\times D[0,\infty[$, is a continuous function of X in $D[0,\infty[$. Since any finite set of values of $Q(e^{\phi}\delta)/A(e^{\phi}\delta;Q)$ is a continuous function of the vector r.f. $[U(f,\delta),V(f,\delta)]$, it converges to a limit that is the corresponding function of $[U(f,\infty),V(f,\infty)]$. The stationarity of the limit results from self-similarity of $[U(f,\infty),V(f,\infty)]$; the required generalization of Doob's representation was proved in Lamperti 1962, p. 64. Finally, $H\geq 0$ is necessary to insure that $Q\geq 1$.

Comment on Conjecture 5. The difficulty here is that $V(f,\infty)$ may be discontinuous, in which case the continuous mapping theorem ceases to apply. Nevertheless, the conclusion in Conjecture 5 seems correct. Thus, even though it has no practical importance, it is a challenge to the mathematician. If the conditions in Theorem 5 were made stronger, the proof could be carried out. (For example, if one adds a condition suggested by Whitt: $V(f,\infty)$ is a.s. continuous and for all a and b such that $0<a<b<1$, one has $\Pr\{\inf_{a\leq f\leq b}V(f,\infty)>0\}=1$.) But such conditions appear as both unnatural and unnecessary. An alternative is to use the first variant of Q described in Section 13, but for practical needs this would be too costly to consider.

6. DIRECT RELATIONSHIP BETWEEN INDEPENDENCE AND THE EXPONENT VALUE $H=1/2$

Theorem 6. If $\mathscr{E}X^2=\infty$ and x is iid and lies in a stable $(0<\alpha<2)$ domain of attraction, then $f\text{-}\lim_{\delta\to\infty}Q(e^{\phi}\delta)/\sqrt{e^{\phi}\delta}$ is a nondegenerate stationary discontinuous r.f. of ϕ.

Proof of Theorem 6. Assuming that X is iid with $\mathscr{E}X^2=\infty$, it follows from Skorohod 1957 that the conditions of Theorem 6 are necessary and sufficient for the validity of Lemma 5. By well-known limit theorems, $A(\delta;X_\Sigma)=\delta^{1/\alpha}L'(\delta)$, with L determined by $\Pr[|X|>A(\delta;X_\Sigma)]\delta\to 1$; see

Feller 1971 (pp. 314-315) Also, $B(\delta;[X^2]_\Sigma) = A^2(\delta; X_\Sigma) = \delta^{2/\alpha}[L'(\delta)]^2$ because $L''(\delta)$ is determined by $\Pr[X^2 > B(\delta;[X^2]_\Sigma)]\delta \to 1$. Hence $H = 1/2$ and $L(\delta) = 1$. (We need not worry about the convergence of the bridge of X_Σ to the stable bridge. This is fortunate, since the latter – although true – is difficult to prove; see Liggett 1968.) For the applications, the central feature of Theorem 6 is that, in contrast to $A(\delta; X_\Sigma)$, $A(\delta; Q)$ is independent of the exponent α of X and of its skewness parameter Β.

Conjecture 6. In Conjecture 5 one can replace *f*-lim by *w*-lim.

Comment on Conjecture 6. The numerator and the denominator of Q are both discontinuous; their jumps occur at the same moments and are dependent. This should be the key factor in the proof.

***Proposition 6. The case* α = 2.** Assume that $\mathscr{E}X^2 = \infty$ and that X is iid and lies in the Gaussian domain of attraction; this attraction is necessarily not normal. Then $w\text{-}\lim_{\delta\to\infty} Q(e^{\phi}\delta)/\sqrt{e^{\phi}\delta}$ is the same r.f. as in Theorem 4.

Proof. The case α = 2 is based on Theorem 5 in a case where – like in Theorem 4 – $V(f, \infty) = fV(1, \infty)$ but – contrary to Theorem 4 – the convergence to this limit is weak, not strong. In other words, we need the (not quite familiar) form of Theorem 5 that the weak law of large numbers takes in the case of iid infinite variance r.v.. Using the U and μ notations in Feller 1950 (Volume 2, second edition, p. 236 and pp. 314-315), the equations $\delta A^{-2}(\delta; X_\Sigma)U[A(\delta; X_\Sigma)] \to 1$ and $\delta B^{-1}(\delta;[X^2]_\Sigma)\mu[B(\delta;[X^2]_\Sigma)]$ continue to yield $B = A^2$. By Skorohod 1957, $w\text{-}\lim_{\delta\to\infty} U(f, \delta) = B(t)$ and, by Feller *op.cit.* (p. 236),

$$\Pr\{|V(f, \delta)/(f\delta) - 1| > \varepsilon\} \to 0.$$

Generalization of Theorem 6 to the case when* α = *0. The definitions of the stable r.v. and of their domains of attraction exclude the limiting value $\alpha = 0$. Also, Theorem 5 assumes that the scaling factors $A(\delta; X_\Sigma)$ and $B(\delta;[X^2]_\Sigma)$ are nonrandom. Now, we shall allow these factors to be random. This brings a bit of new generality to the study of Q because the set of values of the Lévy exponent α is closed by adding a limit one can view as corresponding to "$\alpha = 0$." This limit is encountered when $X > 0$ and when $\Pr(|X| > x)$ itself is slowly varying (and necessarily, nonincreasing).

Indeed, Darling 1952 has shown that, in this case, the limit of order 1 satisfies $1\text{-}\lim_{\delta\to\infty} X_\Sigma(\delta)/\max_{1\le u\le\delta} X(u)=1$, from which it follows that $1\text{-}\lim_{\delta\to\infty} R(\delta)/\max_{1\le u\le\delta} X(u)=1$; similarly,

$$\frac{1\text{-}\lim_{\delta\to\infty}[X^2]_\Sigma(\delta)}{\max_{1\le u\le\delta} X^2(u)}=\frac{[X^2]_\Sigma(\delta)}{\sqrt{[\max_{1\le u\le\delta} X(u)]^2}}=1$$

from which it follows that $1\text{-}\lim_{\delta\to\infty} S^2(\delta)\delta/[\max_{1\le u\le\delta} X(u)]^2=1$. In this case, $1\text{-}\lim_{\delta\to\infty} R^{2/}S^2\delta=1$, which implies that $A(\delta;Q)=\sqrt{\delta}$. To get closer to the notation in Theorems 5 and 6, one can view $\max_{1\le u\le\delta} X(u)$ as both $A(\delta;X_\Sigma)$ and $\sqrt{B(\delta;[X^2]_\Sigma)}$. The resulting Vectorial r.f. $\{U(f,\delta), V(f,\delta)\}$ will converge weakly to $\{U(f,\infty), V(f,\infty)\}$, where $U(f,\infty)=V(f,\infty)=0$ for $0\le f<f_0$ and $U(f,\infty)=V(f,\infty)=1$ for $f_0<f<1$. The crossover value f_0 is a r.v. uniformly distributed between 0 and 1.

7. CONJECTURES CONCERNING THE IID CASE FOR DIFFERENT DOMAINS OF ATTRACTION OF X_Σ

Contrary to Logan et al 1973, we are unable to characterize the limit in Theorem 6. But the following conjectures have been suggested in part by computer simulations, and their proof would justify current practice. For example, Conjecture A combined with Conjecture 2, would show Q to be readily estimated from samples.

A) $0<\mathscr{E}\{\lim_{\delta\to\infty}\delta^{-0.5}Q(\delta)\}^h<\infty$ for every $h>0$.

B) For every B, sup adjust(u 2) $_{\alpha\in[0,2]}$ Pr $\{\lim_{\delta\to\infty} Q(\delta)/\sqrt{\delta}>x\}$ is a nondegenerate distribution.

C) For every B, $\inf_{\alpha\in[0,2]}$ Pr $\{\lim_{\delta\to\infty} Q(\delta)/\sqrt{\delta}>x\}$ is another nondegenerate distribution.

D) The supremum in conjecture B) is effectively attained and is the distribution corresponding to $\alpha=2$.

E) The infimum in conjecture C) is effectively attained and is the distribution corresponding to some $\alpha\in[0,2]$.

F) $\mathscr{E}[\lim_{\delta\to\infty} Q(\delta)/\sqrt{\delta}]$, considered as a function of α, is (for every B) monotonically decreasing with α. Its $\lim_{\alpha\to 0}$ is 1, and its $\lim_{\alpha\to 2}$ is 1.2533.

G) $Var[\lim_{\delta\to\infty} Q(\delta)/\sqrt{\delta}]$, considered as a function of α, is monotonically decreasing with α. (We already know its $\lim_{\alpha\to 0}$ is 0.)

8. CONJECTURES CONCERNING THE IID CASE WHEN X_Σ DOES NOT LIE IN ANY DOMAIN OF ATTRACTION

In this case, $Q(\delta)/\sqrt{\delta}$ has no d-lim, and a fortiori $Q(e^{\Phi}\delta)/\sqrt{e^{\Phi}\delta}$ has no w-lim. Nevertheless, the following conjectures come to mind.

H) The fact that X is iid suffices to establish that

$$\limsup_{\delta\to\infty} \Pr\{\delta^{-0.5}Q(t,\delta) < x\} = Q_2(x) < \sup_\alpha \lim_{\delta\to\infty} \Pr\{\delta^{-0.5}Q(\delta) < x\},$$
$$\liminf_{\delta\to\infty} \Pr\{\delta^{-0.5}Q(t,\delta) < x\} = Q_1(x) > \inf_\alpha \lim_{\delta\to\infty} \Pr\{\delta^{-0.5}Q(\delta) < x\}.$$

I) Corollary of H): X is iid, then $as\text{-}\lim_{\delta\to\infty} \log Q/\log\delta = 1/2$.

Example of a process for which $Q(\delta)/\sqrt{\delta}$ has no d-lim. We select a function $X(t)$ for which the behavior of $X_\Sigma(\delta)$ has been documented in an early paper by Paul Lévy. It is a symmetric iid r.f. with the following marginal distribution:

For $1 \le x < 1000$:

$$\Pr(|X| > x) = x^{-1}.$$

For $10^3 \le x < 10^6$:

$$\frac{\Pr(|X| > x)}{\Pr(|X| > 10^3)} = x^{-5}.$$

For $10^{3n} \le x < 10^{3(n+1)}$, where n is even:

$$\frac{\Pr(|X| > x)}{\Pr(|X| > 10^{3n})} = x^{-1}.$$

For $10^{3n} \le x \le 10^{3(n+1)}$, where n is odd:

$$\frac{\Pr(X > x)}{\Pr(X > 10^{3n})} = x^{-5}.$$

For values of δ up to 1000, $X_\Sigma(\delta)\delta^{-1}$ seems to converge to a Cauchy motion, but this apparent convergence will stop sometime after δ exceeds 1000. Later, for values of δ up to 10^6, $X_\Sigma(\delta)\delta^{-0.5}$ seems to converge to a Brownian motion, but this tendency also eventually stops. The ostensible rate of convergence $A(\delta; X_\Sigma)$ continually flips between the two different analytic forms, δ^{-1} and $\delta^{-0.5}$. The ostensible limit of $X_\Sigma / A(\delta; X_\Sigma)$ also continually flips. This $X(t)$ is an example of a r.f. for which there exists no $A(\delta;Q)$ such that $Q(\delta)/A(\delta;Q)$ has a limit for $\delta \to \infty$. Nevertheless, it appears that the r.v. $Q(\delta)/A(\delta; Q)$ remains positive and finite for all δ. This behavior differs from the situation for X_Σ insofar as $A(\delta; Q) = \delta^{-0.5}$ throughout yet $Q(\delta)\delta^{-0.5}$ first seems to tend to the limit corresponding to $\alpha = 1$, then switches to tend to the limit corresponding to $\alpha = 2$ and so continues to flip between the two different limits.

9. THERE EXIST GLOBALLY DEPENDENT PROCESSES FOR WHICH $H = 1/2$

In broad terms, Section 8 asserts that, even if X is iid (or is locally dependent) but Theorem 5 fails to apply, the relation $A(\delta, Q) \sim \sqrt{\delta}$ may continue to hold with a less demanding interpretation of ~. The converse, however, is false: when the conditions of Lemma 5 are denied, $A(\delta,Q) = \sqrt{\delta}$ is compatible with very global dependence.

First example. In M 1966b, 1969e, I introduced a martingale model of certain kinds of price variation, wherein successive increments are very globally correlated. Theorem 5 is inapplicable, but the proper $A(\delta, Q)$ for those price increments turns out to be $\sqrt{\delta}$.

Second example. Assume that $X(t) = Y(t) - Y(t-1)$, with $Y(t)$ iid for $t = 1, 2, ...$ For every t, $X(t)$ and $X(t+2)$ are independent, that is, $X(t)$ is "finitely dependent." Since $X_\Sigma(t) = Y(t) - Y(0)$, the function $X_\Sigma(t)$ has the same distribution for all t, and it satisfies no limit theorem in which the limit has any property of "universality." Since the notion of local (short-range) dependence implies that the limit is Gaussian or stable, one must view the present finitely dependent $X(t)$ as globally (long-range) dependent.

{P.S. 2000. This paradox occurs under much wider conditions. For example, the same is true of all moving average processes for which the averaging kernel $\mathrm{K}(s)$ satisfies $\sum \mathrm{K}(s) = 0$ and $\mathrm{K}(s) = 0$ for $|s| > \tilde{s}$, where $\tilde{s} < \infty$. }

Note that, for large δ, $R(\delta) \sim (\max - \min)_{0<u<\delta} Y(u)$. If the essential minimum of Y vanishes, then $R(\delta) \sim \max_{0<u<\delta} Y(u)$. Hence, for large δ, the ratio between the maximum and the sum of δr.v. X^2 is, given by $Q^2/2\delta = R^2/2S^2\delta$. Assume further that Y lies in the domain of normal attraction of a stable r.v. of exponent $\alpha < 2$. Then the conditions of a well-known theorem of Darling 1952 are satisfied, and it follows that $d\text{-}\lim_{\delta\to\infty} Q^2/2\delta$, hence $d\text{-}\lim_{\delta\to\infty} Q(\delta)/\sqrt{\delta}$ is nondegenerate. The same is true (the details have been worked out by W. Whitt) of $w\text{-}\lim_{\delta\to\infty} Q(e^{\phi}\delta)/\sqrt{e^{\phi}\delta}$. The generalization when $\Pr(Y<0)>0$ is easy. This shows that the equality $A(\delta, Q) = \sqrt{\delta}$ is conceivable even when X is globally dependent.

This conclusion in no way contradicts Theorems 5, because in the present case the abscissa r.f. $U(f, \delta)$ is $Y(f\delta) - Y(0)$. Hence, $\{U(f, \infty), f \geq 0\}$ is conditionally independent given $Y(0)$ and thus is not in D. Incidentally, the requirement that $\{U(f, \delta), V(f, \delta)\}$ converge weakly in D, rather than from the viewpoint of finite joint distributions, is shown to be more than a technicality.

10. SOME ROLES OF THE UPPER BOUND $H = 1$

First example. Assume $X(t) = G(t - t_0)$ where G is a Gaussian r.v. independent of t. In this case, $X_\Sigma(u) - (u/\delta)X_\Sigma(\delta) = Gu(u-\delta)/2$, and $R(\delta) = |G|\delta^2/8$. Also $S^2(\delta) = G^2\delta^2/12$. Hence, $Q(\delta) = \delta A(\delta; Q) = \delta\sqrt{3/4}$.

Second example. Let $X(t)$ itself be, either a Brownian motion or a Lévy motion of exponent $0 < \alpha < 2$ (stable process) without drift (meaning that $\mathcal{E}X = 0$) when $\mathcal{E}X < \infty$. Now, $Q(e^{\phi}\delta)/e^{\phi}\delta$ is a stationary r.f. of ϕ and dependent on α. When $X(t)$ is a Brownian motion with drift, $Q(\delta)/\delta$ behaves for small δ as if the drift were absent and for large δ as if the Brownian motion were absent.

Typically, $A(\delta; Q) \sim \delta$ extends to other integrals of a stationary r.f.. Consequently, if one wants to distinguish between such an integral and a fractional noise (Section 11) with H nearly equal to 1 (a stationary r.f.), R/S testing is useless. R/S estimation is possible but delicate. Finding that $A(\delta; Q) \sim \delta$ means little but can be interpreted by also performing an R/S analysis on $X'(t)$, and on higher derivatives if necessary, until one reaches H below 1.

11. $0 < H < 1$ WITH $H \neq 1/2$ IS RELATED TO STRONG DEPENDENCE: THE CASE OF GAUSSIAN $X_\Sigma(t)$ ATTRACTED TO FBM

Assume the limit r.f. $V(f, \infty)$ of Lemma 5 is uniformly constant so that $B(\delta;[X^2]_\Sigma) = \delta$. To have $A(\delta;Q) \neq \sqrt{\delta}$, a necessary condition is that the ratio $X_\Sigma(dh)/\sqrt{\delta}$ must not be attracted by Brownian motion. In particular, the ratio $X_\Sigma(\delta)/\sqrt{\delta}$ must not have a Gaussian limit.

Definition. For $0 < H < 1$, the fractional Brownian motion (FBM) of exponent H is the r.f. $B_H(t)$ in C, and hence in D, such that $B_H(0) = 0$ and

$$\mathscr{E}B_H(t')B_H(t'') = (1/2)[\,|t'|^{2H} - |t' - t''|^{2H} + |t''|^{2H}].$$

In particular, $\mathscr{E}B_H^2(t) = t^{2H}$. The term "FBM" was coined in M & Van Ness 1968 because B_H is a "fractional integral" of the Brownian motion $B(t)$. The case $H = 0.5$ degenerates to ordinary Brownian motion. The cases where $H \neq 0.5$ are therefore referred to as "properly fractional;" the main characteristic is that their increments are not strongly mixing (in the sense of Rosenblatt 1960) but rather globally dependent. The intensity of long dependence is measured by the single parameter H, to be called the "exponent of dependence." It is natural to measure the sign of the dependence by the sign of the correlation $C(\delta)$ between $B_H(t) - B_H(t-1)$ and $B_H(t+\delta) - B_H(t+\delta-1)$, where δ is large. If so, the dependence has the same sign as $H - 0.5$. Indeed,

$$\begin{aligned}&\mathscr{E}\{[B_H(t) - B_H(t-1)][B_H(t+\delta) - B_H(t+\delta-1)]\}\\ &\quad = \frac{1}{2}[\,|\delta+1|^{2H} - 2|\delta|^{2H} + |\delta-1|^{2H}].\end{aligned}$$

For $H \to 1$ the limit of $B_H(t)$ is a r.f. with fully correlated Gaussian increments, of the form $G(t - t_0)$; this limit is in the space D. For $H \to 0$ the limit of $B_H(t)$ is $X_\Sigma(t) = G(t) - G(0)$; it lies outside D.

Theorem 11. One class of r.f. such that $A(\delta; Q) = \delta^H L(\delta)$ is the class of functions for which X^2 is ergodic and $w\text{-}\lim_{\delta\to\infty} X_\Sigma(f\delta)/\delta^H L(\delta) = B_H(t)$.

Proof. This is an immediate corollary of Theorem 5. Observe that the distribution of $\lim_{\delta\to\infty} Q(e^\phi\delta)/A(e^\phi\delta)$ is, contrary to $A(\delta)$ itself, independent of $L(\delta)$.

Lemma 11. When $X(t) = B_H(t) - B_H(t-1)$, $w\text{-}\lim_{\delta \to \infty} X_\Sigma(t\delta)/\delta^H = B_H(t)$.

Clearly, every generalization of Lemma 11 leads to an immediate generalization of Theorem 11.

First generalization. Theorem 11 applies to Gaussian processes whose covariance $C(\delta)$ behaves sufficiently like the covariance of $B_H(t) - B_H(t-1)$. Taqqu 1975 shows that $w\text{-}\lim_{\delta \to \infty} X_\Sigma(f\delta)/\delta^H L(\delta) = B_H(f)$ when X is a stationary Gaussian r.f. such that the function $C(\delta)\delta^{-2H+2} = L^2(\delta)$ varies slowly for $\delta \to \infty$.; However, when $0 < H < 0.5$, one must add the condition that $\lim_{t \to \infty} [C(0) + 2\sum_{s=1}^{t} C(s)] = 0$.

Second generalization. This generalization relies on the fact that $B_H(t) - B_H(t-1)$ can be represented by a moving average. Davydov 1970 has proved that $w\text{-}\lim_{\delta \to \infty} X_\Sigma(f\delta)/\delta^H L(\delta) - B_H(f)$ for all r.f. of the form $X(t) = \sum_{s=-\infty}^{0} K(t-s)Y(s)$, where the $Y(s)$ are iid with $\mathscr{E}Y = 0$ and $\mathscr{E}|Y|^{2h} < \infty (h \geq 2)$, where $\sum_{-\infty}^{\infty} K(s-t) < \infty$ and also where $\mathscr{E}[X^2]_\Sigma(t) = t^{2H}L^2(t)$ if $1/(h+2) < H \leq 1$ and if $L(t)$ is a slowly varying function.

Third generalization. Start with the Gaussian $X = B_H(t) - B_H(t-1)$ and study the nonlinear function $Z(X)$. The case where $Z(X) = X^2$ was used briefly as a counter-example in Rosenblatt 1960. The most general function Z was studied in Taqqu 1975, which relates $A(\delta; Q)$ to the "Hermite rank" of Z, defined as the order of the first nonvanishing term in the development of $Z(X)$ in Hermite polynomials.

When $H > 0.5$, $A(\delta; Q)$ can be the same for $Z(X)$ as for X itself, meaning that the intensity of long-run dependence, as measured by H, can be invariant by the transformation from X to $Z(X)$. The necessary and sufficient condition on $Z(X)$ is that its Hermite rank must be one, whereas a transformation of higher rank can decrease the value of H. This means that X is not characterized by one exponent of dependence but rather by a spectrum of exponents including the original H as a maximum. As a result, the experimental situation is likely to be confused: for example, a function $Z(X)$ of rank 3, if disturbed slightly, is likely to become a function of rank 1, with the result that the analytic form of $A(\delta; Q)$ will seem to change with δ.

When $H < 0.5$, to the contrary, one finds in general that $A(\delta; Q) = \sqrt{\delta}$. In other words, the preservation of the long-run dependence expressed by $H < 1/2$ depends very sensitively on the form of Z.

12. GENERALIZATION OF THE SCOPE OF $H \neq 0.5$ AND RELATION BETWEEN THE SIGNS OF $H - 0.5$ AND OF STRONG DEPENDENCE

The following conjecture is fundamental to the following claim of R/S analysis: that Q picks up the rule of long-run dependence in $X(t)$ irrespectively of the marginal distribution.

Conjecture 12. It is conjectured that Theorem 5, with $A(\delta; Q) = d^H$ independent of α, also applies to the fractional integral of order $H - 0.5$ of Lévy's stable process of exponent α. In particular (conjectural lemma), one has $B(\delta,[X^2]_\Sigma) \propto \delta^{\alpha/2}$, independently of H. The range of admissible H depends on α. This paper is not the appropriate place to elaborate this conjecture.

In a different vein, it may happen that Theorem 5 fails to hold, but, in some weakened sense, $Q(\delta)/A(\delta; Q)$ should continue to have a limit, with $A(\delta; Q) = \delta^H L(\delta)$ and $L(\delta)$ varying slowly, or at least should continue to have $as\text{-}\lim_{\delta\to\infty} \log Q(t, \delta)/\log d = H$. In either case, it is tempting to say that the long-run dependence of $X(t)$ is "regular" and to take $H - 0.5$ as a measure of its intensity. However, we know from Section 9 that $H = 0.5$ is compatible with very global dependence. To illuminate the issue, we explore further the second example of Section 9, where $X(t) = Y(t) - Y(t-1)$ with $Y(t)$ iid.

When $|Y(t)|$ is bounded, $\lim_{\delta\to\infty} R(\delta) = \text{ess max } Y(t) - \text{ess min } Y(t)$, which is finite. The $\lim_{\delta\to\infty} S$ is also finite. Hence, $1 \leq \lim_{\delta\to\infty} (R/S) < \infty$, and $H = 0$ with $L(\delta) = 1$. This example conforms to the notion that for $X(t)$ the dependence is negative and as extreme as possible. This is also true in the case when Y is Gaussian. Then, for large δ, $R(\delta)$ is well-known to behave like $\sqrt{\log \delta}$ (Cramèr & Leadbetter 1967). Since $S(t, \delta)$ is practically nonrandom, $H = 0$ with $L(\delta) = \sqrt{\log \delta}$.

When $Y > 1$ and $\Pr\{Y > y\} = y^{-\alpha}$ with $\alpha > 2$ $\mathscr{E}X^2 = 2\mathscr{E}Y^2 < \infty$ and therefore $S^2 \to 2 \text{ Var } X$. In this case, S^2 is a numerical factor of no consequence. On the other hand, it is clear that, for large δ, $R(\delta) \sim \max_{0\leq u\leq\delta} X(u)$. Using the theory of maxima of iid, one finds that for $\delta \to \infty$, $R(\delta)/\delta^{1/\alpha}$ converges to the Fréchet r.v. of exponent α, which is the r.v. Φ_α, which is defined by $\Pr(\Phi_\alpha < x) = \exp(-x^{-\alpha})$. As a result, $Q(\delta)/\delta^{1/\alpha}$ converges toward a Frechet r.v., and $A(\delta,Q) = \delta^{1/\alpha}$. Thus, although intuitively the dependence in X has the same strength for all α, the behavior of $Q(\delta)/A(\delta, Q)$ mimics a strength dependent upon α.

13. TRANSFORMS OF *R*, OTHER THAN *Q*

The variants of Q to be discussed have different purposes. The reason for looking at the first variant is technical: it avoids – at a price – the gap in Theorem 5 that led to Conjecture 5. The reasons for looking at the other variants are esthetic: the first is disappointing, the second is indifferent and the third is probably an improvement over Q.

13.1 The transform $Q^*(\delta) = R(\delta)/\tilde{S}(\delta)$ suggested by Ward Whitt

Here, $\tilde{S}^2(\delta) = \delta^{-1}\int_0^\delta \tilde{S}^2(u)du$. There is no need for a counterpart to Conjecture 5 because the counterpart of Theorem 5, with $A(\delta, Q^*) = A(\delta, Q)$ and $V(f, \infty)$ replaced by $f^{-1}\int_0^f V(u, \infty)du$, can be strengthened from f-lim to w-lim. Indeed, division on $D(0, \infty) \times D(0, \infty)$ will become a continuous function since Whitt's conditions (as stated in the comments on Conjecture 5) are satisfied. In practice, the continuity of the denominator $\tilde{S}$ is achieved at great cost. The virtue of $Q(\delta)$ is that the jumps in $R(\delta)$ are to a large extent counterbalanced by the simultaneous jumps in $S(\delta)$, but they cannot be counterbalanced by the jumps of $\tilde{S}(\delta)$ because $\tilde{S}(\delta)$ is continuous. As a result, the behavior of $Q^*(\delta)$ is likely to be very wild compared to the behavior of $Q(\delta)$, as exemplified in the simulations found in M 1972c{H30}.

13.2 The transform $R(\delta)/S_p(\delta)$

This transform is defined as

$$Q_p(\delta) = R(\delta)/S_p\delta(\delta), \text{ with } S^p(\delta) = \delta^{-1}\sum_{u=1}^{\delta} |X(u) - \delta^{-1}X_\Sigma(\delta)|^p.$$

Thus, S_p generalizes $S_2 = S$, and R/S_p generalizes $R/S_2 = R/S$. To show that the values $p \neq 2$ have no known virtue that is not shared by R/S, it suffices to make the comparison when X is iid. When X is Gaussian, or more generally has finite absolute moments of every order, then $S_p^p(\delta)$ a.s. converges as $\delta \to \infty$ to a nonrandom, positive, finite constant. Therefore, the statistics R, R/S or R/S_p are asymptotically identical, except for numerical factors. Next, when $X(t)$ are iid with Pr $(X > x) = x^{-\alpha}$, it is easy to see that $A(\delta;Q_p) = \sqrt{\delta}$ if either $\alpha > \max(2,p)$ or $\alpha < \min(2,p)$; otherwise, $A(\delta; Q_p)$ depends upon both α and p. This dependence is highly unsatisfactory. Therefore, the most robust value of p is 2, which corresponds to R/S.

13.3 The transform $R/\tilde{R}$

The definition of the ratio R/S suffers from a lack of symmetry in the sense that its numerator and denominator are obtained by different operations. A first symmetric alternative definition to R/S is obtained by taking as the denominator some permutation-invariant range linked to X. The range of $X_R(t)$, obtained by reordering $X(1) \ldots X(\delta)$ at random, is permutation invariant in distribution, and $M\tilde{R}$, defined as the mean value of this range over all permutations, is permutation invariant. Either can be used as the denominator. It seems that $R/\tilde{R}$ and $R/M\tilde{R}$ would, by and large, serve the same purpose as R/S, but these alternatives have not been studied fully.

13.4 The transform $\tilde{S}/S$

In this variant of R/S, the numerator is a moment rather than a range, namely the square root of

$$\tilde{S}^2 = \delta^{-1} \sum_{u=1}^{\delta} [X_\Sigma(u) - (u/\delta) X_\Sigma(\delta)]^2.$$

In general, the behavior of $\tilde{S}/S$ is indistinguishable from that of R/S. However, in the limiting example where $X(t) = Y(t) - Y(t-1)$, with the $Y(t)$ iid, the behavior of $\tilde{S}/S$ does *not* mimic independence.

13.5. Alternatives to R inspired by the Kolmogorov-Smirnov tests

With obvious changes, all our theorems continue to hold if $R(\delta)$ is replaced by $\sup_{0 \le u \le \delta} |X_\Sigma(u) - (u/\delta) X_\Sigma(\delta)|$, $\sup_{0 \le u \le \delta} \{X_\Sigma(u) - (u/\delta) X_\Sigma(\delta)\}$ or $\inf_{0 \le u \le \delta} \{X_\Sigma(u) - (u/\delta) X_\Sigma(\delta)\}$. But replacing $R(\delta)$ by $\sup_{0 \le u \le \delta} |X_\Sigma(s)|$ or $\sup_{0 \le u \le \delta} X_\Sigma(s)$ would not be acceptable.

CONCLUSION

Statistical expressions tend to become more complicated with an increase in the range of possibilities among which one has to discriminate. Thus far, nearly all statistical techniques relative to r.f. have assumed dependence to be local. Now, Hurst's phenomenon (as I interpret it) forces us to face the possibility of global dependence. Most conveniently, the statistic Q

provided by Hurst turns out to be an excellent tool to study these new possibilities. (P.S.1999: However, once again, it is not the last word.)

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