On an Eigenfunction Expansion and on Fractional Brownian Motions.

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Summary. A discussion of a modified fractional Brownian motion, which appeared in this journal, is largely incorrect.

The fractional-Brownian-motion process (fBm) is the Weyl’s fractional integro-differential of the ordinary Brownian-motion process of Wiener. It is originally defined by Mandelbrot and Van Ness (1) as

$$B_{WH}(t) = \int_{-\infty}^{t} \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H + \frac{1}{2})} dB(s),$$

where $B(t)$ is Wiener’s Brownian process. The index $W$, which stands for Weyl, has been added here to avoid ambiguity in the sequel. Together with a bilateral version, $B_{WH}(t)$ has proven very valuable (2).

For reasons he does not disclose, Maccone (3) chooses to substitute the Riemann-Liouville fractional integro-differential, thus forming the function

$$B_{LH}(t) = \int_{0}^{t} \frac{(t-s)^{H-\frac{1}{2}} dB(s)}{\Gamma(H + \frac{1}{2})}.$$

Again, the index $L$, for Liouville, is added here to avoid ambiguity (Maccone preserves my original notation, while changing its meaning). The function $B_{LH}(t)$ had been written down in passing by Lévy, who did not explore it.

The central claim in ref. (3) resides in its eq. (4.1): in the present notation

$$\langle B_{LH}(t_1), B_{LH}(t_2) \rangle \propto \min\{t_1, t_2\}^{2H}.$$

(2) B. B. Mandelbrot: Fractals: Form, Chance and Dimension (San Francisco, Cal., 1977); The Fractal Geometry of Nature (San Francisco, Cal., 1982).
This assertion is incorrect. Indeed, denote by (4.1--) the nonnumbered displayed formula which precedes (4.1) in ref. (9), and by (4.1--) the displayed formula which precedes (4.1--). The claim that (4.1-) follows from (4.1--) involves an error of calculus.

In any event, the expression (4.1) could not possibly be valid, because a Gaussian random function \( X(t) \) that satisfies \( \langle X(t_1)X(t_2) \rangle = G[\min(t_1, t_2)] \) takes the form \( X(t) = B[\sqrt{G(t)}] \): its increments are independent but highly nonstationary. To the contrary, fractional integration is an integral operation that injects infinite dependence, but the increments of \( B_{WH}(t) \) are stationary, and those of \( B_{LH}(t) \) become asymptotically stationary as \( t \to \infty \), because, for large \( t \), \( B_{LH}(t) \sim B_{WH}(t) \).

The correlation being inapplicable, the eigenfunction expansions of §8 and §10 are also inapplicable to \( B_{LH}(t) \). If correct (which I did not check), they apply to the function \( B(t^H) \).

Furthermore, the restriction of the exponent \( H \) to satisfy \( 0 < H < 1 \) (which does apply to \( B_{WH}(t) \), as shown in ref. (2)) is not applicable to \( B_{LH}(t) \). Any value \( H > 0 \) is suitable.

The meaning of the "Papoulis spectrum" described in sect. 6 escapes me entirely, but the result to which it leads happens to coincide with the well-defined spectrum that applies to \( B_{WH}(t) \). Due to this spectrum's forms, \( B'_{WH}(t) \) is a "1/f noise".

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