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NOTE ON THE DEFINITION AND THE STATIONARITY OF FRACTIONAL GAUSSIAN NOISE

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INTRODUCTION

Fractional Gaussian noises (fGn) are a family of simple but peculiar random processes which, following Mandelbrot (1965), many authors have used to their satisfaction as a tool of hydrologic modeling. They are statistically stationary, but an unfortunate error in a well-known textbook has thrown doubt on this fact. Also, the evaluation of the moments of fGn involves a few technicalities which I am sorry I did not touch on in earlier papers on the subject. These technicalities have been a source of concern to some students. Both points have recently been brought to my attention by Mr. Dong H. Kim, and I think my clarifying remarks to him may be of wider usefulness. The fGn comes in many variants; the simplest — to be adopted here — is an infinite moving average of the form:

$$G_H(t) = \sum_{s=0}^{\infty} K_H(s) G(t-s)$$

where the G are a sequence of independent Gaussian random variables of zero expectation and unit variance; and where K_H is a prescribed weighting kernel such that, for $s >> 1$, $K_H(s) \sim s^{H-3/2}$.
The parameter H satisfies $0.5 < H < 1$, from which it follows that $\Sigma K_H(s) = \infty$. The value of H is to be fitted to the data — preferably through the R/S statistic. It is the parameter occurring in Hurst's phenomenon.

STATIONARITY

Moving averages of the form $X(t) = \Sigma K(s) G(t-s)$ are mentioned in Box and Jenkins (1970), where it is claimed repeatedly that, according to Grenander and Rosenblatt (1957), $X(t)$ is stationary if, and only if:

$$\sum_{s=0}^{\infty} K(s) < \infty$$

and the generating function, defined as the sum of the Taylor series:

$$\sum_{s=0}^{\infty} K(s)z^s$$

is an analytic function for $|z| \leqslant 1$. If this claim had been correct, fGn would have failed to be stationary. In fact, on p. 70 of Grenander and Rosenblatt, it is clearly stated that said Taylor series is "analytic for $|z| < 1$ since $\Sigma |K(s)|^2 < \infty$". Therefore, the claim in Box and Jenkins is merely the result of an incorrect transcription, and the stationarity of fGn raises no problem whatsoever.

This issue deserves amplification because the condition:

$$\sum_{s=0}^{\infty} K(s) < \infty$$

does indeed occur in the theory of moving average processes, albeit in a different role. It expresses that the spectral density $S(f)$ of said process is bounded, and in particular that it is finite for $f = 0$. Processes for which such is the case do not have much power at low frequencies. For fGn, on the contrary, the fact that $\Sigma K(s) = \infty$ implies $S(O) = \infty$, which expresses the presence of substantial power near $f = 0$. At the same time, it follows from $\Sigma K^2(s) < \infty$ that the total power of a discrete time fGn is finite, because it is governed by the integrated spectral density $_0\int^1 S(f)\mathrm{d}f$. Finally, nonstationary processes are typically characterized either by an infinite power at exactly $f = 0$, or (in the case of moving averages) by the condition that $\Sigma K^2(s) = 0$ and therefore $_0\int^1 S(f)\mathrm{d}f = \infty$. Before the publication of Mandelbrot (1965), most model makers thought they were limited to a choice between the first and the third of the above possibilities, namely between $S(0) < \infty$ and $^0_0\int^1 S(f)\mathrm{d}f = \infty$. I feel that — quite apart from the detail of its definition — the most basic contribution of fGn has been to point out that there also exist processes in which low-frequency components are sufficiently rich to account for the observed strong long-run effects, and at the same time sufficiently weak for the process to remain perfectly stationary. It is most regrettable, therefore, that transcription errors in Box and Jenkins should make anyone doubt the validity of this new intermediate possibility, often the most suitable one.

THE EXPECTATION

A naive evaluation of $EX(t)$ yields:

$$EG_H(t) = E \sum_{s=0}^{\infty} K_H(s)G(t-s) = \left[\sum_{s=0}^{\infty} K_H(s)\right] EG(t-s) = \infty . 0$$

The fact that this last expression is indeterminate seems to suggest that $EG_H(t)$ is somehow undefined. In fact, a less naive evaluation of $EG_H(t)$ shows there is no problem whatsoever. Indeed, the sequence of approximating Gaussian random variables:

$$G_H(t;n) = \sum_{s=0}^{n} K_H(s)G(t-s)$$

converges to a limit which is, in agreement with intuition, a Gaussian random variable of zero mean and expectation $\Sigma K_H(s)^2 < \infty$.

Proof

The continuity theorem 2 of Feller (1971, p. 508) applies (trivially!) to the sequence:

$$G_H(t;n) \left[\sum_{s=0}^{n} K_H{}^2(s) \right]^{-1/2}$$

Indeed, using Feller's notation, the sequence of the corresponding characteristic functions φ_n remain for all n identical to the characteristic function of the reduced Gaussian. Hence, this sequence has a "limit", which — like every term in the sequence — is the reduced Gaussian.

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