On an Eigenfunction Expansion and on Fractional Brownian Motions.

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(ricevuto il 7 Gennaio 1982)

Summary. A discussion of a modified fractional Brownian motion, which appeared in this journal, is largely incorrect.

The fractional-Brownian-motion process (fBm) is the Weyl's fractional integro-differential of the ordinary Brownian-motion process of Wiener. It is originally defined by MANDELBROT and VAN NESS (1) as

$$B_{\mathrm{WH}}(t) = \int_{-\infty}^{t} \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\, \mathrm{d}B(s)\,,$$

where $B(t)$ is Wiener's Brownian process. The index W, which stands for Weyl, has been added here to avoid ambiguity in the sequel. Together with a bilateral version, $B_{\mathrm{WH}}(t)$ has proven very valuable (2).

For reasons he does not disclose, MACCONE (3) chooses to substitute the Riemann-Liouville fractional integro-differential, thus forming the function

$$B_{\mathrm{LH}}(t) = \int_{0}^{t} \frac{(t-s)^{H-\frac{1}{2}}\, \mathrm{d}B(s)}{\Gamma(H+\frac{1}{2})}\,.$$

Again, the index L, for Liouville, is added here to avoid ambiguity (MACCONE preserves my original notation, while changing its meaning). The function $B_{\mathrm{LH}}(t)$ had been written down in passing by LÉVY, who did not explore it.

The central claim in ref. (3) resides in its eq. (4.1): in the present notation

$$\langle B_{\mathrm{LH}}(t_1)\cdot B_{\mathrm{LH}}(t_2)\rangle \propto \{\min\,[t_1, t_2]\}^{2H}\,. \tag{4.1}$$

(1) B. B. MANDELBROT and J. W. VAN NESS: *SIAM Rev.*, **10**, 442 (1968).
(2) B. B. MANDELBROT: *Fractals: Form, Chance and Dimension* (San Francisco, Cal., 1977); *The Fractal Geometry of Nature* (San Francisco, Cal., 1982).
(3) C. MACCONE: *Nuovo Cimento B*, **61**, 229 (1981).

This assertion is incorrect. Indeed, denote by (4.1^-) the nonnumbered displayed formula which preceeds (4.1) in ref. [3], and by (4.1^{--}) the displayed formula which preceeds (4.1^-). The claim that (4.1^-) follows from (4.1^{--}) involves an error of calculus.

In any event, the expression (4.1) could not possibly be valid, because a Gaussian random function $X(t)$ that satisfies $\langle X(t_1)X(t_2)\rangle = G[\min(t_1, t_2)]$ takes the form $X(t) = B[\sqrt{G(t)}]$: its increments are independent but highly nonstationary. To the contrary, fractional integration is an integral operation that injects infinite dependence, but the increments of $B_{\mathrm{WH}}(t)$ are stationary, and those of $B_{\mathrm{LH}}(t)$ become asymptotically stationary as $t \to \infty$, because, for large t, $B_{\mathrm{LH}}(t) \sim B_{\mathrm{WH}}(t)$.

The correlation being inapplicable, the eigenfunction expansions of § 8 and § 10 are *also inapplicable* to $B_{\mathrm{LH}}(t)$. If correct (which I did not check), they apply to the function $B(t^H)$.

Furthermore, the restriction of the exponent H to satisfy $0 < H < 1$ (which does apply to $B_{\mathrm{WH}}(t)$, as shown in ref. [3]) is *not* applicable to $B_{\mathrm{LH}}(t)$. Any value $H > 0$ is suitable.

The meaning of the « Papoulis spectrum » described in sect. **6** escapes me entirely, but the result to which it leads happens to coincide with the well-defined spectrum that applies to $B_{\mathrm{WH}}(t)$. Due to this spectrum's forms, $B'_{\mathrm{WH}}(t)$ is a « $1/f$ noise ».

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24 Aprile 1982
Lettere al Nuovo Cimento
Serie 2, Vol. **33**, pag. 549-550