Brill–Noether special cubic fourfolds

Asher Auel

Department of Mathematics
Yale University

AMS Summer Institute in Algebraic Geometry
University of Utah, Salt Lake City
Monday 27 July 2015
Cubic fourfolds

$X \subset \mathbb{P}^5$ smooth cubic hypersurface over $\mathbb{C}$

**Torelli Theorem (Voisin).** The integral polarized Hodge structure on $H^4(X, \mathbb{Z})$ recovers $X$ up to isomorphism.

**Integral Hodge Conjecture (Voisin).** The cycle class map is an isomorphism $\text{CH}^2(X) \rightarrow H^4(X, \mathbb{Z}) \cap H^{2,2}(X) = A(X)$.

$A(X)$ odd positive definite free $\mathbb{Z}$-lattice
$h^2 \in A(X)$ distinguished element of norm 3

**Fact.** $A(X) = \mathbb{Z}h^2$ for very general $X$
Noether–Lefschetz loci

$\mathcal{C}$ moduli space of cubic fourfolds

Noether–Lefschetz locus

\[ \{ X \in \mathcal{C} : \text{rk} \ A(X) > 1 \} = \bigcup_d \mathcal{C}_d \]

\[ X \in \mathcal{C}_d \iff \exists T \in A(X) \text{ such that } \langle h^2, T \rangle \subset A(X) \text{ is a primitive sublattice of rank 2 and discriminant } d \]

\[ \iff X \text{ special cubic fourfold of discriminant } d \]

(Hassett) $\mathcal{C}_d \neq \emptyset$ irreducible divisor $\iff d > 6$ and $d \equiv 0, 2 \ (6)$

$\mathcal{C}_d$ called Hassett divisors
The general $X \in C_d$ contains:

$\begin{align*}
    d = 8 & \quad \text{a plane} \\
    d = 12 & \quad \text{a cubic scroll} \\
    d = 14 & \quad \text{a quartic scroll or a quintic del Pezzo surface} \\
    d = 20 & \quad \text{a Veronese surface} \\
    12 \leq d \leq 38 & \quad \text{certain smooth rational surfaces (Nuer)} \\
    d = 44 & \quad \text{the Fano model of an Enriques surface (Nuer)}
\end{align*}$
Geometry of $C_d$

(Li/Zhang) Compute the generating function of the degrees of $C_d$ as a modular form of weight 11 and level 3. These degrees get large: 3402, 196272, 915678, …

(Nuer) $C_d$ is unirational for $d \leq 38$ and has $C_{44}$ has negative Kodaira dimension.

(Tanimoto/Várilly-Alvarado) $C_d$ is of general type for $d \gg 0$. Current state of the art is $d \geq 264$.

Tony’s talk on Tuesday, 4:40–5:30 pm in SFEBB 170!
Rationality of cubic fourfolds

Conjecture. The very general cubic fourfold is not rational.

Example. $X$ contains disjoint planes $\implies X$ is rational

(Hassett) $X \in C_8$ is rational on a countable union of divisors.

\[
\begin{align*}
X \in C_8 & \iff \mathbb{P}^2 \cong P \subset X \\
\quad & \implies \quad \text{Bl}_P X \leftarrow \text{Bl}_P \mathbb{P}^5 \\
\pi \downarrow & \quad \downarrow \\
S & \rightarrow \mathbb{P}^2
\end{align*}
\]

$\pi$ quadric surface bundle degenerating along sextic $D \subset \mathbb{P}^2$

$S$ moduli space of rulings, $\beta_X \in \text{Br}(S)$ class of universal ruling

(Hassett) $\beta_X = 0 \implies X$ is rational

(A./Bernardara/Bolognesi/Várilly-Alvarado) There exist $X \in C_8$ with $X$ rational but $\beta_X \neq 0$. 
Rationality of cubic fourfolds

(Beauville/Donagi, Bolognesi/Russo/Staglianò, A.)
Every $X \in C_{14}$ is rational.

Challenge. Give new rationality constructions for cubic fourfolds.

(Katzarkov) HMS $\implies$ every $X \in C_{26}$ is rational
Associated K3 surface

\[ H^2(S, \mathbb{Z}) \text{ weight 2 signature (2, 20)} \quad 1 \quad 20 \quad 1 \]
\[ H^4(X, \mathbb{Z}) \text{ weight 4 signature (21, 2)} \quad 0 \quad 1 \quad 21 \quad 1 \quad 0 \]

Polarized K3 surface \((S, H)\) choice of ample \(H \in \text{Pic}(S)\)

Marked cubic fourfold \((X, K_d)\) choice of rank 2 \(K_d \subset A(X)\)

Primitive cohomology \(H^2(S, \mathbb{Z})_0 = H^\perp \subset H^2(S, \mathbb{Z})\)

Nonspecial cohomology \(H^4(X, \mathbb{Z})_0 = K_d^\perp \subset H^4(X, \mathbb{Z})\)

(Hassett) Exists a polarized K3 surface \((S, H)\) of degree \(d\) with \(H^4(X, \mathbb{Z})_0 \cong H^2(S, \mathbb{Z})_0(-1) \iff 4 \nmid d, 9 \nmid d, p \nmid d\) for \(p \equiv 2 \mod 3\)

\(d = 14, 26, 38, 42, 62, 74, \ldots\)

\(S\) is an associated K3 surface to \(X\)

(Hassett) \(\mathcal{C}^\text{mar}_d \hookrightarrow \mathcal{K}_d\) embedding of moduli spaces
Associated K3 category

Semiorthogonal decomposition of the derived category
\[ D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle \]

\[ \mathcal{A}_X = \{ E \in D^b(X) : \mathrm{Ext}^i(\mathcal{O}_X(i), E) = 0, \ i = 0, 1, 2 \} \]

\( \mathcal{A}_X \) looks like the derived category of a K3 surface

Example. \( X \in \mathcal{C}_8 \implies \mathcal{A}_X \cong D^b(S, \beta_X) \)

(Huybrechts) There are finitely many \( X' \) such that \( \mathcal{A}_X \cong \mathcal{A}_{X'} \).

If \( X \) is very general, then \( \mathcal{A}_X \) determines \( X \) uniquely.
Suspicions and conjectures

**Suspicion (Harris, Hassett).** $X$ is rational $\iff X$ has an associated K3 surface

**Conjecture (Kuznetsov).** $X$ is rational $\iff \mathcal{A}_X \cong \mathbb{D}^b(S)$ for a K3 surface $S$

**(Addington/Thomas)** $\mathcal{A}_X \cong \mathbb{D}^b(S) \implies X$ has an associated K3 surface $S$. The converse holds generically on $C_d$ if $4 \nmid d$, $9 \nmid d$, $p \nmid d$ for $p \equiv 2 \pmod{3}$.

**(Voisin)** $4 \nmid d \implies$ every $X \in C_d$ has universally trivial $CH_0$

Voisin’s plenary talk from week 1!
Alena Pirutka’s talk on Tuesday, 2:00–2:50 pm, SFEBB 180!
Brill–Noether general cubic fourfolds

(Mukai) Polarized K3 surface \((S, H)\) is *Brill–Noether general* if

\[
h^0(S, N) h^0(S, M) < h^0(S, H) = 2 + d/2 = g + 1
\]

for any nontrivial decomposition \(H = N \otimes M\).

**Example.** \(\text{Pic}(S) = \mathbb{Z}H \iff (S, H)\) is BN general

(Lazarsfeld) \(\text{Pic}(S) = \mathbb{Z}H \iff C \in |H|\) is BN general curve

**Fact.** \(C \in |H|\) is BN general curve \(\implies (S, H)\) is BN general K3

**Open question.** What about the converse?
Checked for \(g \leq 10\) and \(g = 12\) by Mukai.

**Definition.** \((X, K_d)\) marked cubic fourfold is *BN general* if associated K3 surface \((S, H)\) is BN general
Brill–Noether special cubic fourfolds

**Definition.** The complement of BN general is *BN special.*

\[ C_d^\text{mar} \leftrightarrow K_d \rightarrow K_d^\text{BN} \]

\[ C_d^\text{BN} \leftrightarrow K_d^\text{BN} \]

The BN special loci are contained in the union of finitely many Noether–Lefschetz divisors, indexed by *Clifford index.*

*(Saint-Donat, Reid, Donagi/Morrison, Green/Lazarsfeld, Mukai, Ciliberto/Pareschi, Knutsen, Johnsen, Lelli-Chiesa)*

Classification of BN special K3 surfaces via vector bundles and lattice theory. Completely done for \( g \leq 12.\)

*(Program.)* Carry this out for cubic fourfolds.
Brill–Noether special cubic fourfolds

**Theorem (A).**

- $X \in C_{14}$ has a BN general marking of discriminant 14 if and only if $X$ is pfaffian.
- $X \in C_{14}$ has a BN special marking of discriminant 14 if and only if $X$ contains disjoint planes.

The image of $C_{14}^{BN} \hookrightarrow K_{14}^{BN}$ is contained in only one of five K3 Noether–Lefschetz divisors, with maximal Clifford index.

**Corollary.** Every $X \in C_{14}$ is pfaffian or contains disjoint planes (or both), hence is rational.

Next frontier is $d = 26$. Need good constructions of BN general K3 surfaces of degree $d = 26$. Input from moduli theory of curves of $g = 14$?