WHEN CAN PRICE BE ARBITRAGED EFFICIENTLY? A LIMIT TO THE VALIDITY OF THE RANDOM WALK AND MARTINGALE MODELS [1]

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ROUGHLY speaking, a competitive market of securities, commodities or bonds may be considered efficient if every price already reflects all the relevant information that is available. Arrival of new information causes imperfection, but it is assumed that every such imperfection is promptly arbitraged away. In the special case where there is no risk aversion and the interest rate is zero, it can be shown that if an arbitraged price is implemented, it must follow a "martingale random process" — the definition of which will be recalled shortly.

However, even on the theoretical level, the problem of market efficiency is not settled by writing the preceding definition. There exists indeed a class of important cases where useful implementation of arbitraging is *impossible*. The principal purpose of this paper is to describe these cases and to show why I consider them interesting. Moreover, numerous related issues will be solved along the way.

Since my purpose is merely to illustrate, I am allowed to restrict the scope of the problem drastically to avoid extraneous complications. I assume, first, that the process of arbitraging starts with a well-defined single price series $P_0(t)$ — which presumably summarizes the interplay of supply, demand, etc., in the absence of arbitraging. Specifically, the increments of $P_0(t)$ will be assumed to be generated by a stationary finite variance process. Further — unless $P_0(t)$ is itself a martingale.

I assume the purpose of arbitraging is to replace $P_0(t)$ by a different process $P(t)$ that is a) *a martingale and* b) *constrained* not *to drift from $P_0(t)$ without bound.* Had not $P(t)$ and $P_0(t)$ been constrained in some such way, the problem of selecting $P(t)$ would have been logically trivial and economically pointless. Specifically, we shall seek to achieve the smallest possible mean square drift: the variance of $P(t) - P_0(t)$ must be bounded for all t's and as small as possible. In addition, we shall assume that the martingale $P(t)$ is linear, that is related to $P_0(t)$ linearly — we shall explain how and why.

Under these restrictions, the results of this paper fall under three main headings:

A) A necessary and sufficient condition for the existence of the required $P(t)$ is roughly as follows: as the lag s increases, the strength of statistical dependence between $P_0(t)$ and $P_0(t+s)$ must decrease "rapidly," in a sense to be characterized later on. If an arbitraged price series $P(t)$ exists, its intertemporal variability depends upon the process P_0, and may be either greater or smaller than the variability of $P_0(t)$. More often, arbitraging is "destabilizing," but under certain circumstances, it is stabilizing.[2] Note that a market specialist, in order to "insure the continuity of the market," must stabilize the variation of price. Under the usual circumstances under which perfect arbitraging would be destabilizing, the specialist prevents arbitraging from working fully and prevents prices from following a martingale.

B) The case when the strength of statistical dependence of $P_0(t)$ decreases very slowly must be examined. In this case, the belief that perfect arbitraging is possible and leads to a martingale is unfounded. Contrary to what one might have thought, such cases are much more than a mathematical curiosity. Indeed, most economic time series exhibit a "Joseph Effect"

[1] Presented to the Annual Winter Meeting of the Econometric Society, December 28, 1969. This text has benefited greatly from discussions with Mr. Murad Taqqu.

Closely related work is being currently pursued as part of a project sponsored jointly by IBM and the National Bureau of Economic Research.

[2] After this paper had been presented to the Econometric Society, I became aware that considerable literature — including contributions by M. Friedman, W. J. Baumol, and L. Telser — had already been devoted to the question of whether or not speculation is stabilizing. Between these works and my own there are obvious similarities and differences, but, rather than attempt to survey the field at this stage, I have decided to tell my story straight.

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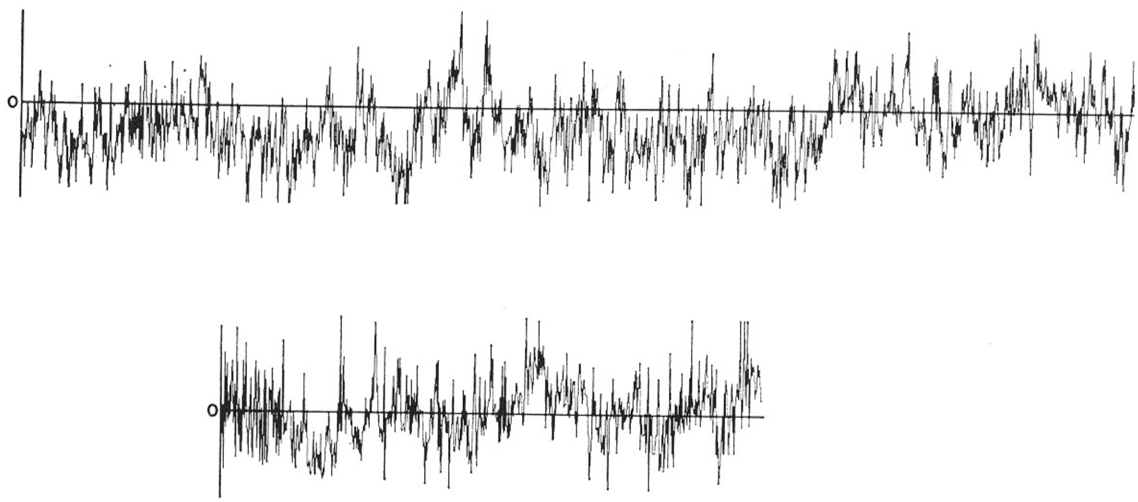
[8,9,10,11,12,13], which is a colorful way of saying that they appear as ruled by a hierarchy of "cycles" of different durations (see figure 1). The simplest way of characterizing such processes is to assume that their spectral density at zero frequency either is infinite, or at least is extremely large. When the spectral density of $P_0(t)$ at zero frequency is infinite, one can show that the distribution of the daily changes of the arbitraged price $P(t)$ would have to be some known distribution scaled by an infinite constant, which outcome is absurd and demonstrates the impossibility of arbitraging.

When the spectral density of $P_0(t)$ at zero frequency is finite but its memory is very long, in a sense to be described, a finite $P(t)$ could be defined, but it would not be acceptable because the variance of $P(t)-P_0(t)$ would have to increase without bound.

An interesting feature of the above "Joseph Effect" is that it is intimately related to the "infinite variance syndrome" ("Noah Effect") I had shown in 1963 [5] to be exhibited by price changes. A full empirical description of prices must take account of both Effects. See [9].

C) Imperfect arbitraging will also be examined in this paper. It never leads to prices following a martingale. We shall describe the effect on arbitraged prices of a gradually increasing imperfection, especially in the case when perfect arbitraging is impossible.

FIGURE 1. —



The cards from which these graphs have been drawn were mislaid; they may be either an empirical record of precipitation [13], or a computer simulated pseudo sample function of a fractional Gaussian noise [12], or an empirical economic record interpretable as a function of causes [9].

This uncertainty of origin happens to underline conveniently the striking resemblance that exists between those possible sources. The intensity of the low frequency components is manifested in each case through an astonishing wealth of "features" of every kind. These functions swing up and down, sometimes irregularly, but also sometimes in a near periodic fashion. In the latter case, irrespective of the total sample size, the number of apparent cycles is approximately the same. In other words, the apparent wavelength is near proportional to the sample size, say, it equals about 10 in a sample of 30, about 100 in a sample of 300, etc. As to the generating mechanism, it is known for the fractional Gaussian noise and it has no built in periodic structure, which means the cycles in question are "perceptual artifacts."

When a process such as this acts as a function of causes, prediction of the future from the known past must involve even the distant past and presents many peculiarities, as seen in the body of the paper.

Introduction

Classical preliminary definitions: $P(t)$ will designate a price at time t, so $P(t+s)-P(t)$ is the random price change between the present time t and the future time $t+s$. The simplest assumptions about a market are that the rate of interest is zero and there is no risk aversion on the part of either the buyers or the sellers. In this case, the *random walk model* asserts that the probability distribution of $P(t+s)-P(t)$ is independent of the current price $P(t)$ and of all past prices. This assertion expresses that, in selecting investments, knowledge of past prices is of no assistance. In addition, it is nearly always assumed that the expectation $E[P(t+s)-P(t)]$ vanishes. If it does not, one speaks of "random walk with a drift."

The *martingale model* is less demanding, being content with assuming that the conditional expectation of $P(t+s)-P(t)$, knowing the present price and/or any number of past prices, vanishes. This simply expresses that no policy exists for buying and selling that has an *expected return* greater than the average return of the market. On the other hand, the martingale model does allow the *actual distribution* of $P(t+s)-P(t)$ to depend on past and present prices, and therefore *it does not deny* that past and present prices can serve in the selection of portfolios of different desired degrees of riskiness. For example, the martingale model allows for buying and selling policies which have much better than an even chance of being superior to the average of the market, but also have a highly appreciable chance of being enormously worse. All that is required is that these situations be mutually balanced so that the expected price change vanishes.[3]

Less special than random walks but more special than general martingales are processes with uncorrelated ("othogonal") increments. When $E[P(t+s)-P(t)]^2$ is finite for all t and s, and $P(t)$ is a martingale, price increments are uncorrelated and spectrally "white." If — in addition — the process $P(t)$ is Gaussian, othogonality becomes synonymous with independence and we see that a *Gaussian martingale can only be a Gaussian random walk.*

Combining the last result with the definitions that precede it, it is clear that *every random walk without drift is a martingale.* The converse is also true in a Gaussian universe. But these results do not exhaust the problem of the relation between random walks and martingales.

Attainment of Market Efficiency or of Approximations Thereto: One reason why the problem remains open is that market efficiency is an aspect of economic equilibrium. It is widely agreed among economists that it does not suffice to affirm that equilibrium must be realized and to study its properties; one must moreover show how equilibrium is either achieved or approached. Compromising between generality and tractability, studies of this type must be addressed to some fully specified model of a competitive market, which combines two assumptions. (A) An assumption about the prices — determined by the exogenous variables — that would have prevailed in the absence of arbitraging. We shall assume they would have followed a finite variance stationary process — not necessarily Gaussian but satisfying a mild restriction of nondeterminism. (B) An assumption about the chosen criterion of arbitraging. We shall assume the martingale is linear and the mean square drift is minimized. We shall only briefly comment upon other methods. A model having been specified, the questions to be raised fall into several categories.

A) As we have already recalled, in cases when our form of perfect arbitraging does lead to well-defined prices, such prices necessarily follow a martingale. But the notion that a specific method of arbitraging *necessarily* leads to well-defined prices is *unwarranted*. Roughly speaking, fully arbitraged prices are well defined if and only if price changes before arbitraging $P_0(t)$ satisfy a certain special condition expressing that statistical dependence decreases rapidly. In addition, the drift of the fully arbitraged prices around $P_0(t)$ has a finite variance if and only if $P_0(t)$ satisfies a second special condition expressing rapidly decreasing dependence.

B) One must investigate the partly arbitraged prices prevailing when anticipation is less than perfect. Assuming linear least squares arbitraging with a finite horizon, one would expect that arbitraging is in general less than perfect. Indeed, the changes of the arbitraged prices generally remain correlated, so prices do not follow a martingale. As anticipation

[3] A correct distinction between the concepts of martingale and random walk is made by Bachelier [1], where one finds an informal statement of the modern concept that price in an efficient market is a martingale, and a near-definitive statement of the Gaussian random walk. A correct distinction is also made in Mandelbrot [5,6,7], in Samuelson [14], and in Fama [4]. In this last paper, it is shown that Fama's earlier claim [3] that evidence supports the random walk model was unwarranted; it is also compatible with more general martingales.

improves, the correlation between successive arbitraged price changes decreases and the changes come ever nearer to being uncorrelated, but the process does not necessarily have a finite limit.

Increments of a martingale process are spectrally "white," so perfect anticipation can be called "spectrally whitening," and increasingly well anticipated prices are increasingly close to whiteness. On the other hand, we shall see that in general improvement in the perfection of the anticipation leads to increase in the variance of price changes. Such a "variance increasing" transformation can be considered "destabilizing."

C) Last but not least, it is important to investigate what is going on in arbitraging, that is, who is doing it and how do people in the market behave who are not. In what way are actual prices the result of a *general* equilibrium in the assets market?" I feel, however, that the discussion of the above issues is best separated from the mathematical comments that follow, and, anyhow, I am not especially qualified to carry it out.

Market Efficiency and the Syndrome of Infinite Variance and H-Spectrum: The preceding reasons for being concerned about the approach to efficiency through arbitraging lie in the mainstream of conventional finite variance or Gaussian econometrics, but there are other more personal reasons to my interest: a desire to find out what arbitraging can tell us about the relations between two syndromes in which I am greatly interested, the Infinite Variance Syndrome (Noah Effect) and the H-Spectrum Syndrome (Joseph Effect).

The term Joseph Effect is of course inspired by the Biblical story of the seven fat and seven lean years (*Genesis* 6:11–12). Pharaoh must have known well that yearly Nile discharges stay up and then down for variable and often long periods of time, so they exhibit strong long-run dependence and a semblance of "business cycles," but without either visible or hidden sinusoidal component (figure 1). As to the total size of crops in Egypt, it closely depends on the Nile levels; were it not for Joseph's ability to forecast the future by interpreting Pharaoh's dream and to arbitrage through storage, crop prices would have plummeted through the fat first seven years and for the lean next seven years they would have soared. Unfortunately, political economists lack Joseph's gift, so the question arises, how does perfect anticipation perform when the exogenous variables and the resulting nonarbitraged prices exhibit the kind of long run dependence described by the familiar intuitive notion of "nonsinusoidal business cycles"? Elsewhere [8,9,11], such dependence has been shown to be associated with the "H-spectrum syndrome" which expresses a very slow decay of statistical dependence between $P_0(t)$ and $P_0(t+s)$. In the present paper, it will be shown that, if one had insisted on perfect least square anticipation, the distribution of the arbitraged price changes would have to be a known distribution rescaled by an infinite constant, for example, a Gaussian with divergent variance, *which is absurd*. Therefore, our assumptions (the stationary finite variance model of unanticipated prices combined with perfect linear least squares arbitraging) lead nowhere.

One way out is to be content with imperfect (finite horizon) least squares arbitraging. This way out will be explored in this paper. Earlier, another way out has been investigated [6]. There, assuming an even more special exogenous variable, nonGaussian but having a finite variance, I had shown that perfect least squares arbitraging is not linear and that absurd divergence of $P(t)$ is avoided. That is, perfectly arbitraged prices are well defined and their changes follow a nondegenerate non-Gaussian distribution with an *infinite* variance, which means they exhibit a "Noah Effect." This last result brings us to my finding [5] that the actual distributions of price increments tend to be stable Paretian ("Pareto-Levy"), meaning that the variance is infinite.[4] In this paper, however, further pursuit of this line of thought would be out of place.

Digression Concerning the Use of the Logarithm of Price: Both the random walk and the

[4] My original discovery of the Noah Effect for prices resulted from theoretical insight and from empirical study of commodity prices (cotton, wheat and other grains), security prices (rails) and various interest and exchange rates. Fama [3] has extended the stable Paretian model to a new case study, the thirty securities of the Dow Jones index. Since my work and Fama's, the empirical evidence in favor of the reality of the infinitive variance syndrome has been broadening considerably.

martingale models of price variation conflict with the basic fact that a price is necessarily positive. A first example of such conflict is that if price followed a stationary random walk, it would almost surely eventually become negative, which is an impossibility. Second example of conflict: If it were rigorously true that price itself, which is positive, follows the martingale model, one could invoke the "martingale convergence theorem" of [2, p. 319] and one would conclude that — almost surely — such a price would eventually converge, that is, would cease to fluctuate. A commodity or security such that its price converges must eventually cease to be the object of speculation. That feature is acceptable in the special case of finite horizon commodity futures (Samuelson [14]), but in general it constitutes an excessively stringent restriction.

The first of the above examples of conflict is well known and has suggested to many authors that the random walk model should not be applied to price itself, but rather to some nonlinear function of price that can tend to either plus or minus infinity — usually the logarithm of price. This function also avoids the second conflict. Reliance on log price raises many issues, however. In particular, one can write price = exp (log price) and the exponential function is convex, so when log price is a martingale, price itself increases on the average, but how is it possible for all the prices in an economy to increase on the average?

However, the issues relative to log price are entirely distinct from those tackled in this paper. Therefore, for the sake of notational simplicity, all arguments will be carried out in terms of price itself.

Perfect Arbitraging

Nonarbitraged Prices and the Function of Causes: In order to study the mechanism of arbitraging, we shall assume that the price $P_0(t)$ that would have prevailed in the absence of arbitraging is well defined. This assumption is admittedly artificial. The function

$$\Delta P_0(t) = P_0(t) - P_0(t-1) = C(t)$$

will be called the "function of causes"; it is supposed to summarize in dollar units all the effects of supply and demand and of anything else that can conceivably affect price — with the exception of arbitraging itself. For a heavily traded security, $C(t)$ may be dominated by significant information. For a lightly traded security, timing of large block sales and a variety of other comparatively insignificant circumstances we may call "market noise," may be dominant. In order to avoid mathematical complications, our discussion will be carried out under the assumptions that the cause $C(t)$ appears in discrete integer-valued time, and that the price change $\Delta P(t) = P(t) - P(t-1)$ follows immediately.

Independent Causes and the Random Walk of Prices: If successive causes are independent, there is nothing to arbitrage and in particular the arbitraged price $P(t)$ satisfies $P(t) = P_0(t)$ and $\Delta P(t) = C(t)$. Successive increments of the price $P(t)$ are independent and $P(t)$ follows a random walk.

Dependent Causes with Finite Variance and No Deterministic Component: In general, successive causes of price change cannot be assumed independent. At any moment, "something" about the future development of $C(t)$, although, of course, not everything, may be extrapolated from known past and present values. But an efficiently arbitraged market should eliminate any possibility that a method of buying and selling based on such extrapolation be systematically advantageous. When setting up prices, everything that is extrapolable from present and past values of the causes should be taken into account. To study such extrapolation, assume that the process $C(t)$ is generated as a moving average of the general form

$$C(t) = \sum_{s=-\infty}^{t} L(t-s)N(s).$$

The quantities $N(s)$ in this expression, called "innovations," are random variables with finite variance and are othogonal (uncorrelated) but are not necessarily Gaussian. The function $L(m)$, called the "lagged effect kernel," must satisfy the relation $\sum_{m=0}^{\infty} L^2(m) < \infty$, which implies that $L(m) \to 0$ as $m \to \infty$. If, and only if, $L(m) = 0$ for $m > m_0$, the moving average is finite. (Note that N is the first letter of "new," and L is the first letter of "lagged.")

Moving average processes are less special than it might seem, since every "purely nondeterministic" stationary random process is of this form [2, section 12.4]. (The definition of "nondeterministic" is traditional and need not be repeated.) Our assumption about $C(t)$ only implies that deterministic elements have been taken out from $C(t)$.

For the random function $C(t)$, define $E_cC(t+n)$ as the conditional expected value of $C(t+n)$ knowing $C(s)$ for $s < t$, that is, knowing the present and past causes. $E_cC(t+n)$ is also known to be an optimal "least squares" estimator of $C(t+n)$. Wold has shown that, in terms of the $N(s)$,

$$E_cC(t+n) = \sum_{s=-\infty}^{t} L(t+n-s)N(s)$$

which is a linear function of the $N(s)$ for $s \leq t$.[5] We shall now study the effect of this form of $E_cC(t+n)$ upon arbitraging.

Search for the Arbitraged Price Series $P(t)$: A linear function of the values of $P_0(t)$ or $\Delta P_0(t)$ for $s < t$ can always be expressed as a linear function of the past values of $N(s)$, and conversely. Therefore, the price series $P(t)$ we seek must be such that $\Delta P(t)$ is a linear function of the values of $N(s)$ for $s < t$. For $P(t)$ to be a finite variance martingale, and a linear function of past $P_0(t)$, it is necessary that $\Delta P(t)$ be proportional to $N(t)$. We shall now seek by an indirect argument the value of this coefficient of proportionality.

Formalism of Infinite Horizon Linear Least Squares Arbitraging: At time t, potential arbitragers will know $E_cC(t+n) = E_cP_0(t+n) - E_cP_0(t+n-1)$ for all time instants in the future $(n > 0)$. We suppose the arbitragers' horizon is infinite, interest rates are zero and there is no risk aversion.

$E_cC(t+n)$ being non-zero for some n implies that prices are expected to go up or down. On the average, arbitragers will bid so as to make expected arbitraged price changes vanish. One may argue then that an arbitrager should take account of the $E_cC(t+n)$ at any instant in the future as if it were a current cause. That is, he should add up the expected future lagged effects of each innovation $N(t)$. Clearly, the total lagged effect of $N(t)$ is $N(t)[\sum_{m=0}^{\infty} L(m)]$, so our arbitrager will attempt to achieve prices whose increments satisfy

$$(*) \quad \Delta P(t) = N(t)[\sum_{m=0}^{\infty} L(m)].$$

Many questions arise: Can this attempt be successful, that is, does the preceding formal expression have a meaning? If it has a meaning, then $P(t)$ is a martingale, but how does $P(t)-P_0(t)$ behave with increasing time: is the mean square price drift $E[P(t)-P_0(t)]^2$ bounded for $t\to\infty$, and — among martingales — does $P(t)$ minimize this drift? More basic but less urgent questions: are the assumptions of the present discussion realistic? All the answers will be shown now to be in the affirmative if the moving average is finite, that is, if $L(m) = 0$ for large enough values of m. Otherwise, the answers depend upon the rapidity of the decrease of $L(m)$ as $m\to\infty$. It will momentarily be shown that three cases must be distinguished:

The case where $V = \sum_{u=1}^{\infty} [\sum_{m=u}^{\infty} L(m)]^2 < \infty$.

The case where $V = \infty$ but $|\sum_{m=0}^{\infty} L(m)| < \infty$.

The case where $|\sum_{m=0}^{\infty} L(m)| = \infty$.

The Classical Case: The classical case for $P_0(t)$ is defined by the condition

$$|\sum_{m=0}^{\infty} L(m)| < \infty$$

which is necessary and sufficient to make (*) meaningful and finite. If this condition is satisfied, the succession of price changes $\Delta P(t)$ is a sequence of orthogonal random variables with zero expectation and a finite and positive variance. These properties define the most general martingale having finite variance increments. In summary: *Under infinite horizon least squares anticipation in a finite variance universe, arbitraged prices ordinarily follow a*

[5] Wiener and Kolmogoroff have given an alternative expression of $E_cC(t+n)$ as a linear function of the past and present values of $C(t)$ itself. However the Wiener-Hopf technique used in implementation requires that $\sum_{m=0}^{\infty} L(m) < \infty$, which assumption is not inocuous and is in fact invalid in the most interesting case to be studied in this paper. "Control theoretical" tools, many of them based on "Kalman filters" have begun to draw the economists' attention. Their basic ideas are borrowed from the Kolmogoroff-Wiener theory.

martingale whose increments have finite variance so they are orthogonal.

Subcase of the classical case: Gaussian causes. In a Gaussian universe, orthogonality is synonymous with independence. Therefore: *Infinite horizon least squares anticipation in a Gaussian universe ordinarily generates prices that follow the prototype martingale, namely the Gaussian random walk without drift.*

Observe that if $\Sigma L(m) = 0$, $P(t)$ is identically constant, which is degenerate but nominally remains a martingale.

Mutual price drift $P(t) - P_0(t)$ in the classical case. For all t,

$$\begin{aligned} P(t) - P_0(t) &= \sum_{s=-\infty}^{t} N(s) \sum_{m=0}^{\infty} L(m) \\ &\quad - \sum_{s=-\infty}^{t} N(s) \sum_{m=0}^{t-s} L(m) \\ &= \sum_{s=-\infty}^{t} N(s) \sum_{m=t-s+1}^{\infty} L(m). \end{aligned}$$

As a result, $E[P(t) - P_0(t)]^2$ is independent of t and equal to V, with the following definition (note that the dummy variable $t-s+1$ is rewritten as u)

$$V = \sum_{u=1}^{\infty} [\sum_{m=u}^{\infty} L(m)]^2.$$

This expression introduces a second criterion.

If $V < \infty$, the martingale $P(t)$ wanders on both sides of $P_0(t)$, but does remain in a band of finite variance. If one replaces $P(t)$ by any other martingale, that is, by any martingale proportional to $P(t)$, the mean square drift is increased, which shows that if $P(t)$ is defined, it is a linear least squares martingale.

If $V = \infty$, on the contrary, $P(t) - P_0(t)$ will drift away without bound, which according to our criteria is not admissible.

The "Nonclassical Case." The nonclassical case of $P_0(t)$ is characterized by the condition $\sum_{m=0}^{\infty} L(m) = \infty$. In this case, the perfectly arbitraged price changes should have an infinite variance because the total price changes triggered by each innovation should be infinite. What this conclusion means is that perfect arbitraging in the nonclassical case is impossible.

Discussion: Role of the Fractional Noises: Of the two conditions $V < \infty$ and $|\Sigma L(m)| < \infty$, the condition $V < \infty$ is the more demanding, and both are more demanding than the condition $\Sigma L^2(m) < \infty$ which $L(m)$ must satisfy in order to be acceptable as a kernel. But, one might ask, why should one go through all of this complicated series of conditions? Cannot every decent and useful $L(m)$ be trusted to satisfy any condition one may demand? The answer is that in fact there exist processes called fractional noises [10], for which either or both conditions fail, and which happen to be widely encountered. One specific subfamily of these processes is called discrete fractional Gaussian noises (dfGn) and is characterized by having the covariance

$$C_H(s) = (1/2)[|s+1|^{2H} - 2|s|^{2H} - |s-|^{2H}]$$

where the parameter H lies between 0 and 1. The value $H = 0.5$ corresponds to the independent Gauss process, so the interesting cases are H between 0 and 0.5 or H between 0.5 and 1. If $P_0(t)$ is a dfGn with $0.5 < H < 1$, $L(m)$ is such that $\Sigma L(m) = \infty$, so perfect arbitraging is impossible even if one allows the drift to be infinite. If $P_0(t)$ is a dfGn with $0 < H < 0.5$, $\Sigma L(m) = 0$ but $V = \infty$, so perfect arbitraging is possible only if one allows the drift to be infinite.

Since nonarbitraged prices are — by definition — not observable directly, to claim that any actual $P_0(t)$ function behaves like the above mentioned dfGn is not verifiable directly. But there is much indirect evidence of such behavior in economic time series and also in branches of physics such as meteorology, hydrology, etc., which provide many among the more important exogeneous economic variables. An excellent example is provided by the fluctuations in the level of the Nile River. Though they are devoid of sinusoidal components, the series I have in mind typically exhibit a multitude of different cycles of different apparent wavelengths: short cycles, middle cycles, and long cycles whose wavelength has the same order of magnitude as the total sample duration. For such behavior, the most economical model is dfGn, as I have found in many subject matter fields (see [9,10,11,13]). To be on the cautious side, let me simply say that all this suggests that $P_0(t)$ series of economics that resemble fractional noise behavior are not exceptional. Hence, the fulfillment of the conditions $\Sigma L(m)$ and $V < \infty$ is not trivial. Exceptions to a good, martingale, behavior for $P(t)$ should be expected.

Imperfect Arbitraging

The Need for Discounting of the Distant Future: Even in the classical case $\sum_{m=0}^{\infty} L(m) < \infty$, distant forward predictions are so risky that infinite horizon least squares arbitraging would give to future lagged effects of past innovations an excessive weight. Unless $L(m)$ vanishes or becomes negligible when m is still small, one must assume the rate of interest is positive and the horizon is finite. The horizon decreases with increased risk aversion. Let us now show that under these more restrictive conditions, market efficiency is no longer expressed by the martingale condition.

Finite Horizon Anticipation: In the present section, the lagged effect of each innovation in the causes $C(t)$ will be followed up to some finite horizon, beyond which it will be neglected. This expresses that, for every past innovation $N(s)$, one only adds up its lagged effects up to time $t+f$, with t designating the present and f the depth of the future. Thus, the total effect of the innovation $N(s)$ will be considered as equal to

$$N(s) \sum_{n=0}^{t+f-s} L(n).$$

The resulting price $P_f(t)$ satisfies

$$\begin{aligned} \Delta P_f(t) &= \sum_{s=-\infty}^{t} N(s) \sum_{m=0}^{t+f-s} L(m) \\ &\quad - \sum_{s=-\infty}^{t-1} N(s) \sum_{m=0}^{t-1+f-s} L(m) \\ &= N(t) \sum_{m=0}^{f} L(m) \\ &\quad + \sum_{s=-\infty}^{t-1} N(s) L(t+f-s). \end{aligned}$$

Since $\lim_{n\to\infty} L(n) = 0$, it is easy to verify that

$$\text{as } f \to \infty,\ \Delta P_f(t) \to \Delta P(t) \text{ and}$$
$$P_f(t) - P_f(0) \to P(t) - P(0),$$

which expresses that the martingale process $P(t)$ of the preceding section can be considered as identical to $P_\infty(t)$. But for finite f, $\Delta P_f(t)$ is a new moving average of the form

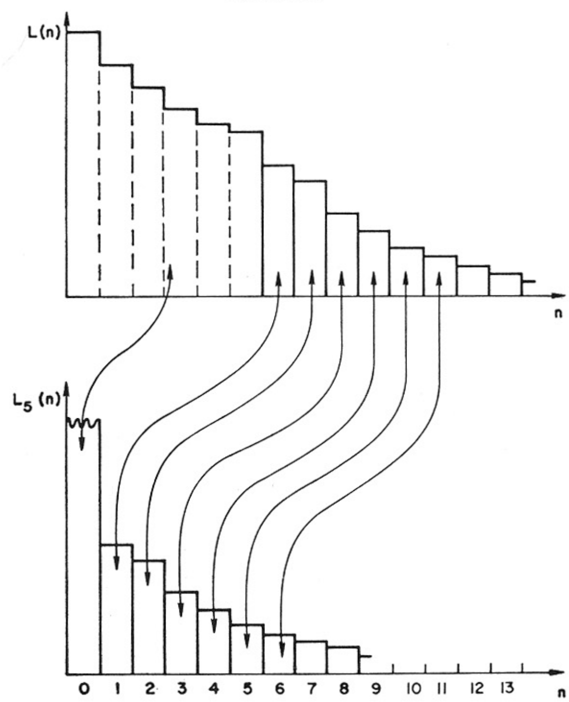
$$\Delta P_f(t) = \sum_{s=-\infty}^{t} N(s) L_f(t+1-s).$$

The function $L_f(n)$ is defined by

$$L_f(n) = \begin{cases} \sum_{m=0}^{f} L(m) & \text{for } n = 0 \\ L(f+n) & \text{for } n \geq 1. \end{cases}$$

The relationship between the two kernels $L(n)$ and $L_f(n)$ is illustrated in figure 2. The formula for $L_f(n)$ shows that the effect of finite horizon anticipation takes different forms depending upon whether or not the lagged effect function becomes strictly zero for large enough lags.

FIGURE 2. —



This is an illustration of the relationship between an original lag effect kernel $L(n)$, and the lag effect kernel $L_5(n)$ corresponding to finite horizon arbitraging of horizon $f = 5$. The heights of the bars equal the values of the kernels; areas of contours linked by arrows are identical.

A) In case the after-effects have a finite span f_0, meaning that $L(n)$ vanishes for all lags n satisfying $n > f_0$, we have

$$L_f(n) = L(f+n) = 0 \text{ for all } n \geq 1 \text{ and } f > f_0.$$

Hence, as soon as $f > f_0$, $P_f(t)$ becomes identical to the martingale $P(t) = P_\infty(t)$.

We may recall that the assumption of nearly independent causes appears most reasonable when such causes are dominated by "market noise." This suggests that, among arbitraged markets, those closest to efficiency are of two kinds: some in which anticipatory horizon is infinite, and others where the "market noise" is so overwhelming that prediction is impossible and the assumption of efficiency cannot be disproved!

B) In the case where lagged effects continue indefinitely (meaning that — however large the value of f — there exists at least one value of $n > f$ such that $L(n) \neq 0$), *the arbitraged price $P_f(t)$ is not a martingale.* That is, it remains possible to forecast that price will increase or decrease on the average. Good market analysts would obviously know of such instances, and they could trade accordingly, but they will be tempted to do so only if their horizon of forecasting exceeds that of the rest of the market; that is, only if the degree of risk they find acceptable — and hence the resources available to them — exceeds those of the rest of the market.

A danger is that one may proceed to partial arbitraging of an already arbitraged price, which would involve a longer horizon and greater risks than are wished. (One is reminded of Keynes's sarcastic remark about competitive prices being based on expectation about expectations, or on expectations about expectations of expectations.)

Mutual Drift of $P_f(t)$ versus $P_0(t)$: For all t,

$$P_f(t) - P_0(t) = \sum_{s=-\infty}^{t} N(s) \sum_{m-0}^{t-s} L_f(m) - \sum_{s=-\infty}^{t} N(s) \sum_{m-0}^{t-s} L(m) = \sum_{s=-\infty}^{t} N(s) \sum_{m-0}^{t-s} [L_f(m) - L(m)].$$

Thus,

$E[P_f(t) - P_0(t)]^2$ is again independent of t and equal to

$$V_f = \sum_{u=1}^{\infty} \left\{ \sum_{m=0}^{u-1} [L_f(m) - L(m)] \right\}^2 .$$

If $u > f$, the u^{th} term of this infinite series reduces to $[\sum_{m=u}^{u+f-2} L(m)]^2$, which is also the u^{th} term of the infinite series yielding $E[P_0(t+f) - P_0(t)]^2$. For every acceptable $L(m)$, that is, for every $L(m)$ satisfying $\Sigma L^2(m) < \infty$, the latter series converges, so that one has $V_f < \infty$. This proves that the drift of $P_f(t)$ from $P_0(t)$ is bounded without any additional assumption. Of course, as $f \to \infty$, $V_f \to V$, a quantity we know may be finite or infinite.

Alternative Forms of Finite Horizon Anticipation: One could also consider the lagged effects of all past and present innovations up to a lag of f. This leads to a price series $P^*_f(t) = N(s) \sum_{m=0}^{f} L(m)$, meaning that $P^*_f(t)$ is a martingale. But as $t \to \infty$, the mutual drift, now defined as $P^*_f(t) - P(t)$, increases without bound for every $L(m)$.

A third form of imperfect anticipation may attribute a decreased weight $W(f)$ — for example, an exponential discount factor — to the lagged effect the innovation $N(s)$ will have at the future instant $t+f$. If so, the innovation $N(s)$ has at time t a total weighted effect equal to

$$N(s) [\sum_{n=1}^{t-s} L(n) + \sum_{f=1}^{\infty} L(t-s+f) W(f)].$$

In this case, $P(t)$ does not diverge even if $\Sigma L(m) = \infty$, but otherwise the spirit of the conclusions reached in preceding subsections remains unchanged.

Effect of Arbitraging Upon Variances, Correlations and Spectra

The effect of arbitraging on the variance of price changes: Under the three basic conditions, namely infinite horizon least square anticipation, finite horizon anticipation, and absence of anticipation, the prices $P_\infty(t) = P(t)$, $P_f(t)$ and $P_0(t)$ satisfy

$$\mathrm{E}[\Delta P(t)]^2 = \left[\sum_{n=0}^{\infty} L(n) \right]^2 E(N^2).$$

$$E[\Delta P_f(t)]^2 = \left[\sum_{n=0}^{\infty} L_f^2(n) \right] E(N^2).$$

$$E[\Delta P_0(t)]^2 = \left[\sum_{n=0}^{\infty} L^2(n) \right] E(N^2).$$

These formulas show that the effect of arbitraging depends on the shape of the function $L(n)$.

A) *In the case when the lagged effect kernel $L(n)$ remains positive for all values of n and decreases monotonically to 0, arbitraging is variance increasing and can be called "de-*

stabilizing." Indeed, the variance of finite horizon anticipatory price changes $\Delta P_f(t)$ increases from a minimum for $P_0(t)$ $(f=0)$ to an asymptotic maximum for $f=\infty$.[6]

On the other hand, the *lag correlation of* $P_f(t+1) - P_f(t)$ *decreases monotonically with* f, which expresses that price increments become less and less strongly interdependent as the horizon of forecasting lengthens.[7]

When $\Sigma(m) < \infty$, the limit for $f = \infty$ is an independent process, but when $\Sigma L(m) = \infty$, $P_f(t)$ remains forever correlated. One should expect therefore to find that many actual price series — even on actively arbitraged markets — are correlated. This expectation constitutes one the main results of this paper, and it is indeen confirmed by experience [9].

B) *In the case when the lagged effect kernel* $L(n)$ *oscillates in such a way that* $\sum_{n=0}^{\infty} L(n)$ *is smaller than* $L(0)$, *arbitraging is variance decreasing and can be called "stabilizing."* Price variability is decreased by infinite horizon anticipation, but as f increases from 0 to ∞, the variance of $\Delta P_f(t)$ does not vary monotonically but oscillates up and down. An example where $\sum_{n=0}^{\infty} L(n) < L(0)$ occurs when the lagged effects of innovation begin with a nearly periodic "seasonal" before they decay for larger lags. A high seasonal is then present in unarbitraged prices but — as one would hope — perfect arbitraging eliminates it.

Special Example: When $\sum_{n=0}^{\infty} L(n) = 0$, perfect arbitraging cancels price variability completely. If unavoidable effects (like spoilage of a seasonal commodity) enter and impose a finite horizon, the seasonal effects in price variance are attenuated but not eliminated.

The Viewpoint of "Harmonic" or "Spectral Analysis." The preceding results can be re-expressed in terms of spectra. The spectral density of $P_f(t+1) - P_f(t) = C(t+1)$ is known to be equal to

$$S'(\lambda) = \Big| \sum_{n=0}^{\infty} L_f(n) \exp(2\pi n\lambda i)\Big|^2.$$

In particular, its value $S'(0)$ at zero frequency λ is equal to $[\sum_{n=0}^{\infty} L_f(n)]^2$. Now observe that the definition of $L_f(n)$ from $L(n)$ (see figure 2) implies that $\sum_{n=0}^{\infty} L_f(n) = \sum_{n=0}^{\infty} L(n)$, independently of the value of f. It follows that the spectral density of $P_f(t+1 - P_f(t)$ at $\lambda=0$ is independent of the value of f.

It must now be recalled that a process is called "white" if its spectral density is independent of the frequency. The values of a white process are mutually orthogonal; if the process is Gaussian, they are independent. Now examine $P_f(t+1) - P_f(t)$ for f varying from 0 to ∞. We start from $P_0(t+1) - P_f(t) = C(t+1)$, which was assumed nonindependent, and hence nonwhite. We end up with $P(t+1) - P(t) = P_\infty(t+1) - P_\infty(t)$ which process is independent (white). Hence, *perfect arbitraging whitens the spectral density.* But the value of the spectral density at $f=0$ stays invariant and constitutes a kind of "pivot point." As f increases from 0 to ∞, and anticipation improves, the spectral density of

[6] Proof: Write

$$\sum_{m=0}^{\infty} L_f^2(m) = \sum_{m=0}^{\infty} L^2(m) + 2 \sum_{0 \leq p < q < 0} L(p)L(q).$$

The second term on the right-hand side takes the form of a sum to which fresh elements are added as f increases.

[7] Proof: In terms of the covariance function

$\mathrm{Cov}_f(1) = E[P_f(t+1) - P_f(t)][P_f(t) - P_f(t-1)]$,

the lag correlation is written as

$\mathrm{Cov}_f(1)/\mathrm{Cov}_f(0) = \mathrm{Cov}_f(1)/E(\Delta P_f)^2$.

As f increases, we have already shown that the denominator $\mathrm{Cov}_f(0)$ increases, so it suffices to prove that the numerator $\mathrm{Cov}_f(1)$ decreases.

Since

$$\begin{aligned}\mathrm{Cov}_f(1) &= \sum_{n=0}^{\infty} L_f(n)L_f(n+1)\\ &= [\sum_{m=0}^{f} L(m)]L(f+1) + \sum_{n=f+1}^{\infty} [L(n)L(n+1)],\end{aligned}$$

where

$$\begin{aligned}\mathrm{Cov}_f(1) - \mathrm{Cov}_{f-1}(1) &= [\sum_{m=0}^{f} L(m)]L(f+1)\\ &- [\sum_{m=0}^{f-1} L(m)]L(f)\\ &+ \sum_{n=f+1}^{\infty} L(n)L(n+1) - \sum_{n=f}^{\infty} L(n)L(n+1).\end{aligned}$$

Rearranging the terms, the preceding expression equals $A+B$, with

$$A = [\sum_{m=0}^{f-1} L(m)][L(f+1) - L(f)], \text{ and}$$

$$B = [L(f)L(f+1) + \sum_{n=f+1}^{\infty} L(n)L(n+1) - \sum_{n=f}^{\infty} L(n)L(n+1)].$$

Term B vanishes, and term A is proportional to $L(f+1) - L(f)$, which is negative as asserted.

$P_f(t+1) - P_f(t)$ becomes increasingly flat. But arbitraging can do absolutely nothing to the amplitude of the really low frequency effect. If $\Sigma L(m) = \infty$, the spectrum of the arbitraged price should be expected to remain unavoidably "red," that is, to include large amounts of energy within very low frequency components. Such is indeed the case [9].

Let us now consider some of our special cases more closely. In the case A) when $L(n) > 0$ for every value of n and $L(n)$ decreases as $n \to \infty$, the spectral density of $P_f(t+1) - P_f(t)$ — considered for $\lambda > 0$ — increases monotonically with f. In other words, the only way in which arbitraging can decrease the correlation of $P_f(t+1) - P_f(t)$ is by making its high frequency effects *stronger*. This is what makes prices more variable. Some authors have proposed to call the expression $\int \lambda S'(\lambda) d\lambda / \int S'(\lambda) d\lambda$ the average frequency of a process. We see that in case A) this quantity *increases* with improved anticipation.

In the case B) when $\Sigma L(m) < L(0)$, improving arbitraging *decreases* the high frequencies effects and the average frequency decreases. In particular in the subclass of B, when $\sum_{n=0}^{\infty} L(n) = 0$, the spectral density of $P_f(t+1) - P_f(t)$ for $\lambda \geqq 0$ tends to zero as $f \to \infty$ (though not necessarily monotonically). (For $\lambda = 0$, we know $S'(\lambda)$ is identically zero for all f.)

Time Increments T Longer Than 1: The spectral density of $P_f(t+T) - P_f(t)$ at $\lambda = 0$ equals $T^2[\Sigma L(m)]^2$, also independently of f. This means that the argument about the origin as "pivot point" continues to hold. But otherwise things are too complicated to be worth describing here.

Alternative Definitions of Imperfect Arbitraging: In all the instances I have examined, the above argument about the pivot at $\lambda = 0$ continues to hold true.

"Price Continuity" and the Specialist

The concepts of "continuity" and "discontinuity" are invoked often in the study of prices, but of course these mathematical terms should not be interpreted textually. Transactions occur at discrete instants of time and are quoted in discrete units, so mathematically speaking a price series is never continuous. But a series that only moves by small steps may be interpolated by a continuous function without violence, while series that move by big steps cannot. So the concepts of continuous and discontinuous price variation — if tackled cautiously — are useful. *Roughly speaking, one can say that improvement in anticipation*, through the resulting increase in high frequency energy, *makes price variation less and less smooth and "continuous."*

Let us now turn briefly to the role of the specialist. If he attempts to insure the "continuity of the market" (to use the words of the Securities Exchange Commission (S.E.C.), he will necessarily smooth away the high frequency price fluctuations, which we have seen acts against good arbitraging. In other words, the S.E.C.'s prescription creates opportunities for systematic gain.

This appears to be a good place to recall which effect price smoothing by the specialist has in those cases where price changes are not Gaussian but have infinite variance. In my first paper on prices [5], I had shown that the big expected gains promised by S. S. Alexander's "filter method" hinged entirely on the assumption that price is "continuous," so one could buy or sell at any price prescribed in advance. My stable Paretian model of price behavior predicts, on the contrary, that price is violently discontinuous. If combined with the smoothing effect of the specialist, my model predicts that every so often prices will go up or down very steeply. Alexander assumed that one could buy or sell during these periods of steep variation, but of course this possibility is not open to ordinary buyers and sellers, so Alexander's results are not in contradiction with market efficiency.

I believe the role of the specialist deserves a more detailed study along the above lines.

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