1 Distributions on R

Definition 1. Let $X$ be a topological space, let $f : X \to \mathbb{C}$, then define the support of $f$ to be the set 

$$\text{supp}(f) = \{ x \in X : f(x) \neq 0 \}.$$ 

Definition 2. Let $\Omega \subset \mathbb{R}^n$, then define $\mathcal{D}(\Omega) \subset \mathcal{C}^\infty(\Omega)$ to be the space of infinitely differentiable (partials of all orders) functions $f : \mathbb{R}^n \to \mathbb{R}$ which have compact support. We can see that $\mathcal{D}(\Omega)$ is a real vector space, an algebra under pointwise multiplication, and an ideal in $\mathcal{C}^\infty(\Omega)$. We can always thing of $\mathcal{D}(\Omega) \subset \mathcal{D}(\mathbb{R}^n)$, and for brevity I will call $\mathcal{D}(\mathbb{R}) = \mathcal{D}$.

Definition 3 (Convergence in $\mathcal{D}(\mathbb{R}^n)$). Let $\{\varphi_i\}$ be a sequence in $\mathcal{D}(\mathbb{R}^n)$ and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then we say that $\{\varphi_i\} \to \varphi$ if

1. $\text{supp}(\varphi_i) \subset K$ for some compact $K \subset \mathbb{R}^n$ for all $i \in \mathbb{N}$.
2. $\{D^p\varphi_i\} \to D^p\varphi$ uniformly for each $p \in \mathbb{N}^n$.

This topology defined by some semi-norm?

Theorem 4. The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$, the space of continuous real function of compact support with topology given by uniform convergence, i.e. for any $f \in C_0(\mathbb{R}^n)$ with $\text{supp}(f) \subset U$ for some open $U \subset \mathbb{R}^n$ and for any $\varepsilon > 0$, there exists a $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(\varphi) \subset U$, and such that for all $x \in \mathbb{R}^n$

$$|f(x) - \varphi(x)| < \varepsilon.$$ 

Definition 5. A distribution $T$ is a continuous linear map $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$, i.e.

1. $T(\alpha \varphi + \beta \psi) = \alpha T(\varphi) + \beta T(\psi)$ for all $\alpha, \beta \in \mathbb{C}, \varphi, \psi \in \mathcal{D}$.
2. $\{\varphi_i\} \to \phi$ in $\mathcal{D} \Rightarrow \{T(\varphi_i)\} \to T(\varphi)$ in $\mathbb{C}$.

The set of all distributions on $\mathbb{R}^n$ will be denoted $\mathcal{D}'(\mathbb{R}^n)$. We can see that $\mathcal{D}'(\mathbb{R}^n) \supset \mathcal{D}^\ast(\mathbb{R}^n)$, and assuming the axiom of choice one can show that there exist linear maps $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ which are not continuous in the topology of $\mathcal{D}(\mathbb{R}^n)$.
Definition 6. Let \( \mathcal{L}(\mathbb{R}^n) \) be the space of all locally integrable functions \( f : \mathbb{R}^n \to \mathbb{C} \), i.e. for all compact sets \( U \subset \mathbb{R}^n \), \( f \) is integrable on \( U \).

Example 7. For every \( f \in \mathcal{L}(\mathbb{R}^n) \), we can define a distribution \( T_f \in \mathcal{D}'(\mathbb{R}^n) \) given by

\[
T_f(\varphi) = \int_{\mathbb{R}^n} f \varphi,
\]

for all \( \varphi \in \mathcal{D}(\mathbb{R}^n) \), where the integral is the Lebesgue integral, and where really, the integral is finite since the support of \( \varphi \), hence of \( f \varphi \), is bounded.

For any \( f \in \mathcal{L}(\mathbb{R}^n) \), the map \( T_f \) is obviously linear from the properties of the integral, now let \( \{ \varphi_i \} \to \varphi \) in \( \mathcal{D} \), with \( \text{supp}(\varphi_i) \subset K \), then

\[
|T_f(\varphi_i) - T_f(\varphi)| \leq \int_{\mathbb{R}^n} |f(\varphi_i - \varphi)| \leq \left( \int_K |f| \right) \sup_{x \in K} |\varphi_i(x) - \varphi(x)| \to 0.
\]

So in fact \( T_f(\varphi_i) \to T_f(\varphi) \), so \( T_f \) is continuous.

Claim 8. Let \( f, g \in \mathcal{L} \), then \( T_f = T_g \iff f = g \) almost everywhere.

So there is an injection

\[
\widetilde{\mathcal{L}} \to \mathcal{D}', \quad f \mapsto T_f,
\]

where \( \widetilde{\mathcal{L}} = \mathcal{L}/\{ f \in \mathcal{L} : f = 0 \text{ almost everywhere} \} \). So we can think of locally integrable functions as distributions. There are distributions which do not accord to locally integrable functions.

Example 9. The Dirac delta distribution \( \delta \in \mathcal{D}' \) is given by,

\[
\delta(\varphi) = \varphi(0), \quad \text{for all } \varphi \in \mathcal{D}'.
\]

Also for each \( a \in \mathbb{R}^n \) we can define \( \delta_a \) in the obvious way. Then obviously \( \delta \) forms a distribution.

In fact there exists no \( f \in \mathcal{L} \) such that \( \delta = T_f \), i.e. such that

\[
\int_{\mathbb{R}^n} f(x)\varphi(x)dx = \varphi(0), \quad \text{for all } \varphi \in \mathcal{D}.
\]

2
2 Derivatives of Distributions

We want to develop the notion of the derivative of a distribution, so, we start by looking at derivative of derivatives corresponding to $C^1$ functions.

**motivation**

Let $f \in C^1(\mathbb{R})$, then for all $\varphi \in \mathcal{D}$,

$$T_f'(\varphi) = \int_{\mathbb{R}} f'(x)\varphi(x)dx$$

$$= f(x)\varphi(x)|_{-\infty}^{\infty} - \int_{\mathbb{R}} (f(x)\varphi'(x))dx$$

$$= -\int_{\mathbb{R}} (f(x)\varphi'(x))dx = -T_f(\varphi').$$

This provides the motivation for defining the derivative of an arbitrary distribution $T \in \mathcal{D}'$ by

$$DT(\varphi) = -T(D\varphi), \quad \text{for all } \varphi \in \mathcal{D},$$

and for each $k \in \mathbb{N}$,

$$D^kT(\varphi) = (-1)^kT(D^k\varphi), \quad \text{for all } \varphi \in \mathcal{D}.$$  

We can see how this generalizes to $\mathbb{R}^n$, for all $p \in \mathbb{N}^n$,

$$D^pT(\varphi) = (-1)^{|p|}T(D^p\varphi), \quad \text{for all } \varphi \in \mathcal{D},$$

where

$$D^p = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{p_n} \quad \text{and} \quad |p| = p_1 + \cdots p_n$$

Note that for any $f \in C^1$, $DT_f = T_{Df}$.

**Example 10.** Let the Heavyside function $H : \mathbb{R} \to \mathbb{R}$ be given by

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}.$$  

Then for any $\varphi \in \mathcal{D}$,

$$DT_H(\varphi) = -T_H(\varphi') = -\int_{\mathbb{R}} H(x)\varphi'(x)dx$$

$$= -\int_0^\infty \varphi'(x)dx = -\varphi(x)|_0^\infty = \varphi(0) = \delta(\varphi),$$
thus we have found out that as distributions $DH = \delta$. Now we want to see what $D\delta$ turns out to be, for any $\varphi \in \mathcal{D}$,

$$D\delta(\varphi) = -\delta(\varphi') = -\varphi'(0),$$

and so in general,

$$D^k\delta(\varphi) = (-1)^k\varphi^{(k)}(0).$$

**Example 11.** Though the function $f(x) = 1/x$ is not locally integrable, the distribution given by, for every $\varphi \in \mathcal{D}$,

$$T(\varphi) = \text{pv} \int_{\mathbb{R}} \frac{1}{x} \varphi(x) dx = \lim_{\varepsilon \to 0} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \left( \frac{1}{x} \varphi(x) dx \right),$$

does make sense. Let $\varphi \in \mathcal{D}$, and suppose $\text{supp}(\varphi) \subset (-a, a)$, then

### 3 Multiplication

In general there is no way of multiplying two arbitrary distribution, this is because, even in the case of $\mathcal{L}$, the product of locally integrable function is not necessarily locally integrable, an example is the function $f(x) = 1/\sqrt{|x|}$ which is locally integrable, however, $f^2(x) = 1/|x|$ is not locally integrable. But we can multiply distributions by infinitely differentiable functions. First, for any $f \in \mathcal{L}$, $\alpha \in \mathcal{C}^\infty$, $\varphi \in \mathcal{D}$,

$$T_{\alpha f}(\varphi) = \int_{\mathbb{R}^n} \alpha f \varphi = \int_{\mathbb{R}^n} f(\alpha \varphi) = T_f(\alpha \varphi),$$

since $\mathcal{D}$ is an ideal in $\mathcal{C}^\infty$. Using this as motivation, for any distribution $T \in \mathcal{D}'$, we define

$$(\alpha T)(\varphi) = T(\alpha \varphi), \quad \text{for all } \varphi \in \mathcal{D}.$$  

**Example 12.** Let $\alpha \in \mathcal{C}^\infty$, then for $\varphi \in \mathcal{D}$,

$$(\alpha \delta)(\varphi) = \delta(\alpha \varphi) = \alpha(0) \varphi(0) = \alpha(0) \cdot \delta(\varphi).$$

In particular

$$\text{id}\delta = 0.$$
Also we have,

\[(\alpha D\delta)(\varphi) = D\delta(\alpha \varphi) = -(\alpha \varphi)'(0) = -\alpha(0)\varphi'(0) - \alpha'(0)\varphi(0) = (\alpha(0)D\delta - \alpha'(0)\delta)(\varphi)\].

In particular

\[idD\delta = -\delta, \quad id^2D\delta = 0, \quad idD^k\delta = -kD^{k-1}\delta.\]

**Theorem 13 (Product rule for distributions).** Let \(\alpha \in C^\infty, T \in \mathcal{D}',\) then

\[D(\alpha T) = \alpha(DT) + (D\alpha)T.\]

**Proof.** Let \(\varphi \in \mathcal{D},\) then

\[
D(\alpha T)(\varphi) = -(\alpha T)(D\varphi) = -T(\alpha D\varphi)
\]
\[
= -T(D(\alpha \varphi) - (D\alpha)\varphi)
\]
\[
= -T(D(\alpha \varphi)) + T((D\alpha)\varphi)
\]
\[
= (\alpha(DT))(\varphi) + (D\alpha)T(\varphi).
\]

\[\square\]

4 Convergence of Distributions

**Definition 14.** A sequence of distributions \(\{T_i\}\) converges to \(T \in \mathcal{D}\) if

\[\{T_i(\varphi)\} \to T(\varphi), \text{ for all } \varphi \in \mathcal{D}.\]

This is “pointwise convergence” or “weak” convergence of functionals.

**Theorem 15.** Let \(\{T_i\}\) be a sequence in \(\mathcal{D}',\) then if for each \(\varphi \in \mathcal{D},\) if \(\{T_i(\varphi)\}\) converges in \(C,\) then \(\{T_i\} \to T\) for some \(T \in \mathcal{D}'.\)

**Theorem 16 (Dominated convergence).** Let \(\{f_i\}\) be a sequence in \(\mathcal{L}\) such that \(\{f_i\} \to f\) for some function \(f\) such that for all \(i \in \mathbb{N}, |f_i| \leq g\) for some \(g \in \mathcal{L},\) then \(\{T_{f_i}\} \to T_f.\)

**Theorem 17.** The derivative operator \(D : \mathcal{D}' \to \mathcal{D}'\) is linear and continuous, i.e. if \(\{T_i\} \to T\) then \(\{DT_i\} \to DT.\)

**Proof.** Suppose \(\{T_i\} \to T,\) then for all \(\varphi \in \mathcal{D},\)

\[\{DT_i(\varphi)\} = \{-T_i(D\varphi)\} \to -T(D\varphi) = DT(\varphi).
\]

\[\square\]