The Journal of Business 40, 1967, 393-413 **E15**

The variation of the prices of cotton, wheat, and railroad stocks, and of some financial rates

• ***Chapter foreword***. M 1963b{E14} argues that the description of price variation requires probability models less special than the widely used Brownian, because the price relatives of *certain* prices series have a variance so large that it may in practice be assumed infinite. This theme is developed further in the present chapter, which covers the following topics.

1. Restatement of the M 1963{E14} model of price variation, and additional data on cotton.
2. The variation of wheat price in Chicago, 1883-1936.
3. The variation of some railroad stock prices 1857-1936.
4. The variation of some interest and exchange rates.
5. Token contribution to the statistical estimation of the exponent α.

Much of the empirical evidence in this paper was part of IBM Research Note NC-87 (March 1962), from which M 1963b{E14} is also excerpted •

THIS CHAPTER CONTINUES Chapter E14, to be referred to as VCSP.

{P.S. 1996: In view of the current focus on serial dependence, special interest attaches to Section 4. Indeed, changes in interest and exchange rates cannot possibly be independent, hence cannot follow the M 1963 model. The sole question tested in this chapter is whether or not the marginal distribution is L-stable, irrespective of serial dependence. The variation of those records in time may be best studied by the methods that Chapter E11 uses for personal income.}

1. RESTATEMENT OF THE M 1963 MODEL OF PRICE VARIATION, AND ADDITIONAL DATA ON COTTON

One goal of this restatement is to answer certain reservations concerning my L-stable model of price variation. Trusting that those reservations will be withdrawn and not wishing to fan controversy, I shall name neither the friendly nor the unfriendly commentators.

1.1 Bachelier's theory of speculation

Consider a time series of prices, $Z(t)$, and designate by $L(t, T)$ its logarithmic relative

$$L(t, T) = \log_e Z(t, T) - \log_e Z(t).$$

The basic model of price variation, a modification of one proposed in 1900 in Louis Bachelier's theory of speculation, assumes that successive increments $L(t, T)$ are (a) random, (b) statistically independent, (c) identically distributed, and (d) Gaussian with zero mean. The process is called a "stationary Gaussian random walk" or "Brownian motion."

Although this model continues to be extremely important, its assumptions are working approximations that must not be made into dogmas. In fact, Bachelier 1914 made no mention of earlier claims of the empirical evidence in favor of Brownian motion. (To my shame, I missed this discussion when I first glanced through Bachelie 1914 and privately criticized him for blind reliance on the Gaussian. Luckily, my criticism was not committed to print.)

Bachelier noted that his original model contradicts the evidence in at least two ways: Firstly, the sample variance of $L(t, T)$ varies in time. He attributed this to variability of the population variance, interpreting the sample histograms as being relative to mixtures of distinct populations, and observed that the tails of the histogram could be expected to be fatter than in the Gaussian case. Second, Bachelier noted that no reasonable mixture of Gaussian distributions could account for the sizes of the very largest price changes, and he treated them as "contaminators" or "outliers." Thus, he pioneered not only in discovering the Gaussian random-walk model, but also in noting its major weakness.

However, new advances in theory of speculation continues to be best expressed as improvements upon the Brownian model. VCSP shows that an appropriate generalization of hypothesis (d) suffices to "save" (a), (b), and (c) in many cases, and in other cases greatly postpones the moment when the misfit between the data and the theory is such that the latter must be amended.. I shall comment upon Bachelier's four hypotheses, then come to the argument of VCSP.

1.2 Randomness

To say that a price change is random is not to claim that it is irrational, only that it was unpredictable before that fact *and* is describable by the powerful mathematical theory of probability. The two alternatives to randomness are "predictable behavior" and "haphazard behavior," where the latter term is taken to mean "unpredictable and not subject to probability theory." By treating the largest price changes as "outliers," Bachelier implicitly resorted to this concept of "haphazard." This might have been unavoidable, but the power of probability theory has since increased and should be used to the fullest.

1.3 Independence

The assumption of statistical independence of successive $L(t, T)$ is undoubtedly a simplification of reality. It was surprising to see VCSP criticized for expressing blind belief in independence. For examples of reservations on this account, see its Section VII as well as the final paragraph of its Sections III E, III F, and IV B. In defense of independence, I can offer only one observation; very surprisingly, models making this assumption account for many features of price behavior.

Incidentally, independence implies that no investor can use his knowledge of past data to increase his expected profit. But the converse is *not* true. There exist processes in which the expected profit vanishes, but dependence is extremely long range. In such cases, knowledge of the past may be profitable to those investors whose utility function differs from the market's. An example is the "martingale" model of M 1966b{E19}, which is developed and generalized in M 1969e. The latter paper also touches on various aspects of the spectral analysis of economic time series, another active topic whose relations with my work have aroused interest. For example, when a time series is non-Gaussian, its spectral whiteness, that is, the absence of correlation, is compatible with great departures from the random-walk hypothesis.

1.4 Stationarity

One implication of stationarity is that sample moments vary little from sample to sample, as long as the sample is sufficiently long. In reality, price moments often fail to behave in this manner. The notorious fact is understated in the literature, since "negative" results are seldom published – one exception being Mills 1927.

Figure 1 adds to VCSP, and displays the enormous variability in time of the sample second moment of cotton prices in the period of 1900-1905. The points refer to successive fifty-day sample means of $[L(t, 1)]^2$, where the $L(t, 1)$ are the daily price relatives. In Brownian motion, these sample means would have stabilized near the population mean. Since no stabilization is in fact observed, we see conclusively that the price of cotton did not follow a Gaussian stationary random walk.

The usual accounts for this variability claim that the mechanism for price variation changes in time. We shall, loosely speaking, distinguish systematic, random, and haphazard changes of mechanism.

The temptation to refer to systematic changes is especially strong. Indeed, to explain the variability of the statistical parameters of price variation would constitute a worthwhile first step toward an ultimate explanation of price variation itself. An example of systematic change is given by the yearly seasonal effects, which are strong in the case of agricultural commodities. However, Figure 1 goes beyond such effects: not all ends of season are accompanied by large price changes, and not all large price changes occur at any prescribed time in the growing season.

The most controversial systematic changes are due to deliberate changes in the policies of the Federal Government or of the Exchanges. For an example of unquestionable long-term change of this type, take cotton prices (Section III D of VCSP.) All measures of scale of $L(t, T)$ (such as the interquartile interval) did vary between 1816 and 1958. Indeed, lines 1*a* and 2*a* of Figure 5 of VCSP, which are relative to the 1900's, clearly differ from lines 1*b* and 2*b*, which are relative to the 1950's. This clearcut decrease in price variability must, at least in part, be a consequence of the deep changes in economic policy that occurred in the early half of this century. However, precisely because it is so easy to read in the facts a proof of the success or failure of changes in economic policy, the temptation to resort to systematic nonstationarity must be carefully controlled.

{P.S. 1996. The extremely cautious phrasing of the last sentences was rewarded: the large apparent decrease in volatility from 1904 to 1952 proved in fact to be caused by misleadingly presented data; see M 1972b, i.e., Appendix III of Chapter E14}.

In order to model what they perceived to be a "randomly changing" random process, many authors have invoked a random-walk process in which the sizes and probabilities of the steps are chosen by some other process. If this second "master process" is stationary, $Z(t)$ itself is not a random walk but remains a stationary random process.

The final possibility is that the variability of the price mechanism is haphazard, that is, not capable of being treated by probability theory. This belief is, of course, firmly entrenched among nonmathematical economists. But to construct a statistical model one expects to change before it has had time to unfold, can hardly be viewed as a sensible approach.

Moreover, and more importantly, early resort to haphazard variation *need not* be necessary, as is demonstrated by the smoothness and regularity of the graph of Figure 2, which is the histogram of the data of Figure 1.

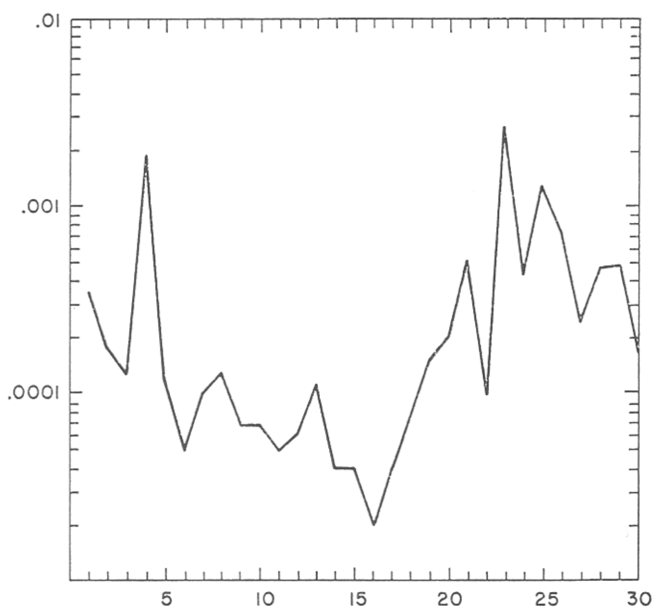

FIGURE C15-1. Sample second moment of the daily change of log $Z(t)$, where $Z(t)$ is the spot price of cotton. The period 1900-1905 was divided into thirty successive fifty-day samples, and the abscissa designates the number of the sample in chronological order. Logarithmic ordinate. A line joins the sample points to improve legibility.

1.5 Gaussian hypothesis

Bachelier's assumption, that the marginal distribution of $L(t, T)$ is Gaussian with vanishing expectation, might be convenient, but virtually every student of the distribution of prices has commented on their leptokurtic (i.e., very long-tailed) character. For an old but eminent practitioner's opinion, see Mills 1927; for several recent theorists' opinions see Cootner 1964. It was mentioned that Bachelier himself regarded $L(t, T)$ as a contaminated mixture of Gaussian variables; see M and Taylor 1967{E22, Sections 1 and 2}.

1.6 Infinite population variance and the L-stable distributions

Still other approaches were suggested to take into account the failure of Brownian motion to fit data on price variation. In all these approaches, each new fact necessitates an addition to the explanation. Since a new set of parameters is thereby added, I don't doubt that reasonable "curve-fitting" is achievable in many cases.

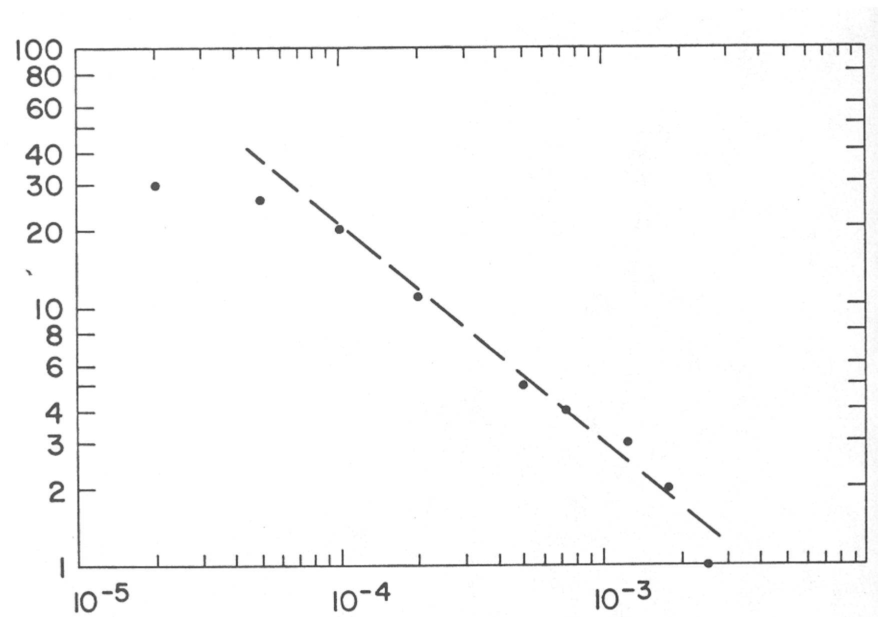

FIGURE C15-2. Cumulated absolute-frequency distribution for the data of Figure 1. Abscissa: log of the sample second moment. Ordinate: log of the absolute number of instances where the sample moment marked as abscissa has been exceeded. The L-stable model predicts a straight line of slope $\alpha/2 \sim 1.7/2$, which is plotted as a dashed line.

However, this form of "symptomatic medicine" (a separate drug for each complaint) could not be the last word! The beauty of science – and the key to its effectiveness – is that it sometimes evolves central assumptions, from which many independently made observations can be shown to follow. These observations are thus organized, and predictions can be made. The ambition of VCSP was to suggest such a central assumption, the infinite-variance hypothesis, and to show that it accounts for substantial features of price series (of various degrees of volatility) without nonstationarity, without mixture, without master processes, without contamination, but with a choice of increasingly accurate assumptions about the interdependence of successive price changes.

When selecting a family of distributions to implement the infinite-variance hypothesis, one must be led by mathematical convenience (e.g.,

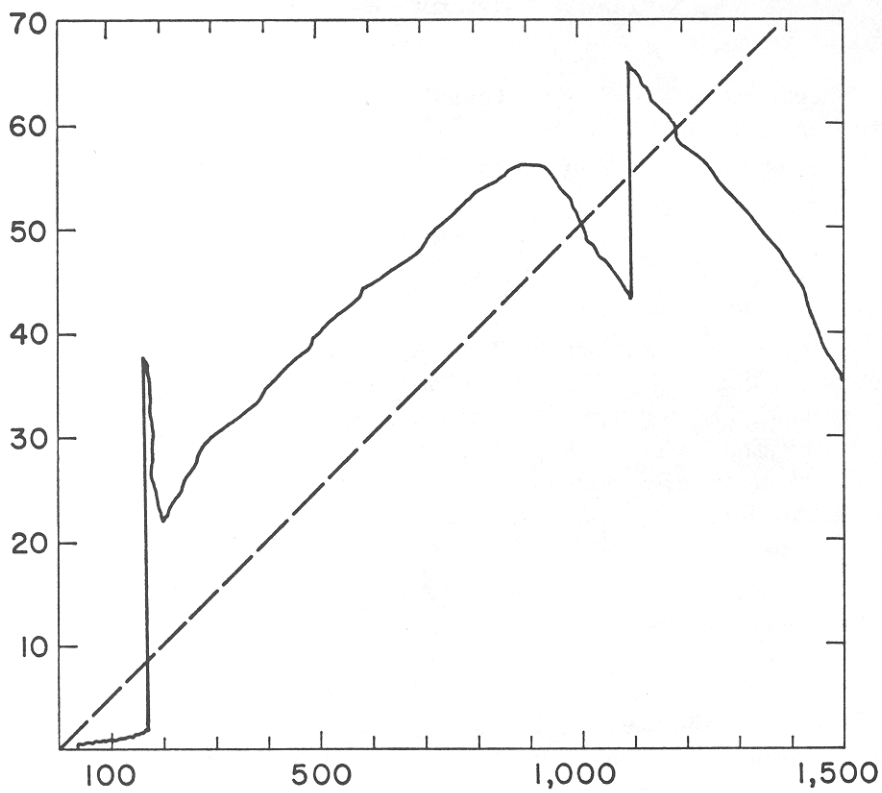

FIGURE C15-3. Sequential variation of the sample kurtosis of the daily changes of $\log Z(t)$, where $Z(t)$ is the spot price of cotton, 1900-1905. The abscissa is the sample size. Linear coordinates.

the existence of a ready-made mathematical theory) and by simplicity. For a probability distribution, one important criterion of simplicity is the variety of its properties of "invariance." For example, it would be most desirable to have the same distribution (up to some – hopefully linear – weighting) apply to daily, monthly, etc., price changes. Another measure of simplicity is the role that a family of distributions plays in central limit theorems of the calculus of probability.

In accordance with this logic, VCSP proposed to represent the marginal distribution of $L(t, T)$ by an appropriate member of a family of probability distributions called "L-stable." The L-stable laws measure volatility by a single parameter α ranging between 2 and 0. The simplest members of the family are the symmetric probability densities defined by

$$\rho_\alpha(u) = (1/\pi)\int_0^\infty \exp(-\gamma s^\alpha)\cos(su)ds.$$

Their limit case $\alpha = 2$ is the Gaussian, but my theory also allows non-Gaussian or "L-stable" cases $\alpha < 2$. Suitable α's turn out to represent satisfactorily the data on volatile prices (see VCSP and Section 2 and 3 below).

In assessing the realm of applicability of the M 1963 model, , one should always understand it as including its classical limit. It is therefore impossible to "disprove" VCSP by identifying out price series for which the Gaussian hypothesis may be tenable.

Now to discuss the fact that L-stable variables with $\alpha < 2$ have an infinite population variance; mathematicians sometimes say that they have "no variance." Firstly, one must reassure those who expressed the fear that the sole reason for my finding $E(L^2) = \infty$ was that I inadvertently took the logarithm of zero! Serious concern was expressed at the implication of this feature for statistics, and surprise was expressed at the paradoxically discontinuous change that seems to occurs when α becomes exactly 2.

This impression of paradox is unfounded. The population variance itself cannot be measured, and every measurable characteristic of a L-stable distribution behaves continuously near $\alpha = 2$ as will be seen later in an example. Consequently, there is no "black and white" contrast between the scaling case $\alpha < 2$ and the Gaussian case $\alpha = 2$, but a continuous shading of gray as $\alpha \rightarrow 2$. The finding that the population second moment is discontinuous at $\alpha = 2$ "only" shows that this moment is not well suited to a study of price variation.

In particular, the applicability of second-order statistical methods is questionable. This word could *not* mean "totally inapplicable," because the statistical methods based upon variances suffer no sudden and catastrophic breakdown as α ceases to equal 2. Therefore, to be unduly concerned with a few specks of "gray" in a price series whose α is near 2, may be as inadvisable as to treat very gray series as white. Moreover, statistics would be unduly restricted if its tools were to be used only where they have been fully justified. (As a matter of fact, the quality of statistical method is partly assessed by its "robustness," i.e., the quality of its performance when used without justification.) However, one should look for other methods. For example, as predicted, least-squares forecasting (as applied to past data) would often have led to very poor inferences; least-sums-of-absolute-deviations forecasting, on the other hand, is always at least as good and usually much superior, and its development should be pressed.

1.7 The behavior of the variance L-stable samples and cotton

Define $V(\alpha, N)$ as the variance of a sample of N independent random variables $U_1, \ldots U_n, \ldots U_N$, whose common distribution is L-stable of exponent α. To obtain a balanced view of the practical properties of such variables, one must *not* focus upon mathematical expectations and/or infinite sample sizes. Instead, one should consider quantiles and samples of large but finite size. Let us therefore select a "finite horizon" by choosing a value of N and a quantile threshold q such that events whose probability is below q will be considered "unlikely." Save for extreme cases contributing to a "tail," of probability q, the values of $V(\alpha, N)$ will be less than some function $V(\alpha, N, q)$. This function's behavior tells us much of what we need to know about the sample variance.

As mentioned earlier, when N is finite and $q > 0$, the function $V(\alpha, N, q)$ varies smoothly with α. For example, over a wide range of values of N, the derivative of $V(\alpha, N, q)$ at $\alpha = 2$ is very close to zero, hence $V(\alpha, N, q)$ changes very little from $\alpha = 2.00$ to $\alpha = 1.99$. This insensitivity is due to the fact that $\alpha = 2.00$ and $\alpha = 1.99$ differ only in the sizes that they predict for some outliers; but those outliers belong to those cases whose effects were excluded by the definition of $V(\alpha, N, q)$. Increasing N or decreasing q, decreases the range of exponents in which α is approximable by 2.

A reader who really objects to infinite variance, and is only concerned with meaningful finite-sample problems, may "truncate" U so as to attribute to its variance a very large finite value depending upon α, N, and q.

The resulting theory may have the asset of familiarity, but the specification of the value of the truncated variance will be *useless* because it will tell nothing about the "transient" behavior of $V(\alpha, N, q)$ when N is finite and small. Thus, even when one knows the variance to be finite but very large (as in the case of certain of my more detailed models of price variation; see M 1969e), the study of the behavior of $V(\alpha, N, q)$ is much simplified if one approximates the distribution with finite but very large variance by a distribution with infinite variance. Similarly, it is well known that photography is simplest when the object is infinitely far from the camera. Therefore, the photographer can set the distance at infinity if the actual distance is finite but exceeds some finite threshold dependent on the quality of the lens and its aperture.

1.8 The behavior of the kurtosis for L-stable samples and cotton prices

Pearson's kurtosis measures the peakedness of a distribution by

$$\frac{E(U^4)}{\left[E(U^2)\right]^2} - 3.$$

The discussion of this quantity in VCSP was called obscure, therefore additional detail may be useful. If U is L-stable with $\alpha < 2$, the kurtosis is undetermined, because $E(U^4) = \infty$ and $E(U^2) = \infty$, One can show, however, that, as $N \rightarrow \infty$, the random variable

$$\sum_{n=1}^{N} U_n^4 \left\{ \sum_{n=1}^{N} U_n^2 \right\}^{-2}$$

tends toward a limit that is finite and different from zero. Therefore, the "expected sample kurtosis," defined as

$$E\left\{ N \sum_{n=1}^{N} U_n^4 \left\{ \sum_{n=1}^{N} U_n^2 \right\}^{-2} - 3 \right\},$$

is asymptotically proportional to N.

The kurtosis of $L(t, 1)$ as plotted on Figure 3 for the case of cotton, 1900-1905, indeed increases steeply with N. While exact comparison is impossible because the theoretical distribution was not tabulated, this kurtosis does fluctuate around a line expressing proportionality to sample size. (For samples of less than fifty, the kurtosis is negative, but too small to be read off the figure.)

1.9 Three approximations to a L-stable distribution: implications for statistics and for the description of the behavior of prices

It is important that there should exist a single theory of prices that subsumes various degrees of volatility. My theory is, unfortunately, hard to handle analytically or numerically, but simple approximations are available in different ranges of values of α Thus, given a practical problem with a finite time horizon N, it is best to replace the continuous range of degrees of “grayness” by the following trichotomy (where the boundaries between the categories are dependent upon the problem in question).

The Gaussian $\alpha = 2$ is the best known and simplest. There is no need to worry about a long-tailed distribution, hence one stands a reasonable chance of rapid progress in the study of dependence. For example, one can use spectral methods and other covariance-oriented approaches. Very close to $\alpha = 2$, Gaussian techniques cannot lead one too far astray.

In the zone far away from $\alpha = 2$, another kind of simplicity reigns. Substantial tails of the L-stable distribution are approximations by the scaling distributions with the same α-exponent ruling both tails. A prime example was provided by the cotton prices studied in VCSP. Sections 2 and 3 examine some other volatile price series: wheat, the prices of some nineteenth-century rail securities and some rates of exchange of interest.

The zone of transition between the almost Gaussian and the clearly scaling cases is by far the most complicated of the three zones. It also provides a test of the meaningfulness and generality of the M 1963 model. If it holds, the histogram of price changes is expected to plot on biogarithmic paper as one of a specific family of inverse-**S**-shaped curves. (Lévy's α-exponent, therefore, is not to be confused with slope of a straight bilogarithmic plot, M 1963e{E3, Appendix III}.) If the M 1963 model fails, the transition between the almost Gaussian and the highly scaling cases would be performed in some other way.

We shall examine in this light the variation of wheat prices and find that it falls into the “light gray” zone of low but positive values for $2 - \alpha$ and medium volatility. Section 2.1 examines wheat data; it is similar in purpose to Fama's 1964 Chicago thesis, Fama 1965, which was the first further test of the ideas of VCSP. To minimize “volatility” and maximize the contrast with my original data, Fama chose thirty stocks of large and diversified contemporary corporations and found their L-stable "grayness" to be unquestionable, although less marked than that of cotton.

2. THE VARIATION OF WHEAT PRICES IN CHICAGO, 1883-1936

Spot prices of cotton refer to standardized qualities of this commodity, but wheat cash prices refer to variable grades of grain. At any given time (say, at closing time), one can at best speak of a *span* of cash prices, and the closing spans corresponding to successive days very often overlap. As a result, Working 1934 chose the week as the shortest period for which one can reasonably express "the" change of wheat price by a single number rather than by an interval. Further interpolation being impossible, the long record from 1883 to 1936 yields a smaller sample than one might have hoped – though a very long one by the standards of economics.

Kendall 1953 suggested that Working's wheat price relatives follow a Gaussian distribution. After all, a casual visual inspection of the histograms of these relatives, as plotted on *natural* coordinates, shows them to be nicely "bell shaped." However, natural coordinates notoriously underestimate the "tails." To the contrary, as seen in Figure 4, probability-paper plots of wheat price relatives are definitely **S**-shaped, though less so than for cotton. As the Gaussian corresponds to $\alpha = 2$ and M 1963b{E14} reports the value $\alpha = 1.7$ for cotton, it is natural to investigate whether wheat is L-stable with an α somewhere between 1.7 and 2.

2.1 The evidence of doubly logarithmic graphs

Figure 3 of VCSP shows that the L-stable distribution predicts that every doubly logarithmic plot of a histogram of wheat price changes should have a characteristic **S**-shape. It would end with a "scaling" straight line of slope near 2, but would start with a region where the local slope increases with u and even begins by markedly exceeding its asymptotic value, M 1963p.

The above conjecture is verified, as seen in Figures 5 and 6. Moreover, by comparing the data relative to successive subsamples of the period 1883-1936, no evidence was found that the *law* of price variation had changed in kind, despite the erratic behavior of the outliers.

When this test was started, one only knew that wheat lies between the highly erratic cotton series and the minimally erratic Gaussian limit.

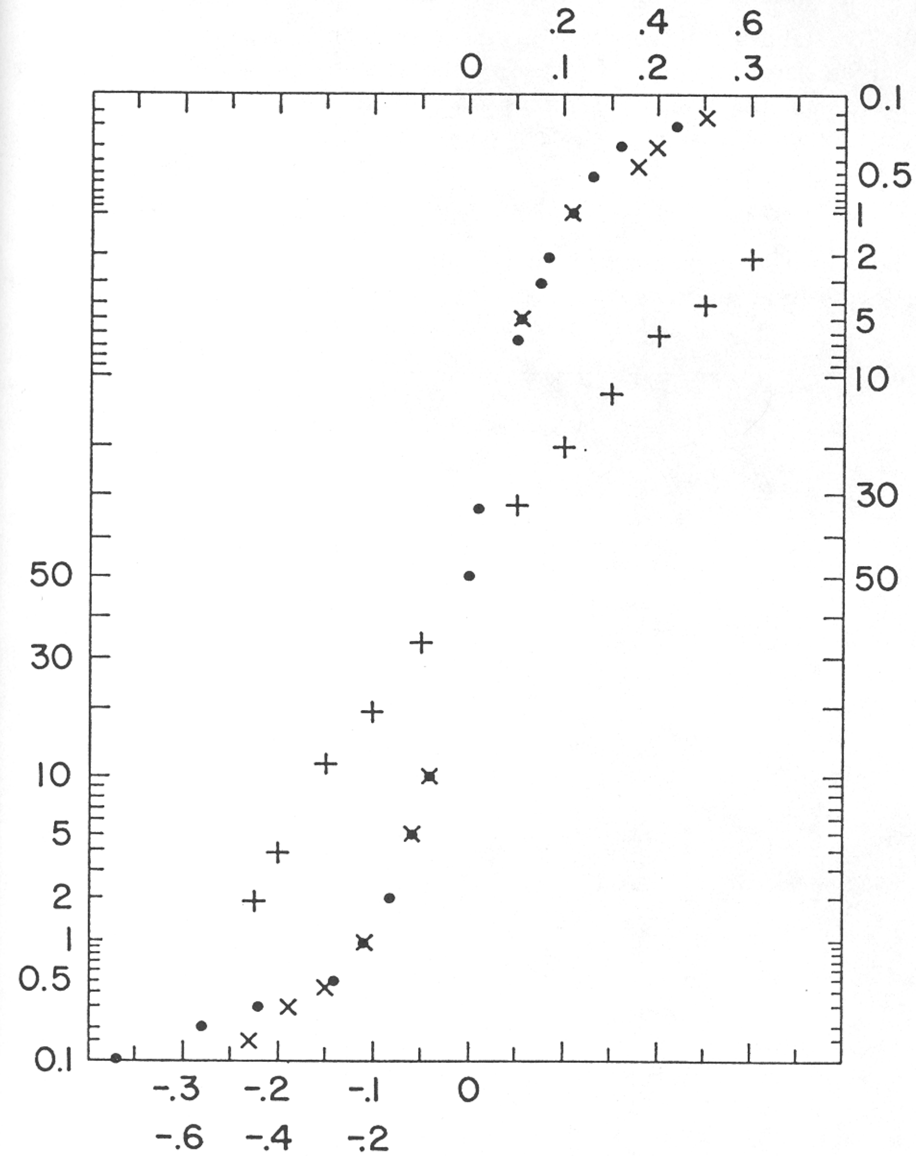

FIGURE C15-4. Probability-paper plots of the distribution of changes of log $Z(t)$, where $Z(t)$ is the spot price of wheat in Chicago, 1883-1934, as reported by Working 1934. The scale -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3 applies to the weekly changes, marked by dots, and to the yearly changes, marked by crosses. The other scale applies to changes over lunar months, marked by ×.

Therefore, Figures 5 and 6 are evidence that the L-stable model *predicted* how the price histogram of wheat price changes "should" behave.

To establish the "goodness of fit" of such an **S**-shaped graph requires a larger sample than in the case of the straight graphs characteristic of cotton. But the available samples are actually smaller. Thus the doubly logarithmic evidence is *unavoidably* less clear-cut than for cotton.

2.2 The evidence of sequential variance

When price series is approximately stationary, one can test whether $\alpha = 2$ or $\alpha < 2$ by examining the behavior of the sequential sample second

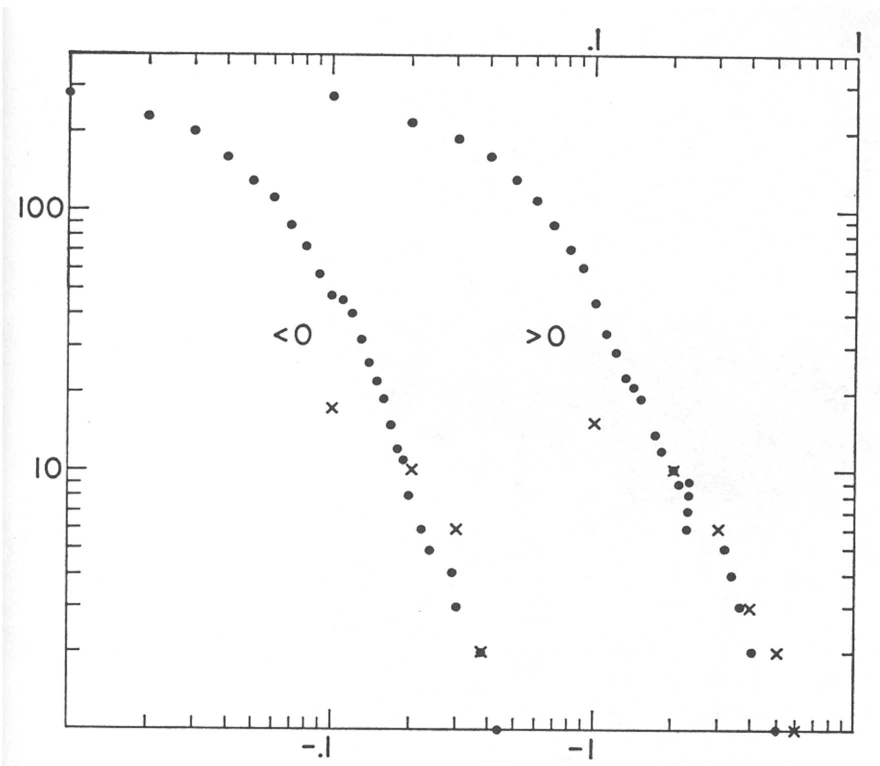

FIGURE C15-5. Weekly changes of $\log Z(t)$, where $Z(t)$ is the price of wheat as reported by Working 1934. Ordinate: log of the absolute frequency with which $L \geq u$, respectively $L \leq -u$. Abscissas: the lower scale refers to negative changes, the upper scale to positive changes.

moment. If $\alpha < 2$, the median of the distribution of the sample variance increases as $N^{-1+2/\alpha}$ for "large" N. If $\alpha = 2$, it tends to a limit. *More importantly*, divide the sample variance by its median value; this ratio's variation becomes increasingly erratic as α moves away from 2. Thus, while the cotton second moment increases very erratically, but the wheat second moment should increase more slowly and more regularly. Figure 7 shows that such is indeed the case.

2.3 Direct test of L-stability

The term "L-stable" arose from the following fact: when N such random variables U_n are independent and identically distributed, one has

$$\Pr\left\{N^{-1/\alpha}\sum_{n=1}^{N} U_n \geq u\right\} = \Pr\{U_n \geq u\}.$$

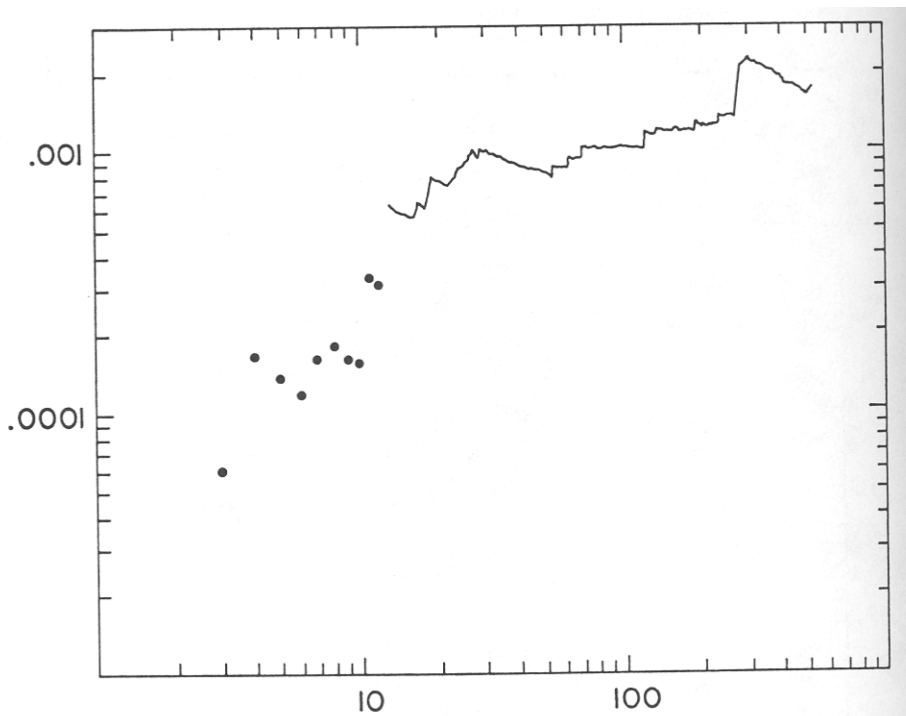

FIGURE C15-6. Changes of log $Z(t)$ over lunar months and years, where $Z(t)$ is the price of wheat as reported in Working 1935. Abscissas and ordinates as in Figure 5.

I settled on $N = 4$. When the random variables U_n are the weekly price changes, $\Sigma_{n=1}^{4} U_n$ is the price change over a "lunar month" of four weeks. Since α is expected to be near 2, the factor $4^{-1/\alpha}$ will be near 1/2.

One can see in Figure 4 that weekly price changes do indeed have an **S**-shaped distribution *indistinguishable* from that of one-half of monthly changes. (The bulk of the graph, corresponding to the central bell containing 80% of the cases, was not plotted for the sake of legibility.) Sampling fluctuations are apparent only at the extreme tails and do not appear systematic. Applied in Fama 1965 to common-stock price changes, this method also came in favor of L-stability.

The combination of Figures 5 and 6 provides another test of L-stability. They were plotted with absolute, not relative, frequency as the ordinate, and the L-stable theory predicts that such curves should be

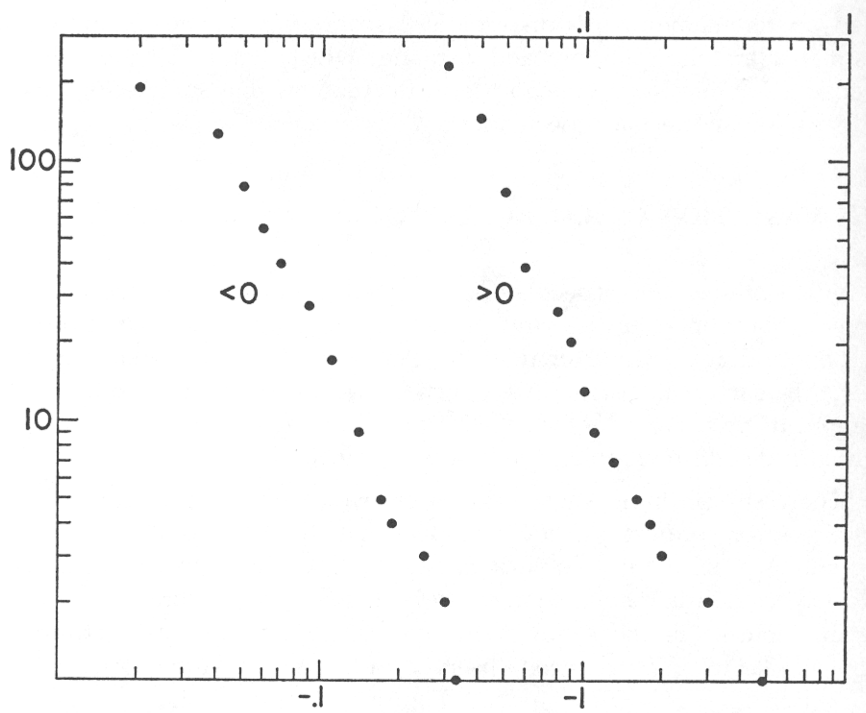

FIGURE C15-7. Sequential variation of the second moment of the weekly changes of log $Z(t)$, where $Z(t)$ is the price of wheat as reported in Working 1934. One thousand weeks beginning in 1896. Bilogarithmic coordinates. For small samples, the sample second moments are plotted separately; for lay samples, they are replaced (for the sake of legibility) by a freely drawn continuous line.

superposable in their tails, except, of course, for sampling fluctuations. (In outline, the reason for this prediction is that, in a L-stable universe, a large monthly price change is of the same order of magnitude as the largest among the four weekly changes adding to this monthly change.) Clearly, wheat data pass this second test also.

It should be stressed that, while the two tests use the same data, they are conceptually *distinct*. Figure 4 compares one-half of a monthly change to the weekly change of the same frequency; Figures 5 and 6 compare monthly and weekly changes of the same size. Stability is thus doubly striking.

2.4 The evidence of yearly price changes

Working 1934 also published a table of average January prices of wheat, and Figure 4 also included the corresponding changes of log $Z(t)$.

Assuming that successive weekly price changes are independent, the evidence of the yearly changes again favors L-stability. It is astonishing that the hypothesis of independence of weekly changes can be consistently carried so far, showing no discernible discontinuity between long-term adjustments to follow supply and demand, which would be the subject matter of economics, and the short-term fluctuations that some economists discuss as "mere effects of speculation."

3. THE VARIATION OF RAILROAD STOCKS PRICES, 1857-1936

For nineteenth-century speculators, railroad stocks were preeminent among corporation securities, and played a role comparable to that of the basic commodities. Unfortunately, Macaulay 1932 reports them incompletely: for each major stock, it gives the mean of the highest and lowest quotation during the months of January; for each month, it gives a weighted index of the high and low of every stock.

I began by examining the second series, even though it is averaged too many times for comfort. If one considers that there "should" have been no difference in kind between various nineteenth-century speculations, one would expect railroad stock changes to be L-stable, and averaging would bring an increase in the slope of the corresponding doubly logarithmic graphs, similar to what has been observed in the case of cotton price averages (Section III E of VCSP). Indeed, Figure 8, relative to the variation of the monthly averages, yields precisely what one expects for averages of L-stable processes with an exponent very close to that of cotton.

Yearly data, to the contrary, are little affected by averaging. Figure 9 should be regarded as made of two parts: the first five graphs concern companies with below average merger activity, the others to companies with above average merger activity.

The first five graphs, in my opinion, proved striking confirmations of the tools and concepts I developed in the study of cotton. Basically, one sees that the fluctuations of the price of these stocks were all L-stable, with the same α-exponent characteristic of the industry and clearly below the critical value 2. (Moreover, they all had practically the same value of the positive and negative "standard deviations," σ' and σ'', defined in VCSP.)

For the companies with an unusual amount of merger activity, the evidence is similar but more erratic.

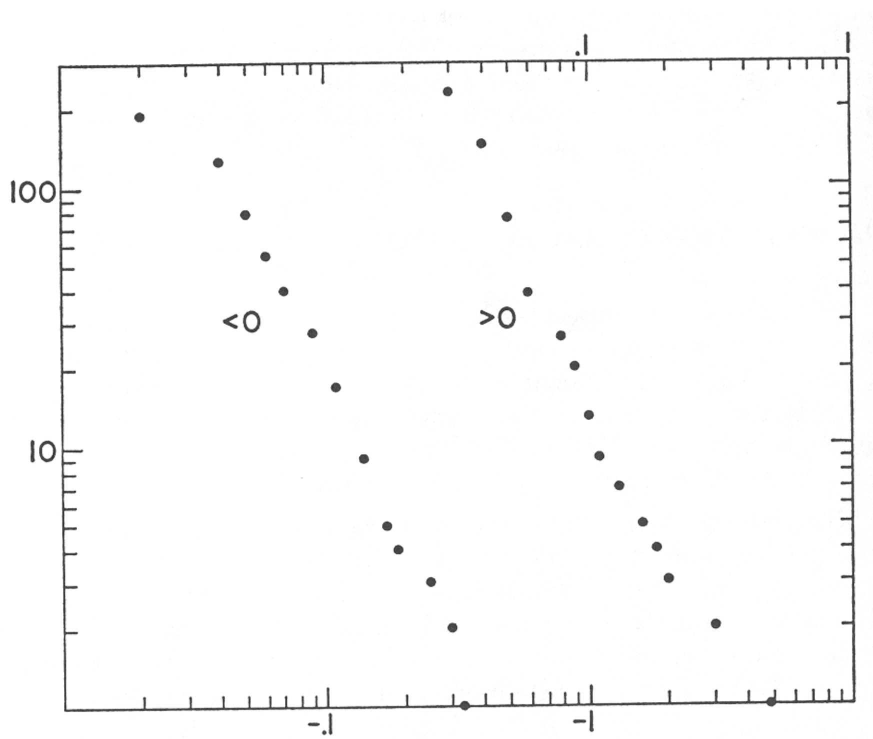

FIGURE C15-8. Monthly changes of log $Z(t)$, where $Z(t)$ is the index of rail stock prices, as reported in Macaulay 1932. Abscissas and ordinates as in Figure 5.

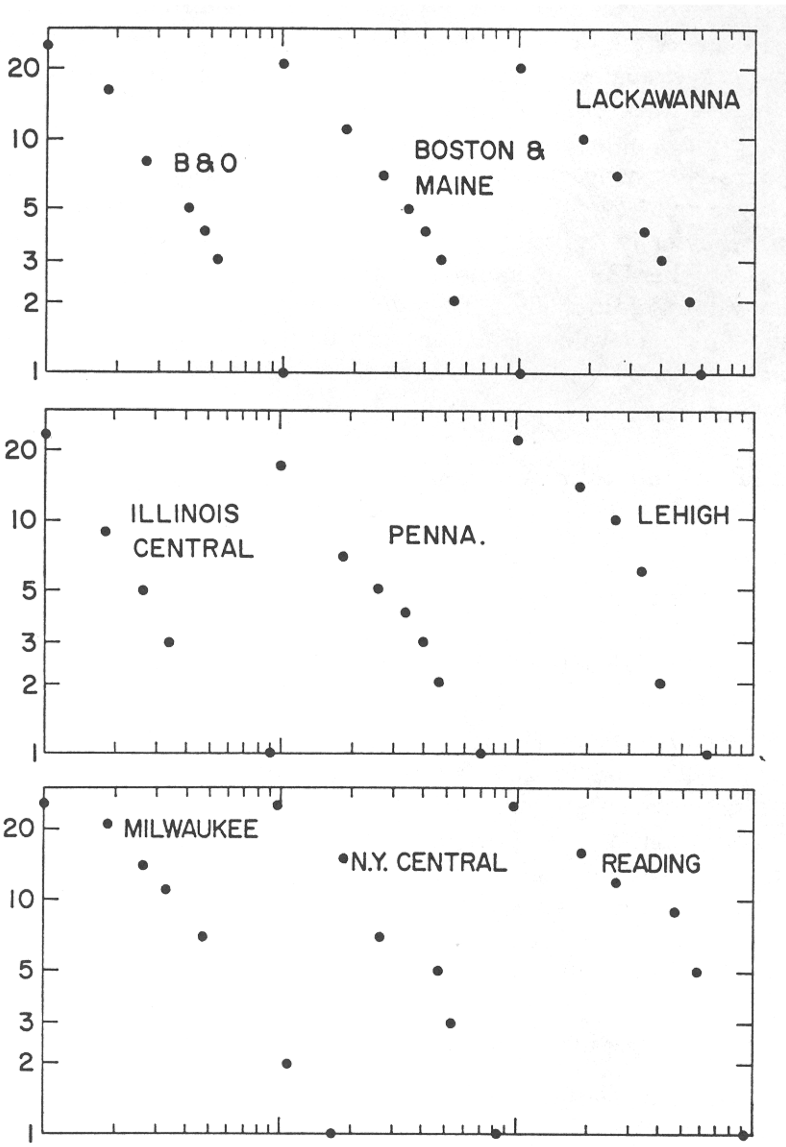

FIGURE C15-9. Yearly changes of $\log Z(t)$, where $Z(t)$ is the index based on the prices of nine selected rail stocks in January, as reported in Macaulay 1932. Ordinates: absolute frequencies. Abscissas are not marked to avoid confusion: for each graph, they vary from 1 to 10.

4. THE VARIATION OF SOME INTEREST AND EXCHANGE RATES

Various rates of money – and especially the rate of call money in its heyday – reflect the overall state of the speculative market. One would therefore expect analogies between the behaviors of speculative prices and of speculative rates. But one cannot expect them to be ruled by identical process. For example, one cannot assume (even as rough approximations) that successive changes of a money rate are statistically independent: Such rates would eventually blow up to infinity, or they would vanish. Neither behavior is realistic. As a result, the distribution of $Z(t)$ itself is meaningless when Z is a commodity price, but it is meaningful when it is a money rate. Additionally, a 5% rate of money is a price near equal to 1.05, and for small values of Z, one has $\log(1+Z) \sim Z$. Therefore, in the case of money rates, one should study $Z(t+T) - Z(t)$ rather than $\log Z(t+T) - \log Z(t)$.

4.1 The rate of interest on call money

In Figure 10, the abscissa is based on the data in Macaulay 1932, concerning the excess of rate of call money over its "typical" value, 6%. I have not even attempted to plot the distribution of the other tail of the difference "rate minus 6%," since that expression is by definition very short-tailed, being bounded by 6 per cent, while the positive value of "rate minus 6%" can go sky high (and occasionally did.)

The several lines of Figure 10 correspond, respectively, to the total period 1857-1936 and to three subperiods. They show that call money rates are single-tailed scaling, with an exponent markedly smaller than 2. Scale factors (such as the upper quartile) have changed – a form of non-stationarity – but the exponent α seems to have preserved a constant value, lying within the range in which the scaling distribution is known to be invariant under mixing of data from populations having the same α and different γ; see M 1963e{E3}.

4.2 Other interesting money rates

Examine next the distribution of the classic data collected by Erastus B. Bigelow (Figure 11, dashed line) relative to "street rates of first class paper in Boston" (and New York) at the *end* of each month from January 1836 to December 1860. (Bigelow also reports some rates applicable at the beginnings or middles of the same months, but I disregarded them to avoid the difficulties due to averaging.) The dots on Figure 11 again represent the difference between Bigelow's rates and the typical 6%; their behavior is what we would expect if essentially the same scaling law applied to these rates and those of call money.

Finally, examine a short sample of rates, reported by Davis 1960, on the basis of records of New England textile mills. These rates remained much closer to 6% than those of Bigelow. They are plotted in such a way that the crosses of Figure 11 represent ten times their excess over 6%. The sample is too short for comfort, but, until further notice, it suggests that the two series have differed mostly by their scales.

4.3 The dollar-sterling exchange in the nineteenth century

The exchange premium or discount in effect on a currency exchange seems to reflect directly the difference between the various "forces" that condition the variations of the values of the two currencies taken separately.

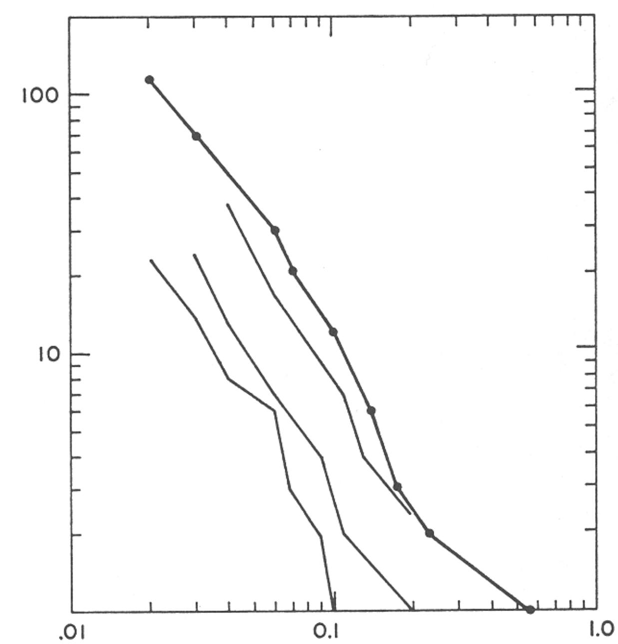

FIGURE C15-10. The distribution of the excess of over 6% of the monthly average of call money rates as reported in Macaulay 1932. Ordinates: absolute frequencies. *Bold line*: total sample 1857-1936. *Thin lines*, read from left to right: subsamples 1877-97, 1898-1936, 1857-76. Note that the second subsample is twice as long as the other two. Thus, the general shape of the curves has not changed except for the scale, and the scale has steadily decreased in time.

This differential quantity has an advantage over the changes of rates: one can consider it without resorting to any kind of economic theory, not even the minimal assumption that price changes are more important than price levels. We have therefore plotted the values of the premium or discount between dollar and sterling between 1803 and 1895, as reported by Davis and Hughes 1960 (Figure 12). This series is based upon operations which involved credit as well as exchange. In order to eliminate the credit component, the authors used various series of money rates. We also plotted the series based upon Bigelow's rates. Note that all the graphs of Figure 11 conform strikingly with the expectations generalized from the known behavior of cotton prices.

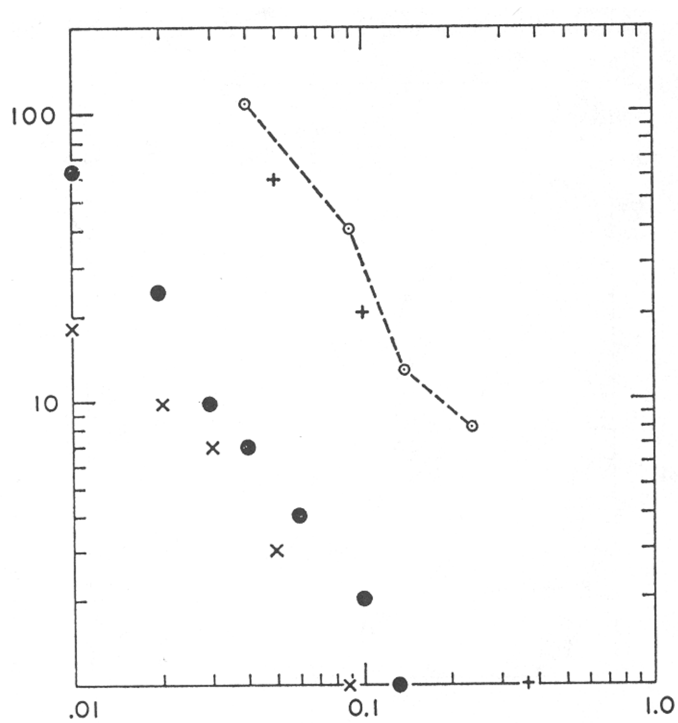

FIGURE C15-11. Four miscellaneous distributions of interest and exchange rates. Reading from the left: two different series of dollar-sterling premium rates. *Crosses*: ten times the excess over 6% of Davis's textile interest rate data. *Dashed line*: excess over 6% of Bigelow's money rates. The four series, all very short, were chosen haphazardly. The point of the figure is the remarkable similarity between the various curves.

5. MAXIMUM LIKELIHOOD ESTIMATION OF α NEAR α= 2

A handicap for the theory of VCSP is that no closed analytic form is known for the L-stable distributions, nor is a closed form ever likely to be discovered. Luckily, the cases where the exponent α is near 1.7 can be dealt with on the basis of an approximating hyperbolic distribution. Now let α be very near 2. To estimate $2-\alpha$ or to test $\alpha=2$ against $\alpha<2$ is extremely important, because $\alpha=2$ corresponds to the Gaussian law and differs "qualitatively" from other values of α. To estimate such an α is very difficult, however, and the estimate will intrinsically be highly dependent upon the number and the "erratic" sizes of the few most "outlying" values of u_n. I hope to show in the present section that simplifying approximations are fortunately available for certain purposes. The main idea is to represent a L-stable density as a sum of two easily manageable expressions, one of which concerns the central "bell," while the other concerns the tails.

5.1 A square central "bell" with scaling tails

The following probability density can be defined for $3/2<\alpha<2$:

$$p(u)=\alpha-3/2 \text{ if } |u|\leq 1 \qquad \text{(adding up to } 2\alpha-3);$$

$$p(u)=(2-\alpha)\alpha|u|^{-(\alpha+1)} \text{ if } |u|>1 \qquad \text{(adding up to } 4-2\alpha).$$

When α is near 2, $p(u)$ is a rough first approximation to a L-stable density that lends itself to maximum-likelihood estimation.

Order $u_1, \ldots u_n, \ldots, u_N$, a sample of values of U, by decreasing absolute size, and denote by M the values such that $|u_n|\geq 1$. The likelihood function is

$$\prod p(u_n)=(\alpha-\frac{3}{2})^{N-M}[(2-\alpha)\alpha]^M\left\{\prod_{n=1}^{M}|u_n|\right\}^{-(\alpha+1)}.$$

The logarithm of the likelihood is

$$L(\alpha)=(N-M)\log(\alpha-\frac{3}{2})+M\log\big[(2-\alpha)\alpha\big]-(\alpha+1)\sum_{n=1}^{M}\log|u_n|.$$

This $L(\alpha)$ is a continuous function of α. If $M=0$, it is monotone increasing and attains its maximum for $\alpha=2$. This is a reasonable answer, since $|U|<1$ for $\alpha=2$.

To the contrary, if $M>0$, $L(\alpha)$ tends to $-\infty$ as $\alpha \to 3/2$ or $\alpha \to 2$. Therefore, $L(\alpha)$ has at least one maximum, and its most likely value $\hat{\alpha}$ is the root of the third-degree algebraic equation

$$\frac{N-M}{\alpha-3/2}-\frac{M}{2-\alpha}+\frac{M}{\alpha}-\sum_{n=1}^{M}\log|u_n|=0.$$

Thus, $\hat{\alpha}$ only depends on M/N and $M^{-1}\Sigma_{n=1}^{M}\log|u_n|=V$.

For the latter $\log|U|$, conditioned by $\log|U|>1$, satisfies

$$\Pr\{\log|U|>u|\log|U|>0\}=\exp(-\alpha u).$$

Its expected value is $1/\alpha$. Therefore, as $M\to\infty$ and $N\to\infty$,

$$\text{the terms } \frac{1}{\alpha}-\frac{1}{M}\sum_{n=1}^{M}\log|u_n| \text{ will tend to 0,}$$

and can be neglected *in the first approximation*. When both M and N are large, the equation in $\hat{\alpha}$ simplifies to the first degree and

$$\hat{\alpha}=2-M/2N.$$

This value depends on the u_n through the ratio of these numbers in the two categories $|U|<1$ and $|U|>1$, i.e., the relative number of the "outliers" defined by $|U|>1$.

For example, if M/N is very small, $\hat{\alpha}$ is very close to 2. As N/M barely exceeds 1, $\hat{\alpha}$ nears 3/2. However, this is range in which $p(u)$ is a very poor approximation to a L-stable probability density.

It may be observed that, knowing N, M/N is symptotically Gaussian, and so is $\hat{\alpha}$ for all values of α.

In a second approximation, valid for α near 2, one will use $\alpha=2$ to compute

$$W = \frac{1}{\alpha} = \frac{1}{M} \sum \log |u_n| .$$

The equation for $\hat{\alpha}$ is now of the second order. One root is very large and irrelevant; the other root is such that $\alpha - (2 - M/2N)$ is proportional to W.

5.2 Scope of estimation based upon counts of outliers

The method of Section 5.1, namely, estimation of $\hat{\alpha}$ from M/N, applies without change under a variety of seemingly generalized conditions:

1. Suppose that the tails are asymmetric, that is,

$$p(u) = (2 - \alpha)\alpha p' u^{-(\alpha + 1)} \qquad \text{if } u > 1$$
$$p(u) = (2 - \alpha)\alpha p'' |u|^{-(\alpha + 1)} \qquad \text{if } u < -1,$$

where $p' + p'' = 1$. In estimating α, one will naturally concentrate upon the random variable $|U|$, which is the same as in Section 5.1

2. The conditional density of U, given that $|U| < 1$, may be non-uniform as long as it is independent of α. Suppose, for example, that for $|u| < 1$, $p(u) = (\alpha - 3/2)\, D \exp(-u^2/2\sigma^2)$, where $1/D(\sigma)$ is defined as equal to $\int_{-1}^{1} \exp(-s^2/2\sigma^2)ds$. The likelihood of α then equals

$$[D(\sigma)2^{-1}(\alpha - 3/2)]^{N-M} \exp\left(- \sum_{n=M+1}^{N} \frac{u_n^2}{2\sigma^2} \right) [(2 - \alpha)\alpha]^M \left(\prod_{n=1}^{M} |u_n| \right)^{-(\alpha + 1)} .$$

As function of the U_n, the maximum likelihood is as in Section 5.1.

Acknowledgement. This text incorporates several changes suggested by Professor Eugene F. Fama.