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ON THE DISTRIBUTION OF STOCK PRICE DIFFERENCES

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Price changes over a fixed number of transactions may have a Gaussian distribution. Price changes over a fixed time period may follow a stable Paretian distribution, whose variance is infinite. Since the number of transactions in any time period is random, the above statements are not necessarily in disagreement. A possible explanation is proposed by TAYLOR, and then shown by MANDELBROT to be intimately related to an earlier discussion of the specialists' function of ensuring the continuity of the market.

THERE ARE at least four schools of thought on the statistical distribution of stock price differences, or more generally, stochastic models for sequences of stock prices. In terms of number of followers, by far the most popular approach is that of the so-called 'technical analyst' phrased in terms of short term trends, support and resistance levels, technical rebounds, and so on. Rejecting this technical viewpoint, two other schools agree that sequences of prices describe a random walk, where price changes are statistically independent of previous price history, but these schools disagree in their choice of the appropriate probability distributions and/or in their choice of the appropriate 'time' parameter (the physical time—days, hours—or a randomized operational time ruled by the flow of transactions). Some authors find price changes to be normal or Gaussian,[1,2,8,14,15] while the other group find them to follow a stable Paretian law with infinite variance.[3,4,9,11] Finally, a fourth group (overlapping with the preceding two) admits the random walk as a first-order approximation but notes recognizable second-order effects.[10,12,13,16]

Basically, our point is this: the Gaussian random walk as applied to transactions is compatible with a symmetric stable Paretian random walk as applied to fixed time intervals.

REVIEW OF STABLE LAWS

LET $\{Z(t),\ t \geqq 0\}$ be a stochastic process with stationary independent increments, that is, a random walk. In order that $Z(t)$ follow a stable law it is necessary and sufficient that the characteristic function $\varphi_{Z(t)}(u) = E[\exp\{iuZ(t)\}]$ have the form:

$$\varphi_{Z(t)}(u) = \exp\{i\delta t u - \gamma t|u|^{\alpha}[1 + i\beta(u/|u|)w(u,\ \alpha)]\}, \tag{1}$$

where

$$w(u,\alpha) = \begin{cases} \tan(\pi\alpha/2) & \text{if } \alpha \neq 1, \\ (2/\pi)\log|u| & \text{if } \alpha = 1, \end{cases}$$

and $|\beta| \leqq 1,\ 0 < \alpha \leqq 2,\ \gamma > 0$ (reference 6, p. 164).

In general α is called the *characteristic exponent* of the stable law (reference 6, p. 171). When $\alpha = 2$ one gets the normal distribution, and $Z(t)$ is a Brownian motion process. The Cauchy law corresponds to a $\alpha = 1$ and $\beta = 0$, (reference 6, p. 171). When $1 < \alpha < 2$ the corresponding distribution has a finite mean but infinite variance and is the stable Paretian law referred to in the introduction. A stable distribution is positive if and only if $\alpha < 1$, $\beta = 1$, and $\delta \geqq 0$ (reference 5, p. 542), and is symmetric when $\beta = 0$.

A RELATION BETWEEN THE GAUSSIAN AND OTHER STABLE LAWS*

LET $\{X(v),\ v \geqq 0\}$ be a Gaussian stochastic process with stationary independent increments, $E[X(v)] = 0$ and $E[X(u)X(v)] = \sigma^2 \min\{u,\ v\}$. The characteristic function $\varphi_{X(t)}(\xi) = E[\exp\{i\xi X(t)\}]$ is given by $\varphi_{X(t)}(\xi) = \exp\{-\tfrac{1}{2}\xi^2\sigma^2 t\}$. Let $\{T(t),\ t \geqq 0\}$ be a positive stable stochastic process with characteristic function

$$\varphi_{T(t)}(\eta) = \exp\{-\gamma t|\eta|^{\alpha}[1 + i(\eta/|\eta|)\tan(\pi\alpha/2)]\}, \tag{3}$$

where $0 < \alpha < 1$ and we have taken $\delta = 0$, $\beta = 1$ in the general form, equation (1). Define a new process $Z(t) = X[T(t)]$. This process is said to be *subordinated* to $\{X(v)\}$ and the process $\{T(t)\}$ is called the *directing process* (reference 5, p. 335).

We interpret $\{X(v),\ v \geqq 0\}$ as the stock prices on a 'time scale' measured in volume of transactions, and consider $T(t)$ to be the cumulative volume or number of transactions up to physical (days, hours) time t. Then $Z(t)$ is the stock price process on the physical time scale. That $Z(t)$ is a stable process with independent increments and with characteristic exponent $2\alpha < 2$, [reference 5, p. 336, example (c)] may be shown by computing the characteristic function.

* An observation by HOWARD M. TAYLOR.

$$\varphi_{Z(t)}(\xi) = E[\exp\{i\xi X(T(t))\}]$$
$$= E[E[\exp\{i\xi X(T(t))\}|T(t)]]$$
$$= E[\varphi_{X(T(t))}(\xi)]$$
$$= E[\exp\{-\tfrac{1}{2}\,\xi^2\sigma^2 T(t)\}].$$

Formally,* this becomes

$$\varphi_{Z(t)}(\xi) = \varphi_{T(t)}(\tfrac{1}{2}\,i\xi^2\sigma^2)$$
$$= \exp\{-\gamma(\sigma^2/2)^\alpha t|\xi|^{2\alpha}[1-\tan(\pi\alpha/2)]\}$$
$$= \exp\{-\hat{\gamma}t|\xi|^{2\alpha}\},$$

where $\hat{\gamma} = \gamma(\sigma^2/2)^\alpha[1-\tan(\pi\alpha/2)]$. We have thus obtained the symmetric stable law with characteristic exponent $2\alpha<2$ for which the β and δ terms in equation (1) are zero.

INTRODUCTION TO THE DISCUSSION†

As I SHOWED in 1963,[9] successive price changes over fixed time periods are approximately independent and their distribution is approximately 'stable Paretian.' This means in particular that their population variance is infinite. Defining $T(t)$ to be the cumulative number of transactions up to time t, one can write $Z(t)=X[T(t)]$. TAYLOR's remark is that, when the distribution of $Z(t)$ is symmetric, and if $T(t)$ is a special random function called 'subordinator,' the observed behavior of $Z(t)$ is compatible with the assumption that price changes between transactions are independent and Gaussian.‡ (This representation of Z appears due to S. Bochner, who conceived the concept of subordination.)

As noted by Taylor, the subordinator is itself a non-Gaussian stable random function, with an infinite mean and a fortiori an infinite variance. His remark, therefore, does not belong to the category of attempts to avoid stable Paretian distributions and to 'save' the finite variance. He basically suggests an alternative way of introducing the infinite-moment hy-

* This step, as presented here, is formal, because it substitutes a complex variable in a characteristic function formula developed for a real argument. But the step is readily justified, *see* reference 5.

† Discussion by BENOIT MANDELBROT.

‡ Other authors have recently shown interest in the relations between price changes over fixed numbers of transactions and over fixed time increments. However, GRANGER[7] only points out that it is conceivable that $X(T)$ be Gaussian even when $Z(t)$ is extremely long-tailed, without noting that this requires $T(t)$ to be approximately a subordinator. BRADA ET AL.[2] belabor the fact that the price changes between transactions are short-tailed and approximately Gaussian, a feature that is ensured by the S. E. C.'s instructions to the specialists (*see* footnote †, p. 1060).

pothesis into the study of price behavior.* I wish to develop his alternative, which is illuminating though restricted to the symmetric case and as yet devoid of direct experimental backing.

ACTUAL AND 'VIRTUAL' TRANSACTIONS

THE BASIC fact is that the subordinator function is (almost surely almost everywhere) discontinuous and varies by positive jumps whose size U is characterized by having the following conditional distribution for each $h>0$: if $u>h$, $\Pr(U \geqq u | U \geqq h)^{-\alpha/2}$.† These jumps mean that Taylor implicitly suggests that, if $U>1$, transactions are performed in 'bunches.' Let the last transaction in the bunch be called 'final' and the other 'virtual.'

If the amounts traded in the 'virtual' transactions were negligible, the price change between successive final transactions would be the sum of U independent reduced Gaussian variables. The variance of that price change would be equal to the expectation of U, which is infinite; more precisely, one can easily verify that—as it should—the distribution of price changes between final transactions is identical to the distribution of the sizes of the jumps of the infinite variance stable Paretian process.

Thus, Bochner's representation of the stable process brings nothing new, unless nonnegligible amounts are traded on the so-called 'virtual' transactions.

SPECIALISTS' TRADES

IN MY 1963 paper (reference 9, section VI.B), I pointed out that the discontinuities of the process $Z(t)$ were unlikely to be observed by examining transaction data, being *either* hidden within periods when the market is closed or the trading interrupted, *or* smoothed away by specialists who, in accordance with S.E.C. instructions, "ensure the continuity of the market" by performing transactions in which they are party.‡

* Another alternative was proferred in my paper.[10] *See also* my "Comment" following reference 7.

† Rigorously, since $T(t)$ must be an integer, it cannot be exactly a subordinator. In the first approximation, we shall assume that the jumps of T have been quantized but not smoothed out. Similarly, $X(T)$ must be a process in discrete time. Its independent Gaussian increments will be assumed to have zero mean and unit variance. Taylor's conclusion is practically unaffected if one so quantizes T.

‡ The distribution of price changes between transactions has a direct bearing upon the 'method of filters,' discussed in Section VI.C of reference 9.

Let me amplify this role of the specialist. He can interpret the 'continuity of the market' in at least two ways. *First,* he will smooth out small 'aimless' price drifts, so that the expression $Z_{i+1}-Z_i$ is equal to zero more often than would have been the case without him. *Second,* the specialist will replace most large price changes by runs of small- or medium-sized changes, whose amplitudes will be so interdependent that almost all will have the same sign. On some markets, this is even ensured by the existence of legal 'limits.' (These are imposed on the price change within a day, but they naturally also impose an upper bound upon price changes between transactions.) On other markets, limits are not fixed by law. Suppose, however, that right

It is tempting to postulate the identity of Bochner-virtual transactions and the specialists' transactions, though the latter presumably see where the prices are aimed and can achieve the desired ΔZ in less than U independent Gaussian steps. Thus, the Bochner representation is concretely plausible and a program for empirical study of specialists.

DIFFICULTY OF USING IN PRACTICE THE DATA RELATIVE TO PRICE CHANGES BETWEEN SUCCESSIVE TRANSACTIONS

LET Z_i and Z_{i+1} be successive quotations. The fact that the distribution of $Z_{i+1}-Z_i$ is short-tailed (or even Gaussian) is now seen to be fully compatible with the stable Paretian behavior of $Z(t+T)-Z(t)$. However, it is difficult to see of what use the distribution of $Z_{i+1}-Z_i$ may be for the individual investor, because individual investors do not have 'transaction clocks' available. Even if they did, they could not be sure that their transactions would be the next to be executed on the market, or the 50th or 100th after the next. The investor's order to buy or sell, registered when price is Z_i, will be executed at a price Z^0 such that Z^0-Z_i is some complicated mixture (carried over all values of j) of quantities of the form $Z_{i+j}-Z_i$. Mixtures of Gaussian distributions are known to have fatter tails than the Gaussian, and these tails can be very fat indeed.

Similarly, it is difficult to see how transaction data can be used by economists who are not primarily concerned with the activities of the specialists. For example, if stock price changes are just one of the variables in a time series model, it would not make sense to measure the price changes over a transaction interval (and thus a variable time interval) if the other variables are measured over fixed intervals of time.

The criticism expressed in the preceding paragraph should not be interpreted as implying that the problem of trading on the NYSE is fully solved by describing the function $Z(t)$ in natural (uniformly flowing) time, because the instant when one's transaction will be executed is also impossible to fix in advance. Therefore, some kind of mixture will again appear. In Bachelier's original model, $Z(t+T)-Z(t)$ being a Gaussian

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after a transaction is performed at time t_i with price Z_i, a major piece of information dries out all supply of that security below a price Z^0. Then the specialist is supposed to sell from his own holdings and he will be one of the parties in the next few or many transactions. As a by-product of the 'time for reflection' thus provided, such smoothing will surely affect the distribution of the quantity $Z_{i+1}-Z_i$ by eliminating most large values. The only large values that will surely remain are those corresponding to the cases where t_i is either the instant of the last transaction within a session, or the instant of the last transaction before an interruption of trading. Although such cases are extremely important, their effect upon the statistics of $Z_{i+1}-Z_i$ will be swamped out by the huge number of transactions performed within any trading session.

random variable, one encounters the same difficulties as when mixtures of $Z_{i+j}-Z_i$ are considered. On the other hand, the stable Paretian model has the interesting property that the asymptotic behavior of the distribution is the same for a mixture of $Z(t+T)-Z(t)$ (carried over a set of values of T), and for each $Z(t+T)-Z(t)$ taken separately. Thus, mixing has no effect upon considerations related to that asymptotic behavior.

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