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THE STABLE PARETIAN INCOME DISTRIBUTION WHEN THE APPARENT EXPONENT IS NEAR TWO, II

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The conjecture presented in the second of the preceding reprints is that the theory of income distribution, which I prposed in the first reprint, is applicable even to these recent data which are represented by doubly logarithmic graphs with a slope greater than two. That is, I believe that one should not be arrested by the facts that Lévy's exponent α at most equals two, and that this limit case corresponds to the Gaussian law, which means that it is not asymptotically Paretian. I pointed out, indeed, that, as Lévy's α nears its limit, the doubly logarithmic graph of the stable law takes a kinked form. Figure 1 demonstrates its behavior in the case of densities. It is therefore too tempting to argue that many "smoothing" mechanisms could lead observers to simplify matters by representing the whole income distribution by a straight line, which would necessarily have a slope greater than two.

One can add that, when Lévy's α equals 1.8, very little smoothing is actually necessary to confuse the stable Paretian law of $\alpha = 1.8$ with a Pareto's law of $\alpha = 2$. It is, in fact, quite possible that I should have always kept the two concepts of alpha separate, even in the first of the preceding reprints. In any event, much more smoothing is required when Lévy's α becomes very close to 2.

One of the fundamental smoothing mechanisms is provided by "mixing" of different distributions. The law of Pareto itself has the following property of invariance under mixing: If the variables U_n are such that $Pr(U>u) \sim C_n u^{-\alpha}$, where α is independent of n, and if one mixes data from those various distributions in the proportions p_n, one obtains a random variable U_W such that $Pr(U_W>u) \sim (\Sigma p_n C_u) u^{-\alpha}$. This simply results from the fact that the doubly logarithmic graphs of Pareto's laws are straight lines and have no "kink" that mixture could smooth off.—

But the stable Paretian laws do have two clear-cut kinks, especially if α is near 2; therefore, the overall distribution would appear Paretian with a high alpha, even if the distribution of income were stable Paretian within sufficiently narrowly defined categories of high-income earners.

As I had no idea of the actual distribution of income within narrow high -income categories, the above conjecture constituted a genuine prediction of my theory. The primary purpose of the present Note is to reproduce data that suggest that my inference was indeed correct and that I was originally too prudent in handling the consequences of my theory. Figures 2 to 4 are based upon graphs communicated to me privately by H. S. Houthakker, who used them (for entirely different purposes) in the preparation of his article "Education and Income", Review of Economics and Statistics , 41 (1959) pp. 24-27. The main fact that emerges from those figures is that these curves which "on the whole" have the largest absolute slope are also the least straight. Q. E. D.

Another consequence of this figure is that Lévy's α of incomes changes during a man's lifetime, thus throwing doubt on all "diffusion models" which obtain Pareto's law as the limit state resulting from the action of various transformations, and throwing doubt also on the model in reference [18] of the first reprint. Reference [19] probably remains valid.

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FIGURE 1. DENSITIES OF CERTAIN MAXIMALLY SKEW STABLE PARETIAN DISTRIBUTIONS. These densities are defined by the fact that their Fourier transform is of the form

$$\exp \{ - |z|^{\alpha} [1-i|z|z^{-1} \tan (\alpha\Pi/2)] \}$$

Reading from the lower left to the upper right corner, the unique parameter α takes the values 2, 1.99, 1.95, 1.9, 1.8, 1.7, 1.6 and 1.5.

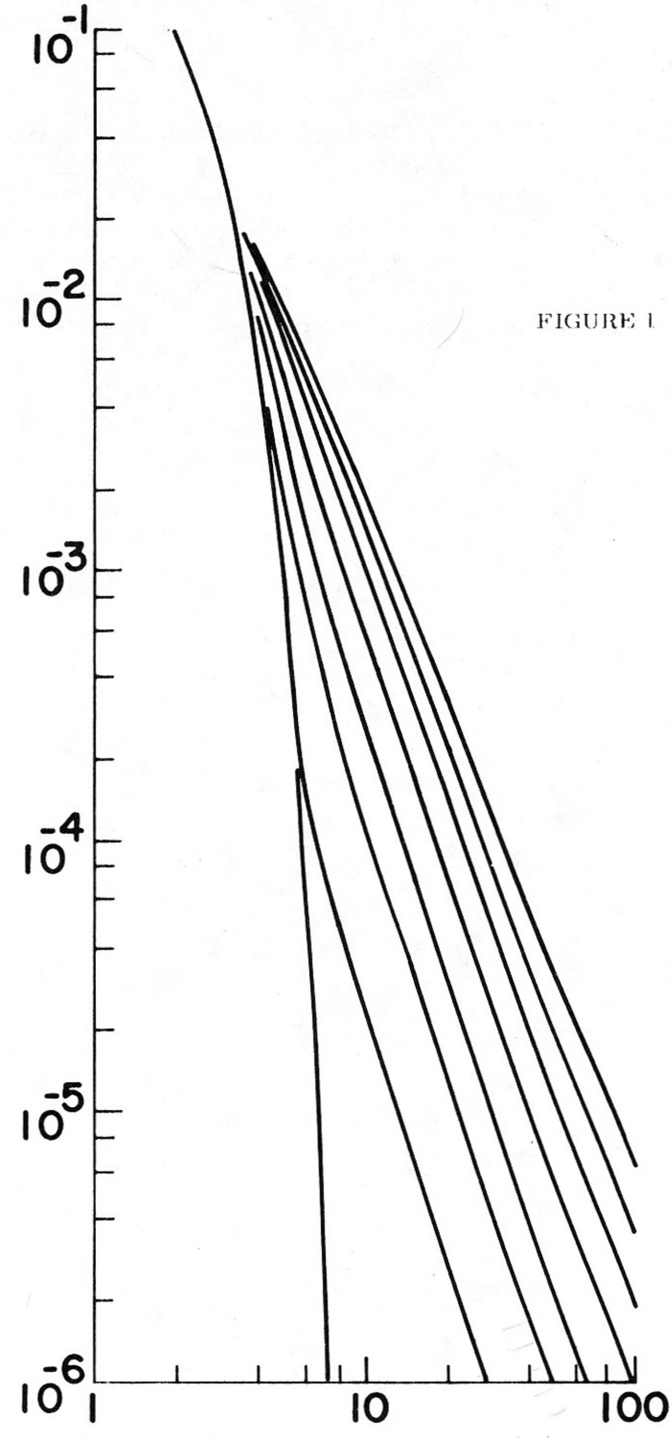

FIGURE 1

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FIGURE 2. DISTRIBUTION OF 1949 INCOMES IN THE U. S. A. AMONG PERSONS IN THE AGE GROUP FROM 35 TO 44 YEARS. Horizontally: income u in dollars. Vertically: number of persons with an income exceeding u. Reading from the lower left to the upper right corner, the bold lines refer to the following levels of education, as measured by the number of years in school: None, 1-4, 5-7, 8, High School 1-3, High School 4, College 1-3, College 4 or plus.

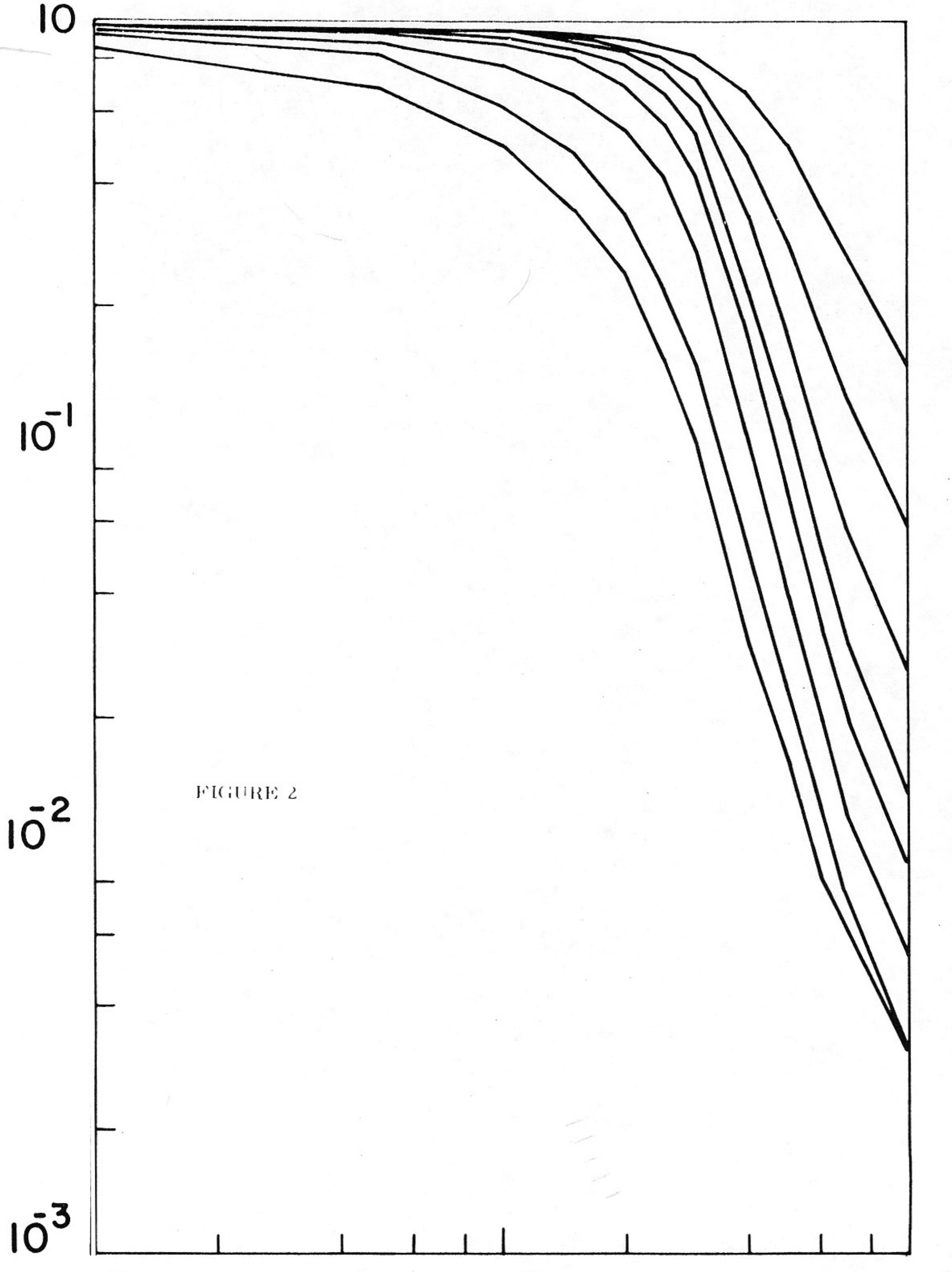

FIGURE 2

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FIGURE 3. DISTRIBUTION OF 1949 INCOMES IN THE U. S. A. AMONG PERSONS WITH 8 YEARS OF SCHOOL. Horizontally: income u in dollars. Vertically: number of persons with an income exceeding u. Reading from the lower left to the upper right corner, the bold lines refer to the following age groups: 14-15, 16-17, 18-19, 20-21, 22-24, 25-29, 30-34, 35-44, 45-54. The dashed line refers to all income-earners with 8 years of school and aged 25 or more.

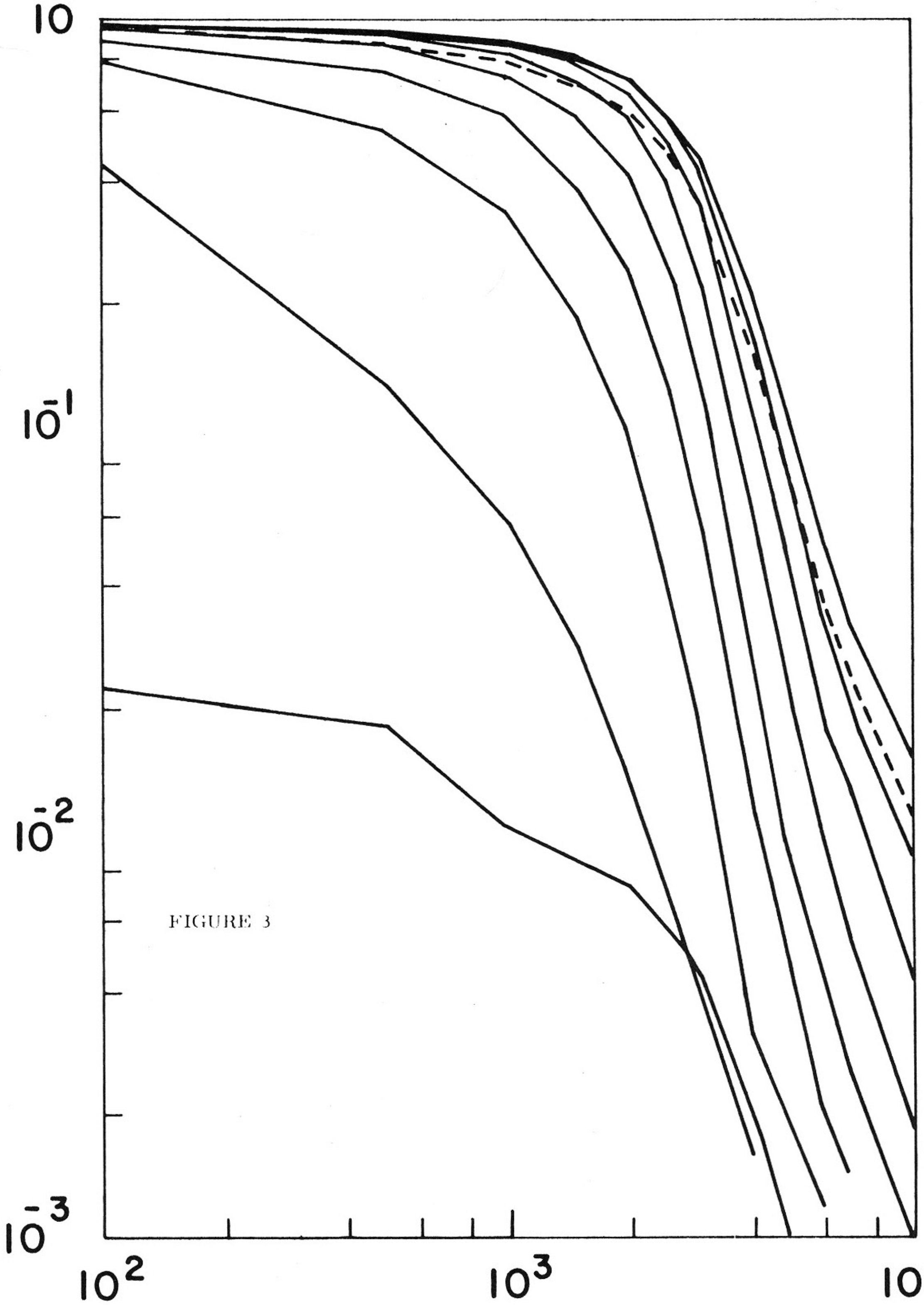
10
10^{-1}
10^{-2}
FIGURE 3
10^{-3}
10^2
10^3
10^4

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FIGURE 4. DISTRIBUTION OF 1949 INCOMES IN U. S. A. AMONG PERSONS WITH FOUR YEARS OF COLLEGE OR MORE. Horizontally: income u in dollars. Vertically: number of persons with an income exceeding u. Reading from the lower left to the upper right corner, the bold lines refer to the following age groups: 20-21, 22-24, 25-29, 30-34, 35-44, 45-54. The dashed line refers to all income-earners with 4 years of college or more and aged 25 or more.

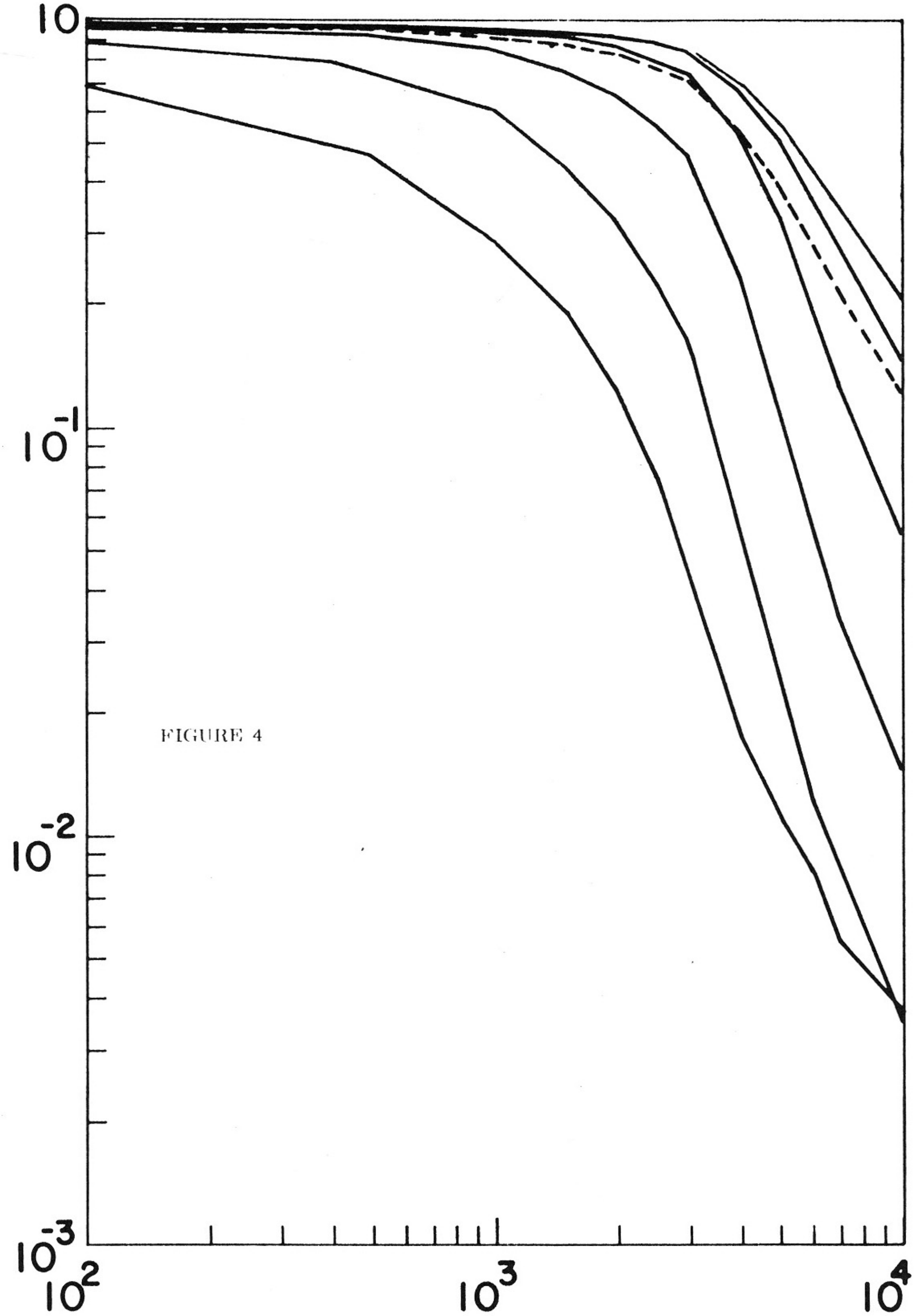

FIGURE 4