A Fast Fractional Gaussian Noise Generator

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IMPORTANT CORRECTION

Page 545, right column, replace $\dfrac{1 - B^{-(1-H)}H(2H-1)}{\Gamma(3-2H)}$

by $1 - \dfrac{B^{-(1-H)}H(2H-1)}{\Gamma(3-2H)}$

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Abstract. By design fast fractional Gaussian noises (ffGn) have the following characteristics: The number of operations needed to generate them is relatively small. The long run statistical dependence that they exhibit is strong and has the form required for self similar hydrology. Their short run properties, as expressed by the correlations between successive or nearly successive yearly averages, are adjustable (within bounds) and can be fitted to the corresponding short run properties of records. Extension to the multidimensional (multisite) case should raise no essential difficulty. Finally, their definition, as sums of Markov-Gauss and other simple processes, fits the intuitive ideas that climate can be visualized as either unpredictably inhomogeneous or ruled by a hierarchy of variable regimes.

Discrete fractional Gaussian noises (dfGn) normalized to have zero mean and unit variance are Gaussian random processes $X_d(t, H)$ defined as having the covariance (Table 1),

$$C(s, H) = 2^{-1} \cdot [|s + 1|^{2H} - 2|s|^{2H} + |s - 1|^{2H}]$$

The dfGn have a single parameter, normalized to vary from 0 to 1, called the exponent H. The dfGn have been tailored to model the variation of natural phenomena characterized by long run effects, that is, by very strong and very durable statistical dependence. In such phenomena the correlation between successive values may be small; large lag correlations are even smaller but are such that their cumulative effect is nonnegligible. Notably the rescaled range R/S is proportional to s^H with $H \neq 0.5$; for definitions and a discussion of R/S, see *Hurst* [1951], *Mandelbrot and Wallis* [1969*a*, *b*, *c*]. If long run dependence is absent, $H = 0.5$ [*Feller*, 1951].

Hydrology is the oldest field in which long run dependence has been observed, and self similar hydrology, which is the statistical model using dfGn or other related fractional noises, has been explored in some detail [*Mandelbrot*, 1965; *Mandelbrot and Wallis*, 1968, 1969*b*]. This model appears to have been received well [*Kottegoda*, 1971; *Askew et al.*, 1971]. In addition, long run dependence is observed in economics, physics, geophysics, and other fields [*Mandelbrot and Van Ness*, 1968; *Mandelbrot*, 1969; *Mandelbrot and Wallis*, 1969*b*; *Mandelbrot and McCamy*, 1970]. Finally, the $1/f$ noises of electronics are examples of fractional noises; they have been known for a long time but have not led to a careful study of dfGn.

Construction of a sample function of dfGn would unfortunately involve an infinite number of operations, and so approximations are needed. The type 1 and type 2 functions of *Mandelbrot and Wallis* [1969*a*] are finite weighted moving averages in which the number of Gaussian variables to average is roughly proportional to the size T of the desired sample. The type 1 approximation follows the theoretical algorithm and both types have been very useful in exploratory investigations. But, as we shall see, subsequent use of the early approximations has turned out to present practical and psychological drawbacks. To avoid these drawbacks I have developed a new approximation to be designated $X_f(t, H)$ and called fast fractional Gaussian noise (ffGn); its construction is described in the next section. Details, including a list of its principal properties and advantages and the mathematical rationale, follow. Brief references will be made to other possible approximations including the use of the fast Fourier transform.

FAST FRACTIONAL GAUSSIAN NOISES

This section describes without motivation the steps in the construction of ffGn. To construct

a sample of the ffGn $X_f(t,H)$, one needs H and two additional convenience parameters, the base $B > 1$ and the quality factor Q. Recommended choices of B and Q will be described below. Typical errors of approximation, corresponding to $T = 10{,}000$, $B = 3$, and $Q = 6$, are given in Table 2.

A sample of T values of $X_f(t, H)$, normalized to have zero mean and unit variance, is then constructed as the sum of a high frequency Markov-Gauss term and a low frequency term that is the weighted sum of a number $N(T)$ of independent Markov-Gauss processes.

Definition of the low frequency term $X_L(t, H)$. By definition

$$X_L(t, H) = \sum_{n=1}^{N(T)} W_n X(t, r_n \mid MG)$$

where $X(t, r_n|MG)$ is the Markov-Gauss process of variance 1 and covariance r_n^t, with $r_n = \exp(-B^{-n})$. The weight given to $X(t, r_n|MG)$ is W_n, with

$$W_n^2 = \frac{H(2H - 1)(B^{1-H} - B^{-1+H})}{\Gamma(3 - 2H)} B^{-2(1-H)n}$$

Γ being the gamma function. Thus each term depends on B and H. The number of terms satisfies

$$N(T) = ||\log (QT)/\log B||$$

which also depends on the desired sample size T; $||X||$ designates the smallest integer above X. The base B and the quality factor (or relative memory) Q together determine the quality of approximation.

Some indications about computation may be useful. To generate a sample from $t = 1$ to $t = T$ of $X(t, r_n|MG)$, one needs a sequence of Gaussian variables of zero mean and unit variance $G_n(t)$. One proceeds by the following steps:

$$X(1, r_n \mid MG) = G_n(1)$$
$$X(2, r_n \mid MG) = r_n X(1, r_n \mid MG) + (1 - r_n^2)^{1/2} G_n(2)$$
$$\vdots$$
$$X(t, r_n \mid MG) = r_n X(t - 1, r_n \mid MG) + (1 - r_n^2)^{1/2} G_n(t)$$
$$\vdots$$

TABLE 1. Values of 10,000 $C(s, H)$ for Various Values of s and H

s	$H = 0.55$	$H = 0.60$	$H = 0.65$	$H = 0.70$	$H = 0.75$	$H = 0.80$	$H = 0.85$	$H = 0.90$	$H = 0.95$
0	10,000	10,000	10,000	10,000	10,000	10,000	10,000	10,000	10,000
1	717	1,486	2,311	3,195	4,142	5,157	6,245	7,411	8,660
2	306	711	1,232	1,887	2,696	3,683	4,874	6,301	7,996
3	208	505	914	1,461	2,180	3,109	4,295	5,792	7,668
4	159	398	743	1,224	1,882	2,765	3,933	5,463	7,447
5	129	332	634	1,069	1,681	2,526	3,676	5,222	7,281
6	110	287	557	957	1,533	2,347	3,479	5,034	7,149
7	95	253	500	872	1,419	2,206	3,321	4,880	7,039
8	84	227	455	805	1,327	2,090	3,190	4,751	6,945
9	76	207	419	749	1,250	1,994	3,079	4,640	6,864
10	69	190	389	703	1,186	1,911	2,983	4,543	6,792
20	37	109	239	464	838	1,448	2,422	3,955	6,336
30	25	78	180	363	684	1,231	2,144	3,646	6,084
40	19	62	147	306	592	1,097	1,967	3,442	5,912
50	16	52	126	267	530	1,003	1,840	3,292	5,781
60	13	45	111	240	484	933	1,742	3,174	5,677
70	12	40	99	218	448	877	1,663	3,078	5,590
80	10	36	90	201	419	831	1,598	2,997	5,516
90	9	32	83	188	395	793	1,542	2,927	5,451
100	8	30	77	176	375	760	1,494	2,866	5,394
200	4	17	47	116	265	576	1,213	2,495	5,033
300	3	12	35	91	216	490	1,074	2,300	4,833
400	2	9	29	76	187	436	986	2,172	4,696
500	2	8	25	67	167	399	922	2,077	4,592
600	1	7	22	60	153	371	873	2,003	4,509
700	1	6	19	54	141	349	833	1,942	4,440
800	1	5	18	50	132	331	800	1,891	4,381
900	1	5	16	47	125	315	773	1,847	4,330

TABLE 2. Values of 10,000 Times the Error of Approximation of $C(s, H)$ for Various Values of s and H for $T = 10{,}000$, $B = 3$, and $Q = 6$

s	$H = 0.55$	$H = 0.60$	$H = 0.65$	$H = 0.70$	$H = 0.75$	$H = 0.80$	$H = 0.85$	$H = 0.90$	$H = 0.95$
0	0	0	0	3	13	53	208	780	2,835
1	0	0	0	3	13	53	208	780	2,835
2	63	111	141	152	153	169	291	828	2,852
3	20	40	57	68	80	114	256	811	2,849
4	4	8	13	18	29	69	222	790	2,841
5	−1	−2	−3	−2	7	48	204	778	2,835
6	−3	−6	−9	−9	0	39	196	773	2,833
7	−3	−7	−10	−11	−2	37	194	772	2,832
8	−3	−7	−10	−10	−2	37	194	771	2,832
9	−3	−6	−9	−10	−2	38	194	771	2,832
10	−2	−5	−8	−9	−1	38	195	771	2,832
20	−1	−3	−5	−5	2	41	196	772	2,832
30	−1	−2	−3	−3	4	43	198	773	2,832
40	0	−1	−2	−2	5	44	199	773	2,832
50	0	−1	−2	−1	6	45	199	773	2,832
60	0	−1	−2	−1	6	45	200	773	2,832
70	0	−1	−1	0	7	46	200	774	2,832
80	0	−1	−1	0	7	46	201	774	2,832
90	0	0	−1	0	8	47	201	774	2,832
100	0	0	−1	0	8	47	201	774	2,832
200	0	0	0	0	9	48	202	775	2,832
300	0	0	0	1	10	49	203	775	2,832
400	0	0	0	1	10	50	203	775	2,832
500	0	0	0	1	11	50	203	775	2,831
600	0	0	0	1	11	50	203	775	2,831
700	0	0	0	2	11	50	204	776	2,831
800	0	0	0	2	11	50	204	776	2,831
900	0	0	0	2	11	51	204	775	2,831

Note that the errors for a given value of H depend very little on the value of s with the exception of $s = 2$. Even the remaining error there could be readily eliminated by replacing the Markov-Gauss high frequency term with an appropriate three-step moving average. However this step appears unnecessary or at least premature.

Each operation requires the latest value $X(t - 1, r_n|MG)$, which must be taken from memory, and a Gaussian random variable independent of the values used previously, which may be generated either in advance or on the spot; then one performs a weighted average that involves two multiplications and one addition. As $B \to 1$ and $Q \to \infty$, the function $N(T)$ increases and the quality of the low frequency approximation improves.

Recommended values of B and Q. I have found it convenient to take B equal to 2, 3, or 4; the recommended value of Q depends on H and on two further convenience parameters *th*2 and *th*3, as shown in Table 3. The role played by TQ is analogous to that of the memory M as defined for type 1 and type 2 approximations in *Mandelbrot and Wallis* [1969*a*].

Definition of the short run, high frequency term $X_h(t, H)$. The construction of the high frequency term is independent of T and can be selected to fit either the data or the dfGn model optimally. The term $X_h(t, H)$ will usually be a Markov-Gauss process, but for H near 1 it is sufficient to settle for an independent Gaussian term that fits dfGn best; its variance depends on H and B as follows:

$$\frac{1 - B^{-(1-H)}H(2H - 1)}{\Gamma(3 - 2H)}$$

If one takes for $X_h(t, H)$ the recommended Markov-Gauss process, its variance should be as described above and its correlation of lag one should be

$$2^{2H-1} - 1 + \sum_{n=1}^{N(T)} W_n(1 - r_n) - \frac{B^{-(1-H)}H(2H - 1)}{\Gamma(3 - 2H)}$$

TABLE 3. Values of Quality Factor Q for Various Thresholds th and Values of H

	$H = 0.55$	$H = 0.60$	$H = 0.65$	$H = 0.70$	$H = 0.75$	$H = 0.80$	$H = 0.85$	$H = 0.90$	$H = 0.95$
$th2 = 0.1$, $th3 = 0.1$	0.5*	1.0	1.7	2.4	3.2	4.0	4.7	5.4	6.0
$th2 = 0.05$, $th3 = 0.1$	0.5*	1.1	1.9	3.0	4.5	6.1	7.9	9.6	11.1
$th2 = 0.1$, $th3 = 0.05$	1.0	2.0	3.1	4.2	5.1	5.9	6.6	7.1	7.5
$th2 = 0.05$, $th3 = 0.05$	1.0	2.3	4.0	6.1	8.2	10.2	12.0	13.5	14.6

* Only Q values for which $1/Q$ is not less than 1.

COMPARISONS OF THREE APPROXIMATIONS TO DISCRETE FRACTIONAL GAUSSIAN NOISE

In this section several characteristics of the ffGn $X_f(t, H)$ are described and contrasted with the corresponding properties of the type 1 and type 2 approximations $X_1(t, H)$ and $X_2(t, H)$. Each characterization is followed by a digression that the reader in a hurry can skip.

First Comparison between Approximations

The first comparison between approximations concerns the quality of fit in the long run, low frequency range. The low frequency properties of ffGn with large Q and of the type 1 approximations with large memory M are practically identical and agree with the empirical s^H law of Hurst, where H may have any value between 0.5 and 1.0.

Digression concerning type 2 approximations. *Mandelbrot and Wallis* [1969a], who introduced the type 2 approximation, have noted that its low frequency behavior is good only when H is near 1. For other values of s the transient regime, a characteristic of graphs of R/S before the s^H law takes over, is extremely long. This remark may explain why several investigators who used type 2 functions for the sake of economy of computation have obtained very odd results. For my part, I now regret that the definition of $X_2(t, H)$ ever got into print.

Second Comparison between Approximations

The second comparison between approximations concerns the number of operations required for simulation. As compared to type 1 and type 2 functions, ffGn requires much smaller numbers of multiplications and of quantities to store. Computing a sample of T values of ffGn requires about $2N(T)T + 2T = [2 \log (QT)/\log B + 2]T$ multiplications and as few as $N(T) \sim \log (QT)/\log B$ quantities to store, whereas type 1 requires $10T^2$ multiplications and $10T$ quantities to store. Thus the cost of an ffGn sample, evaluated as a fraction of the cost of a type 1 sample, is roughly proportional to $\log (QT)/(T \log B)$, which is very small when T is large. This ratio even decreases as the ffGn approximation is made rougher by increasing B.

Digression. One may consider it a general rule that, as the horizon T increases, hydrologic models necessarily become increasingly complex and costly. For example, for multilag autoregressive models [*Fiering*, 1967, p. 85, figure 3.8], the minimum number of lags had to increase roughly proportionately to T. This observation was empirical. For type 1 and type 2 approximations, a similar rule was established semiempirically by the fact that the memory M had to be roughly proportional to T. For the ffGn process the general rule still holds, but T is replaced by $\log T$. Not only is the resulting saving enormous, but the rate at which the complication of the model must increase with T now follows a very precise rule.

Digression concerning the use of fast Fourier transforms. In harmonic analysis, fast Fourier transform (fFt) methods also lead to economies described by the replacement of T by $\log T$. The two instances of such economy seem unrelated.

On the other hand, the current availability of good fFt programs shows that samples of fractional noise can be generated by still another method, which is also much faster than the type 1 algorithm but requires much larger core memories than ffGn (M. Taqqu, private communication, 1970).

Third Comparison between Approximations

The third comparison between approximations concerns the quality of fit in the short run, high frequency range. The present study disre-

gards seasonal yearly variation, and so a short run effect is defined as one concerning the interdependence between successive or nearly successive yearly averages. The high frequency term of ffGn may be selected in either of two ways: (1) it may be fitted to the theoretical dfGn optimally and thus it depends only on H and the convenience parameters B and Q, or (2) it may be fitted to have any other high frequency, low lag properties that may be suggested by empirical records. In the former case ffGn is as nearly self similar as desired; in the latter case ffGn is only asymptotically (not strictly) self similar.

To appreciate this freedom in fitting, we must recall the successive steps that led to the dfGn model. Starting from Hurst's law, *Mandelbrot* [1965] conjectured that hydrologic records are asymptotically self similar for large lags. From the viewpoint of analytical manipulation, the simplest model is uniformly self similar for all lags, but this model exhibits high frequency complications, and so the best alternative is dfGn. Nevertheless, the fact that the high frequency properties of dfGn were built into the type 1 approximation is a serious drawback. An ideal statistical model should give the hydrologist freedom to select a short run interpolation either to be the simplest and most convenient (the dfGn model) or to be as close as possible to the record evidence (the ffGn function).

Fourth Comparison between Approximations

The fourth comparison between approximations concerns the ease of extension to the multidimensional (multisite) case. Study of the temporal variation of the discharge of one river at one spot is usually a preliminary to the design of wider networks. For this task the joint study of the discharges of a number of rivers is necessary. The various discharges taken together may be thought of as a vector, but the study of multidimensional vector processes is much more difficult than that of one-dimensional scalar processes.

In the ffGn model, low, mid, and high frequency effects are effectively separated and the laws of dependence relative to each can be treated independently and assumed to be different. Caution against excessive increase in the number of parameters is of course advised. In this paper the multidimensional theory will not be developed.

Fifth Comparison between Approximations

The fifth comparison between approximations concerns their psychological acceptability and the concepts of nonhomogeneous discharge and of discharges ruled by a hierarchy of regimes. A tool for the scientific model maker, statistical self similarity has the virtue of being simple and the handicap of being unexplained. Explanation may mean either a full and explicit reduction to the basic laws of fluid mechanics or, less ambitiously, an adequate conformity with the practitioners' intuitive vision of hydrologic reality. Each of the competitors of the self similar model embodies such an intuitive notion. The standard Markov models embody the idea that some water is carried over from one year to the next and perhaps to the year after next. Some nonhomogeneous models embody the idea that the discharge is ruled by an independent process fluctuating around a mean level, with the added complication that the mean level is subject to jumps or to variable trends. Other nonhomogeneous models think of climatic change as ruled by a hierarchy of random processes. The simplest models postulate that observations result from fast, short memory fluctuations superimposed on a secular level of slow, long memory fluctuations. More complex models invoke several regimes, the short memory fluctuations being added to slower swings, themselves superimposed on even slower swings, and so forth, down to processes that vary so slowly that within a time span equal to the sample length T, they are constant.

Despite appearances to the contrary, the self similar and random variable regime statistical models are not incompatible. In fact, they can be fitted so that the sample functions that they generate are undistinguishable. (However, the Markov model fitted to the observed small lag characteristic of the data stands apart.) That is, a self similar model of given H can be approximated by any of many different variable models. It is even possible to view the ffGn approximation as a variable model in which each of its Markov-Gauss components is visualized as ruling one regime. Conversely, it is typical for a variable model, if the sources of variability are random and very numerous, to be safely

approximated by some self similar model. This last situation is reminiscent of the fact that the addition of a large number N of random variables smoothes out the idiosyncrasies of the addends, the sum being typically approximated by a Gaussian variable.

Until it becomes possible to study the different components of the variable regime model separately, I do not think such models should be taken too seriously. They are psychologically satisfying, however, and so it is good that the self similar model should not contradict them.

DIGRESSION ON MINIMAL APPROXIMATIONS TO DISCRETE FRACTIONAL GAUSSIAN NOISE

Other authors have also recognized that random processes that include both a rapidly and a slowly varying term are necessary in hydrologic modeling. It is useful to view such combinations as crude (minimal) approximations of ffGn. Examples are provided by (1) the sum of one independent Gauss process and one slowly varying Markov-Gauss process and (2) the sum of a rapidly and a slowly varying Markov-Gauss process. The former can be written as

$$X(t) = EX(t) + \sigma_1 G(t) + \sigma_2 M(t)$$

where $G(t)$ is an independent Gauss process of zero mean and unit variance, $M(t)$ is a slowly varying Markov-Gauss process of zero mean, unit variance, and large correlation r, and finally σ_1 and σ_2 are two weights. Not counting the mean EX and the variance $\sigma_1^2 + \sigma_2^2$, $X(t)$ has two nontrivial parameters, r and σ_1/σ_2. Its covariance (Figure 1) had been considered previously by *Fiering* [1967, p. 74, equation 3.35], who seems, however, to have abandoned it in favor of multilag autoregressive models.

The sum of two Markov-Gauss processes (N. C. Matalas, private conversation, 1969) has three nontrivial parameters.

As compared to either classical alternative, the independent Gauss process (no nontrivial parameter) or the Markov-Gauss process (one nontrivial parameter r), the two approximations described above are considerable improvements. They are even preferable to the two-step Markov-Gauss process (two nontrivial parameters) and to multilag autoregressive processes (three nontrivial parameters or more).

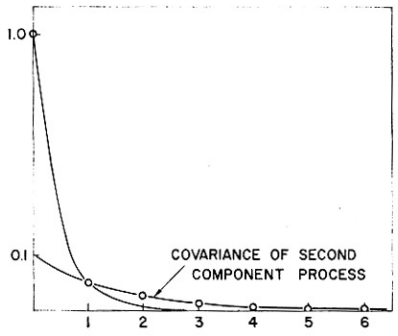

Fig. 1. The circles mark the value of the covariance for the sum of an independent Gauss process of variance 0.9 and a Markov-Gauss process of covariance 0.1 $(0.5)^s$. The Markov-Gauss process fitted to have the value 0.05 for a lag of one would have the steeply falling covariance also marked on this figure.

The reason that the Markov-Gauss process model is poor is that to make its lag one correlation fit the evidence, its large lag correlation must be made absolutely negligible, whereas to give it a nonnegligible covariance for large lags, its lag one correlation must be made absurdly large. That the sum of an independent process and a Markov process avoids this dilemma is shown in Figure 1.

However, the value of the minimal processes should not be exaggerated. Other forms of ffGn, whose cost is only slightly higher, provide a better approximation.

DERIVATION OF THE LOW FREQUENCY TERM OF FAST FRACTIONAL GAUSSIAN NOISE

The function $r^s = e^{-us}$ is known to be the covariance of a Markov-Gauss process of unit variance 1 having a lag one correlation equal to r, and so one can consider its damping time as equal to $1/u = 1/\log(1/r)$. Thus to approximate a process $X(t)$ by a finite sum of Markov-Gauss processes, one must represent the covariance of $X(t)$ in the form $\Sigma\, W_n \exp(-su_n)$. The ffGn approximation will proceed in successive steps, which will be summarized.

In step 0, the covariance $C(s, H)$ of dfGn is examined more closely.

In step 1, $C(s, H)$ is replaced by a covariance C_1 (s, H) (the index 1 refers to step 1) that is not a sum but an integral of the form $\int_0^\infty W(u) e^{-su} du$ so that the approximating proc-

ess is the sum of an infinite number of infinitesimal Markov-Gauss processes.

In step 2, high frequencies in $C_1(s, H)$ are dropped by replacing $u = \infty$ by some finite upper bound of integration. This step leads to an approximate covariance $C_2(s, H)$. More accurate high frequency terms will be introduced in a later section.

In step 3, all very low frequency terms are dropped by replacing $u = 0$ with some positive lower bound of integration. This step leads to an approximate covariance $C_3(s, H)$.

In step 4, the integral $C_3(s, H)$ is replaced by a finite sum $C_4(s, H)$ in the remaining mid frequency range, but only after a change of the variable u. Thus $C_4(s, H)$ is based on increments of u of unequal size. This fact is what will make computation so much more economical.

The detailed description of the five steps follows.

Step 0

In step 0 dfGn is defined. As previously noted, the normalized dfGn of exponent H $(0 < H < 1)$, $X_d(t, H)$, is a zero expectation, unit variance, stationary Gaussian random process, the time t being the discrete integer valued. Its covariance $C(s, H)$ depends on a single parameter H $(0 < H < 1)$ as follows:

$$C(s, H) = E[X_d(t, H) \cdot X_d(t + s, H)]$$
$$= 2^{-1}[|s + 1|^{2H} - 2|s|^{2H} + |s - 1|^{2H}]$$

For all H, $C(0, H) = 1$. In the special case $H = 0.5$, $C(s, 0.5) = 0$ for all $s \geq 1$; therefore $X_d(t, 0.5)$ is a discrete sequence of independent Gaussian values. The properties of dfGn in the two subranges $0.5 < H < 1$ and $0 < H < 0.5$ are very different. We shall be concerned with the subrange $0.5 < H < 1$, which is practically more useful. Values of $C(s, H)$ for H varying from 0.55 to 0.95 by increments of 0.05 and for s varying from 0 to 900 are listed in Table 1.

Digression. Although there are many variants of fractional Gaussian noise, one had to be selected as a basis for ffGn. Which one is chosen does not matter greatly, because I expect that ultimately ffGn will be fitted to the data rather than to any theoretical formula. Nevertheless, dfGn has advantages when one is dealing with total discharge or precipitation within a time unit, because dfGn is the sequence of increments of fractional Brownian motion over successive time units.

Step 1

In step 1 the covariance $C_1(s, H)$ is introduced. First note that $C(s, H)$ is the finite second difference of the function $s^{2H}/2$. For large s, $s^{2H}/2$ does not vary very rapidly with s, and its second difference can be closely approximated by its second derivative, namely,

$$C_1(s, H) = H(2H - 1)s^{2H-2}$$

The approximation error $C(s, H) - C_1(s, H)$ is $-\infty$ for $s = 0$; its values for $s \geq 1$, listed in Table 4, are positive, and as $s \to \infty$, they tend to zero rapidly.

Next, observe that one can write

$$C_1(s, H) = H(2H - 1)s^{2H-2} = \int_0^\infty e^{-su} W(u)\, du$$

with $W(u) = u^{1-2H}/2|\Gamma(-2H)|$. The integrand of $C_1(s, H)$ is the covariance of a Markov-Gauss process for which the correlation of lag one is equal to e^{-u} and the variance is infinitesimal and equal to $W(u)du$.

Digression. A justification of the formal step from $C(s, H)$ to $C_1(s, H)$ is that there is a random process $X_c(t, H)$ of which $C_1(s, H)$ is the covariance, namely, the continuous time fractional Gaussian noise. The random process $X_c(t, H)$ is the derivative of a process $B_H(t)$ called fractional Brownian motion [*Mandelbrot and Van Ness*, 1968]. (In terms of $B_H(t)$, the process $X_d(t)$ is made of the integer time increments of $B_H(t)$.) However, the fact that $C_1(0, H) = \infty$ shows that $B_H'(t)$ is not an ordinary function but requires special interpretation as a Schwartz distribution. Here it is best to forget about $B_H'(t)$ and merely consider $C_1(s, H)$ and $C(s, H)$ as formal approximations of each other.

To represent $C_1(s, H)$ in the form $\int_0^x e^{-su} W(u)du$ is equivalent to writing $W(u)$ as the inverse Laplace transform of $C_1(s, H)$. Standard tables indicate that s^z is the Laplace transform of $u^{z-1}/\Gamma(Z)$, namely,

$$s^z = \frac{\int_0^\infty e^{-su} u^{Z-1}\, du}{\Gamma(Z)}$$

TABLE 4. Values of 10,000 $[C(s, H) - C_1(s, H)]$ for Various Values of s and H

s	$H = 0.55$	$H = 0.60$	$H = 0.65$	$H = 0.70$	$H = 0.75$	$H = 0.80$	$H = 0.85$	$H = 0.90$	$H = 0.95$
1	168	287	362	396	392	358	296	212	111
2	12	22	32	40	45	46	42	34	19
3	4	7	11	13	15	16	16	13	8
4	2	3	5	6	7	9	8	7	4
5	0	1	2	3	4	5	5	4	3
6	1	1	1	2	3	3	4	3	2
7	0	1	1	1	2	3	3	2	1
8	0	0	1	1	2	1	2	1	1
9	0	1	1	0	1	1	2	1	1
10	0	0	0	0	1	1	1	1	1
20	0	0	0	0	0	0	0	1	0

In particular setting $Z = -2H + 2$,

$$s^{2H-2} = \frac{\int_0^\infty e^{-su} u^{1-2H}\, du}{\Gamma(-2H + 2)}$$

and

$$H(2H - 1)s^{2H-2} = \frac{\int_0^\infty e^{-su} u^{1-2H}\, du}{\Gamma(-2H + 2)/H(2H - 1)} = \frac{\int_0^\infty e^{-su} u^{1-2H}\, du}{2\,|\Gamma(-2H)|}$$

Step 2

In step 2 the covariance $C_2(s, H)$ is introduced by dropping the high and very high frequency terms of $C_1(s, H)$. A Markov-Gauss process will be called rapidly varying if its lag one correlation e^{-u} lies below some threshold $th1 < 1$. From the viewpoint of simulation, the sum of such Markov-Gauss components can be approximated by a single much simpler process, to be taken care of later when we deal with the high frequency component. Hence one can replace $C_1(s, H)$ by

$$C_2(s, H) = \frac{\int_0^{-\log(th1)} e^{-su} u^{1-2H}\, du}{2\,|\Gamma(-2H)|}$$

The threshold $th1$ is acceptable only if $C_2(0, H) < 1$. For this condition, it suffices that

$$\int_0^{|\log(th1)|} u^{1-2H}\, du < 2\,|\Gamma(-2H)|$$

which yields

$$[|\log(th1)|]^{2(1-H)} < 4(1 - H)\,|\Gamma(-2H)| = \Gamma(3 - 2H)/H(2H - 1)$$

A sufficient condition valid for all H is $|\log(th1)| < 20$ (i.e., $th1 > e^{-20} \sim 10^{-18}$), which ordinary values of $th1$ will satisfy handily. In Step 4 a special value of $th1$ will impose itself.

Step 3

In step 3 the covariance $C_3(s, H)$ is introduced by dropping the Markov-Gauss components of $C_2(s, H)$ that follow either of two criteria: they vary very slowly or they are negligible to begin with. First the slow variation criterion is considered. Between the instants $t = 1$ and $t = T$, the elementary Markov-Gauss process of covariance $e^{-tu}\, W(u)du$ increases by a random amount of variance

$$W(u)\, du\, \{1 - \exp[-(T - 1)u]\} \sim W(u)[1 - \exp(-Tu)]\, du.$$

Let the relative change in variance be defined as $1 - \exp(-Tu)$. If this quantity lies below a second threshold $th2$, the resulting Markov-Gauss process is essentially constant, and in simulation it need not be considered. In other words one can drop all values of u below u', u' being defined by

$$1 - \exp(-Tu') = th2$$

or

$$u' = T^{-1}\,|\log(1 - th2)|$$

Note that $u'T$ is independent of H. If $th2$ is small, then $u'T \sim th2$.

The next criterion is based on the contribution of different values of u to the variance of ffGn. The idea is to increase the lower bound of u from u' to u'' to weed out the Markov-Gauss contributions that do vary rapidly but are negligible to begin with. The difficulty is that the notion of negligible contribution is not determined uniquely; from the short run viewpoint, a Markov-Gauss process of covariance $e^{-uv}W(u)du$ is negligible if only $W(u)$ is very small, irrespective of the value of u. But from the long run viewpoint we shall require the average of this Markov-Gauss process between times $t = 1$ and $t = T$ to be very small. The expectation of this average obviously vanishes. Its variance is readily evaluated and turns out to be $2T^{-2}u^{-2}(Tu-1 + e^{-Tu})W(u)du$. With our $W(u)$, this expression becomes

$$2T^{-2}u^{-2}(Tu - 1 + e^{-Tu}) \cdot\{u^{1-2H}[2\ |\Gamma(-2H)|]^{-1}\}\ du$$

Even if $W(u)$ is very small, the preceding expression becomes large if u is very small. Our second criterion will be that the ratio between the contribution of values of u between u' and u'' and the contribution of values above u' should lie below a third threshold $th3$. Since high frequency terms with very large u contribute very little, one need not worry about them. Thus our second criterion yields

$$\frac{\int_{u'}^{u''} u^{-1-2H}(Tu - 1 + e^{-Tu})\ du}{\int_{u'}^{\infty} u^{-1-2H}(Tu - 1 + e^{-Tu})\ du} = th3$$

Changing the variable from u to $z = uT$, we transform this condition into

$$\frac{\int_{u'T}^{u''T} z^{-1-2H}(z - 1 + e^{-z})\ dz}{\int_{u'T}^{\infty} z^{-1-2H}(z - 1 + e^{-z})\ dz} = th3$$

When $u'T$ and $u''T$ do not exceed 1, one can approximate $e^{-z}-1 + z$ by $z^2/2$, and the condition becomes

$$\frac{(u''T)^{2-2H} - (u'T)^{2-2H}}{4(1 - H)\ |\Gamma(-2H)| - (u'T)^{2-2H}} = th3$$

Write $u''T = 1/Q$, and transform the denominator by noting first that $4(1 - H)|\Gamma(-2H)| = \Gamma(3 - 2H)/H(2H - 1)$ and next that in the range $0.5 < H < 1$ the term $\Gamma(3 - 2H)$ depends little on H and is reasonably approximated by 1. Solving, we obtain the final criterion,

$$Q^{-2(1-H)} = [-\log(1 - th2)]^{2(1-H)} \cdot(1 - th3) + th3/H(2H - 1)$$

Solutions of this equation for Q, rounded to the nearest tenth, are given in Table 3. One can verify that, except for the two starred values, $1/Q$ is indeed less than 1 as was assumed in deriving the approximation. Step 3 yields the approximation,

$$C_3(s, H) = \frac{\int_{1/TQ}^{-\log(th1)} e^{-su}u^{1-2H}\ du}{2\ |\Gamma(-2H)|}$$

Step 4

In step 4 a finite discrete Markov-Gauss approximation $C_4(s, H)$ is introduced. This step processes the Markov-Gauss terms that were left after the preceding steps eliminated the low frequencies and the high and very high frequencies. In a first substep we select a parameter $B > 1$ and change the variable of integration from u to B^{-v}, v being continuous from $-\infty$ to ∞. Applied to the original C_1, this change yields

$$C_1(s, H) = \frac{\log B}{|2\Gamma(-2H)|} \cdot\int_{-\infty}^{\infty} \exp(-sB^{-v})B^{-2(1-H)v}\ dv$$

Digression. Behind this change of variable lies the idea that when two Markov-Gauss processes have the respective damping times $1/u'$ and $1/u''$, the best measure of the degree of similarity or difference between these processes is not the difference, $u' - u''$, but rather the ratio u'/u''. After the change of variable, $u = B^{-v}$, this degree of similarity or difference is measured by $B^{-(v'-v'')}$ or simply by $v' - v''$ itself. The best approximation is one that relies on uniformly spread values of v.

In a second substep the variable v is made discrete by dividing its domain of variation into

spans of unit length of the form $n - 0.5$ to $n + 0.5$, n being an integer.

At one end the values of n much below 0 correspond to very high frequencies (step 2), and it is simplest to set the boundary between mid and high frequencies by calling $n = 1$ the highest frequency in the mid range. Thus a threshold $th1$ equal to $\exp(-1/B^{1/2})$ is implied, which for ordinary B satisfies the condition on $th1$ obtained in step 2.

At the other end, values of n much above 0 correspond to low frequencies (step 3) and can be truncated as soon as possible beyond the quantity v that satisfies $B^{-v} = 1/TQ$; i.e., $v \sim \log(TQ)/\log B$. Since n must be an integer, one must round off $\log(TQ)/\log B$ to the nearest larger integer, designated $||\log (TQ)/\log B||$.

Within each span of v, the integrand $[\exp(-sB^{-v})][B^{-2(1-H)v}]$ is the product of the two terms in brackets. In a third substep these terms are approximated as follows: $\exp(-sB^{-v})$ is replaced by its mid span value $\exp(-sB^{-n})$, whereas $B^{-2(1-H)v}dv$ is replaced by its integral over the total span. This process leads to the final approximation

$$C_1(t, H) \sim C_4(t, H)$$

with

$$C_4(s, H) \sim \frac{\log B}{2\,|\Gamma(-2H)|} \cdot \sum_{n=0}^{||\log (QT)/\log B||} \exp(-sB^{-n}) \cdot \int_{n-0.5}^{n+0.5} B^{-2(1-H)v}\,dy$$

$$= \sum_{n=0}^{||\log (QT)/\log B||} \exp(-sB^{-n}) \cdot \frac{B^{1-H} - B^{-1+H}}{4(H-1)\,|\Gamma(-2H)|} B^{-2(1-H)n}$$

Finally the characterization sketched at the beginning of the paper is obtained by replacing $|\Gamma(-2H)|$, which is not given in the tables, by $\Gamma(3 - 2H)[(2H)(2H - 2)(2H - 1)]^{-1}$.

Approximation Errors

Steps 1 and 2 have introduced a high frequency error that is important and will be discussed in the next section. Step 3 has introduced a low frequency error that, being an unknown (random) constant of known variance, is not important. Step 4 has introduced a grid error, but it is not very important as long as we avoid cases in which T and B are both large and log $(QT)/\log B$ is small.

DERIVATION OF THE HIGH FREQUENCY TERM OF FAST FRACTIONAL GAUSSIAN NOISE

The approximations in steps 1 and 2 add up to an error term with a variance equal to

$$1 - C_2(0, H) = 1 - \frac{[-\log (th1)]^{2-2H}}{4(1-H)\,|\Gamma(-2H)|}$$

Adjustment to djGn. The crudest correction is to approximate this error by an independent Gaussian process. Since steps 3 and 4 do not affect the high frequency terms, we obtain the approximation

$$X_f(t, H) \sim X_4(t, H) + [1 - C_2(0, H)]^{1/2} G(t)$$

The covariances of $X(t, H)$ and of this last approximation coincide for $s = 0$ and for large s but differ for $s = 1$ by $C(1, H) - C_2(1, H)$, which happens to be positive. A better correction will thus be achieved by a Markov-Gauss process whose covariance equals

$$C_0(0) = 1 - C_2(1, H)$$

$$C_0(1) = C(1, H) - C_2(1, H)$$

$$C_0(s) = [C_0(1)]^s \qquad s \geq 2$$

Adjustment of ffGn to empirical short lag correlations. As long samples of an empirical record become available, the model maker may become increasingly demanding. I have only investigated the case in which the record is reasonably nearly Gaussian and its long run properties are reasonably described by some $H > 0.5$, but its variance and lag one covariance do not fit the variance and lag one covariance predicted by the last obtained approximation. Clearly, change in the short run component will not affect the long run effects but will change the variance and lag one covariance. As a result a more accurate fit may be allowed between model and data. The algebra is straightforward but so lengthy that going through it here appears premature. Details may be provided later.

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