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RANDOM WALKS, FIRE DAMAGE AMOUNT AND OTHER PARETIAN RISK PHENOMENA

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Being one of the oldest branches of operations research, actuarial science has accumulated a substantial store of knowledge about the risks associated with living. The present paper will discuss one such question. Although it is relative to a specific problem of fire casualty, it illustrates more generally why the Paretian distribution of incomes and fortunes should constitute "a source of anxiety for the risk theory of insurance." Very similar mechanisms apply in many other problems.

IN A somewhat simplified form, the following statement summarizes an empirical law established by LARS-G. BENCKERT AND INGVAR STERNBERG:[1]

The damage to a house due to fire reaches different amounts with probabilities that obey a frequency function deduced as follows from Pareto's hyperbolic law: If the damage is greater than a minimum threshold $m=\$20$ and smaller than the maximum destroyable amount M of the building, the probability that the damage be equal to x has the density $s(x)=\alpha x^{-\alpha-1}m^{\alpha}$. The probability that the maximum destroyable amount M be actually destroyed is equal to $(M/m)^{-\alpha}$, the integral from M to infinity of the law of Pareto applicable to damages smaller than M.

This law applies to all classes of Swedish houses outside of Stockholm. The values of α were found to oscillate between 0.45 and 0.56; it seems fair therefore to begin by investigating the consequences of an assumed value $\alpha=\frac{1}{2}$.

A MODEL OF FIRE DAMAGE AMOUNT

THE LAW of Pareto of exponent 0.5 plays classically a central role as the distribution of the returns to equilibrium in the game of tossing a fair coin. This theory is developed in most textbooks of probability (such as reference 3), and it can be translated into insurance terms as follows.

Suppose that the intensity of a fire is characterized by a single number, designated by U, which can only take integral values; there is no fire when $U=0$; a fire starts when U becomes equal to 1, and it will end either when U becomes equal to 0 again, or when all that can possibly burn has already burned out.

Suppose now that, at any instant of time, there is a probability $p=\frac{1}{2}$ that the fire encounter new material so that its intensity increases by 1, and the probability $q=\frac{1}{2}$ that the absence of new materials or the action of fire-fighters decrease the intensity by 1. In the preceding statement, 'time' is to be measured by the extent of damage. If there is no finite maximum extent of damage, and no lower bound to recorded damages, the duration of a fire will be an even number given by a classical result concerning the return to equilibrium in cointossing:

$$\Pr(\text{duration of fire}=x)=2\binom{1/2}{x/2}(-1)^{x/2-1}.$$

Except for the first few values of x, this expression is proportional $x^{-3/2}$. Hence, if one assumes that very small damages, smaller than m, are not even properly recorded, the duration of a fire (i.e., the extent of damage) will be given by the law of Pareto with exponent $\frac{1}{2}$:

$$\Pr(\text{duration of fire}>x>m)=(x/m)^{-1/2}.$$

Suppose finally that one takes account of the fact that the fire *must* end if and when all has burnt out. One then obtains precisely the interpretation of the Benckert-Sternberg findings that we gave above.

RELATIONS INVOLVING THE SIZE OF THE PROPERTY AND THE EXPECTED AMOUNT OF THE DAMAGE DUE TO FIRE

It is easy to compute the expected value of the random variable considered in the opening section.

$$\begin{aligned}\text{Expected amount of damage}&=\int_m^M x(\tfrac{1}{2})x^{-(1/2+1)}m^{1/2}dx+M(M/m)^{-1/2}\\&=2\sqrt{Mm}-m.\end{aligned}$$

This mean value tends to infinity with M.†

On the other hand, according to von Savitsch and G. Benktander,[2] the expected number of fires per house in the course of a year is a linear function of M. If this is indeed so, it would imply that, for large values of M, the rate of isnurance should be proportional to $\sqrt{M}$.

Note also that, when the distribution of property sizes M is known, one has:

$$\Pr(\text{amount of damage}>x)=\Pr(M>x)(x/m)^{-1/2}.$$

If moreover the von Savitsch inference is correct, one has

$$\begin{aligned}d\Pr(\text{amount of damage per year}>x)&=Cxd\Pr(\text{amount of damage}>x)\\&=Cxd[\Pr(M>x)(x/m)^{-1/2}].\end{aligned}$$

† Random variables with infinite population moments are often considered to be mathematical devices with no practical application. Such is certainly not the case; *see* reference 4.

Let the distribution of M be itself Paretian with the exponent α^*, as is the case of all kinds of liability amounts.[5] The distribution of damage in a single fire will then be Paretian with exponent $\alpha^*+\frac{1}{2}$, and the distribution of damage per year will be Paretian with exponent $\alpha^*-\frac{1}{2}$. This demonstrates that mathematical manipulations based on Pareto's law are especially convenient.

GENERALIZATION

THE RANDOM walk with $p=q=\frac{1}{2}$ represents a kind of equilibrium state between the fire and the fire-fighters: if indeed the quantity of combustible property were unbounded, a fire following the rules given above will surely die out, although its expected duration would be infinite. On the contrary, if $p>q$, the fire-fighting efforts would be inadequate, and there would be a nonvanishing probability that the fire continue forever.

If $p<q$, the probability of running forever would again be zero and the expected duration of the fire would be finite. The law giving the duration of the fire would then take the form

$$d\ \Pr(\text{duration of fire} \geqq x > m) \sim c^* \exp(-cx) x^{-3/2}\, dx,$$

where c^* and c are two constants depending upon m, p, and q. If x is actually bounded, and if $q-p$ (and hence c) is small, the above formula will be indistinguishable in practice from a law of Pareto with an exponent slightly greater than $\frac{1}{2}$; this is perhaps an explanation of the more precise experimental results of Benckert and Sternberg.

We shall not stop to rephrase the generalization of the random walk, provided by any other of the many classical models of diffusion. Nor shall we go into the details of a random walk model in which there is the possibility that the intensity of the fire remain unchanged from one instant of time to the next. This would not add anything beyond a change in time scale.

ANOTHER APPLICATION

The results of Benckert and Sternberg strongly resemble those of Lewis F. Richardson,[7] [8] and the model that has been sketched above is easy to translate in the terms of Richardson's problem. It would be fascinating to ponder how relevant that translation may be.

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