FAILURE OF THE LOCAL-GLOBAL PRINCIPLE FOR ISOTROPY OF QUADRATIC FORMS OVER RATIONAL FUNCTION FIELDS

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abstract. we prove the failure of the local-global principle, with respect to all discrete valuations, for isotropy of quadratic forms of dimension $2^n$ over a rational function field of transcendence degree $n$ over $\mathbb{C}$. our construction involves the generalized kummer varieties considered by borcea [6] and cynk and hulek [11].

introduction

the hasse–minkowski theorem states that if a quadratic form $q$ over a number field is isotropic over every completion, then $q$ is isotropic. this is the first, and most famous, instance of the local-global principle for isotropy of quadratic forms. already for a rational function field in one variable over a number field, witt [20] found examples of the failure of the local-global principle for isotropy of quadratic forms in dimension 3 (and also 4). lind [17] andreichardt [18], and later cassels [7], found examples of failure of the local-global principle for isotropy of pairs of quadratic forms of dimension 4 over $\mathbb{Q}$ (see [1] for a detailed account), giving examples of quadratic forms over $\mathbb{Q}(t)$ by an application of the amer–brumer theorem [2], [13, thm. 17.14]. cassels, ellison, and pfister [8] found examples of dimension 4 over a rational function field in two variables over the real numbers.

here, we are interested in the failure of the local-global principle for isotropy of quadratic forms over function fields of higher transcendence degree over algebraically closed fields. all our fields will be assumed to be of characteristic $\neq 2$ and all our quadratic forms nondegenerate. recall that a quadratic form is isotropic if it admits a nontrivial zero. if $k$ is a field and $v$ is a discrete valuation on $k$, we denote by $k_v$ the fraction field of the completion (with respect to the $v$-adic topology) of the valuation ring of $v$. when we speak of the local-global principle for isotropy of quadratic forms, sometimes referred to as the strong hasse principle, in a given dimension $d$ over a given field $k$, we mean the following statement:

if $q$ is a quadratic form in $d$ variables over $k$ and $q$ is isotropic over $k_v$
for every discrete valuation $v$ on $k$, then $q$ is isotropic over $k$.

our main result is the following.

theorem 1. fix any $n \geq 2$. the local-global principle for isotropy of quadratic forms fails to hold in dimension $2^n$ over the rational function field $\mathbb{C}(x_1, \ldots, x_n)$.

previously, only the case of $n = 2$ was known, with the first known explicit examples appearing in [15], and later in [5] and [14]. for a construction, using algebraic geometry, over any transcendence degree 2 function field over an algebraically closed field of characteristic 0, see [3], [4, §6]. this later result motivates the following.

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Conjecture 2. Let $K$ be a finitely generated field of transcendence degree $n \geq 2$ over an algebraically closed field $k$ of characteristic $\neq 2$. Then the local-global principle for isotropy of quadratic forms fails to hold in dimension $2^n$ over $K$.

We recall that by Tsen–Lang theory [16, Theorem 6], such a function field is a $C_2$-field, hence has $u$-invariant $2^n$, and thus all quadratic forms of dimension $> 2^n$ are already isotropic. An approach to Conjecture 2, along the lines of the proof in the $n = 2$ case given in [4, Cor. 6.5], is outlined in Section 4.

Finally, we point out that in the $n = 1$ case, with $K = k(X)$ for a smooth projective curve $X$ over an algebraically closed field $k$, the local-global principle for isotropy of binary quadratic forms (equivalent to the “global square theorem”) holds when $X$ has genus 0 and fails for $X$ of positive genus.

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1. Hyperbolicity over a quadratic extension

Let $K$ be a field of characteristic $\neq 2$. We will need the following result about isotropy of quadratic forms, generalizing a well-known result in the dimension 4 case, see [19, Ch. 2, Lemma 14.2].

Proposition 1.1. Let $n > 0$ be divisible by four, $q$ a quadratic form of dimension $n$ and discriminant $d$ defined over $K$, and $L = K(\sqrt{d})$. If $q$ is hyperbolic over $L$ then $q$ is isotropic over $K$.

Proof. If $d \in K^\times$, then $K = K(\sqrt{d})$ and hence $q$ is hyperbolic over $K$. Suppose $d \notin K^\times$ and $q$ is anisotropic. Since $q_L$ is hyperbolic, $q \simeq <1, -d> \otimes q_1$ for some quadratic form $q_1$ over $K$, see [19, Ch. 2, Theorem 5.3]. Since the dimension of $q$ is divisible by four, the dimension of $q_1$ is divisible by two, and a computation of the discriminant shows that $d \in K^\times$, which is a contradiction. \hfill \Box

For $n \geq 1$ and $a_1, \ldots, a_n \in K^\times$, recall the $n$-fold Pfister form

$$<a_1, \ldots, a_n> = <1, -a_1> \otimes \cdots \otimes <1, -a_n>$$

and the associated symbol $(a_1) \cdots (a_n)$ in the Galois cohomology group $H^n(K, \mu_2^\otimes)$. Then $<a_1, \ldots, a_n>$ is hyperbolic if and only if $<a_1, \ldots, a_n>$ is isotropic if and only if $(a_1) \cdots (a_n)$ is trivial.

For $d \in K^\times$ and $n \geq 2$, we will consider quadratic forms of discriminant $d$ related to $n$-fold Pfister forms, as follows. Write $<a_1, \ldots, a_n>$ as $q_0 \perp <(-1)^n a_1 \ldots a_n>$, then define $<a_1, \ldots, a_n; d> = q_0 \perp <(-1)^n a_1 \ldots a_n d>$. For example:

$$<a, b; d> = <1, -a, -b, abd>$$

$$<a, b, c; d> = <1, -a, -b, -c, ab, ac, bc, -abcd>$$

for $n = 2$ and $n = 3$, respectively. We remark that every quadratic form of dimension 4 is similar to one of this type. If $q = <a_1, \ldots, a_n; d>$, we note that, in view of Proposition 1.1 and the fact that $q_L$ is a Pfister form over $L = K(\sqrt{d})$, we have that $q$ is isotropic if and only if $q_L$ is isotropic, generalizing a well-known result about quadratic forms of dimension 4, see [19, Ch. 2, Lemma 14.2].
2. Generalized Kummer varieties

In this section, we review a construction, considered in the context of modular Calabi–Yau varieties [11, §2] and [12], of a generalized Kummer variety attached to a product of elliptic curves. This recovers, in dimension 2, the Kummer K3 surface associated to a decomposable abelian surface, and in dimension 3, a class of Calabi–Yau threefolds of CM type considered by Borcea [6, §3].

Let $E_1, \ldots, E_n$ be elliptic curves over an algebraically closed field $k$ of characteristic $\neq 2$ and let $Y = E_1 \times \cdots \times E_n$. Let $\sigma_i$ denote the negation automorphism on $E_i$ and $E_i \to \mathbb{P}^1$ the associated quotient branched double cover. We lift each $\sigma_i$ to an automorphism of $Y$; the subgroup $G \subset \text{Aut}(Y)$ they generate is an elementary abelian 2-group. Consider the exact sequence of abelian groups

$$1 \to H \to G \xrightarrow{\Pi} \mathbb{Z}/2 \to 0,$$

where $\Pi$ is defined by sending each $\sigma_i$ to 1. Then the product of the double covers $Y \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is the quotient by $G$ and we denote by $Y \to X$ the quotient by the subgroup $H$. Then the intermediate quotient $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is a double cover, branched over a reducible divisor of type $(4, \ldots, 4)$.

We point out that $X$ is a singular degeneration of smooth Calabi–Yau varieties that also admits a smooth Calabi–Yau model, see [11, Cor. 2.3]. For $n = 2$, the minimal resolution of $X$ is isomorphic to the Kummer K3 surface $\text{Kum}(E_1 \times E_2)$.

Given nontrivial classes $\gamma_i \in H^1_\text{ét}(E_i, \mu_2)$, we consider the cup product

$$\gamma = \gamma_1 \cdots \gamma_n \in H^n_\text{ét}(Y, \mu_2^{\otimes n})$$

and its restriction to the generic point of $Y$, which is a class in the unramified part $H^1_\text{ét}(k(Y)/k, \mu_2^{\otimes n})$ of the Galois cohomology group $H^n(k(Y), \mu_2^{\otimes n})$ of the function field $k(Y)$ of $Y$ (see [9] for background on the unramified cohomology groups). These classes have been studied in [10]. We remark that $\gamma$ is in the image of the restriction map $H^n(k(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1), \mu_2^{\otimes n}) \to H^n(k(Y), \mu_2^{\otimes n})$ in Galois cohomology since each $\gamma_i$ is in the image of the restriction map $H^1(k(\mathbb{P}^1), \mu_2) \to H^1(k(E_i), \mu_2)$.

We make this more explicit as follows. For each double cover $E_i \to \mathbb{P}^1$, we choose a Weierstrass equation in Legendre form

$$y^2 = x_i(x_i - 1)(x_i - \lambda_i)$$

where $x_i$ is a coordinate on $\mathbb{P}^1$ and $\lambda_i \in k \setminus \{0, 1\}$. Then the branched double cover $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is birationally defined by the equation

$$y^2 = \prod_{i=1}^n x_i(x_i - 1)(x_i - \lambda_i) = f(x_1, \ldots, x_n)$$

where $y = y_1 \cdots y_n$ in $\mathbb{C}(Y)$. Up to an automorphism, we can even choose the Legendre forms so that the image of $\gamma_i$ under $H^1_\text{ét}(E_i, \mu_2) \to H^1(k(E), \mu_2)$ coincides with the square class $(x_i)$ of the function $x_i$, which clearly comes from $k(\mathbb{P}^1)$.

The main result of this section is that the class $\gamma$ considered above is already unramified over $k(X)$. We prove a more general result.

**Proposition 2.1.** Let $k$ be an algebraically closed field of characteristic $\neq 2$ and $K = k(x_1, \ldots, x_n)$ a rational function field over $k$. For $1 \leq i \leq n$, let $f_i(x_i) \in k[x_i]$ be polynomials of even degree satisfying $f_i(0) \neq 0$, and let $f = \prod_{i=1}^n x_i f(x_i)$. Then the restriction of the class $\xi = (x_1) \cdots (x_n) \in H^n(K, \mu_2^{\otimes n})$ to $H^n(K(\sqrt{f}), \mu_2^{\otimes n})$ is unramified with respect to all discrete valuations.
In particular, the transcendence degree of $k \leq \gamma$ and linear factors in changing the extension class $u$, $h$ image of $\alpha x$ of definition of $f$ maximal ideal $L$. Let $4 AUEL AND SURESH$ Proposition 2.2. Then the restriction of the class field $K$, field is a subfield of $x$. By definition, $f$ assume that $f$ is a product of linear factors in $k[x_i]$, we have that $x_i - a_i \in p$ for some $a_i \in k$, with $a_i \neq 0$ since $f_i(0) \neq 0$. Thus the image of $x_i$ in $k(p)$ is equal to $a_i$ and hence a square in $k(p)$. In particular, $x_i$ is a square in $L_v$, thus $\xi$ is trivial (hence unramified) at $v$.

Now, suppose that $f_i(x_i) \notin p$ for all $m + 1 \leq i \leq n$. Then for each $1 \leq i \leq m$, we see that since $x_i \in p$ and $f_i(0) \neq 0$, we have $f_i(x_i) \notin p$. Consequently, we can assume that $f = x_1 \cdots x_m u$ for some $u \in k[x_1, \ldots, x_n] \supset p$.

Computing with symbols, we have
\[(x_1) \cdots (x_m) = (x_1) \cdots (x_{m-1}) \cdot (x_1 \cdots x_m) \in H^m(K, \mu_2^m).
\]
By definition, $f = x_1 \cdots x_m u$ is a square in $L$, and thus we have that
\[(x_1) \cdots (x_m) = (x_1) \cdots (x_{m-1}) \cdot (u) \in H^m(L, \mu_2).
\]
Thus it is enough to show that $(x_1) \cdots (x_{m-1}) \cdot (u) \cdot (x_{m+1}) \cdots (x_n)$ is unramified at $v$.

Let $\partial_v : H^n(L, \mu_2^{n+1}) \to H^{n-1}(k(v), \mu_2^{n-1})$ be the residue homomorphism at $v$. Since $x_i$, for all $m + 1 \leq i \leq n$, and $u$ are units at $v$, we have
\[\partial_v((x_1) \cdots (x_{m-1}) \cdot (u) \cdot (x_{m+1}) \cdots (x_n)) = \alpha \cdot (\overline{u}) \cdot (\overline{x}_{m+1}) \cdots (\overline{x}_n)
\]
for some $\alpha \in H^{m-2}(k(v), \mu_2^{m-2})$, where for any $h \in k[x_1, \ldots, x_n]$, we write $\overline{h}$ for the image of $h$ in $k(p)$. Since the transcendence degree of $\kappa(p)$ over $k$ is at most $n - m$ and $k$ is algebraically closed, we have that $H^{n-m+1}(\kappa(p), \mu_2^{n-m+1}) = 0$. Since $\overline{u}, \overline{x}_i \in \kappa(p)$, we have $(\overline{u}) \cdot (\overline{x}_{m+1}) \cdots (\overline{x}_n) = 0$. In particular $\partial_v(\xi) = 0$. Finally, the class $\xi$ is unramified at all discrete valuations on $L$. \hfill \Box

As an immediate consequence, we have the following.

**Proposition 2.2.** Let $E_1, \ldots, E_n$ be elliptic curves over an algebraically closed field $k$ of characteristic $\neq 2$, given in the Legendre form (1), with $K = k[x_1, \ldots, x_n]$. Then the restriction of the class $\gamma = (x_1) \cdots (x_n)$ in $H^n(K, \mu_2^{\infty})$ to $H^n(k(X), \mu_2^{\infty})$ is unramified at all discrete valuations.

Finally, we will need the fact, proved in the appendix by Gabber to the article [10], that if $k = \mathbb{C}$ and the $j$-invariants $j(E_1), \ldots, j(E_n)$ are algebraically independent, then any cup product class $\gamma = \gamma_1 \cdots \gamma_n \in H^n(\mathbb{C}(Y), \mu_2^{\infty})$, with $\gamma_i \in H^1(\mathbb{C}(E_i), \mu_2)$ nontrivial as considered above, is itself nontrivial.
3. Failure of the local-global principle

Given elliptic curves $E_1,\ldots,E_n$ defined over $\mathbb{C}$ with algebraically independent $j$-invariants, presented in Legendre form (1), and $X \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ the double cover defined by $y^2 = f(x_1,\ldots,x_n)$ in (2), we consider the quadratic form

$$q = \langle x_1,\ldots,x_n; f \rangle$$

over $\mathbb{C}(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1) = \mathbb{C}(x_1,\ldots,x_n)$, as in Section 1.

Our main result is that $q$ shows the failure of the local-global principle for isotropy, with respect to all discrete valuations, for quadratic forms of dimension $2^n$ over $\mathbb{C}(x_1,\ldots,x_n)$, thereby proving Theorem 1.

**Theorem 3.1.** The quadratic form $q = \langle x_1,\ldots,x_n; f \rangle$ is anisotropic over $\mathbb{C}(x_1,\ldots,x_n)$ yet is isotropic over the completion at every discrete valuation.

**Proof.** Let $K = \mathbb{C}(x_1,\ldots,x_n)$ and $L = K(\sqrt{f}) = \mathbb{C}(X)$. Let $v$ be a discrete valuation of $K$ and $w$ an extension to $L$, with completions $K_v$ and $L_w$ and residue fields $\kappa(v)$ and $\kappa(w)$, respectively. We note that $\kappa(v)$ and $\kappa(w)$ have transcendence degree $0 \leq i \leq n-1$ over $\mathbb{C}$. By Proposition 1.1, we have that $q \otimes_K K_v$ is isotropic if and only if $q \otimes_K L_w$ is isotropic.

By Proposition 2.2, the restriction $(x_1) \cdots (x_n) \in H^n(L, \mu_2^{\otimes n})$ is unramified at $w$, hence $q \otimes_K L = \langle x_1,\ldots,x_n \rangle$ is an $n$-fold Pfister form over $L$ unramified at $w$. Thus the first residue form for $q \otimes_K L$, with respect to the valuation $w$, is isotropic since the residue field $\kappa(w)$ is a $C_1$-field and $q$ has dimension $2^n > 2^i$. Consequently, by a theorem of Springer [19, Ch. 6, Cor. 2.6], $q \otimes_K L$, and thus $q$, is isotropic.

Finally, $q$ is anisotropic since the symbol $(x_1) \cdots (x_n)$ is nontrivial when restricted to $\mathbb{C}(Y)$ by [10, Appendice], hence is nontrivial when restricted to $\mathbb{C}(X)$. \[\square\]

To give an explicit example, let $\lambda,\kappa,\nu \in \mathbb{C} \setminus \{0,1\}$ be algebraically independent complex numbers. Then over the function field $K = \mathbb{C}(x,y,z)$, the quadratic form

$$q = \langle 1, x, y, z, xy, xz, yz, (x-1)(y-1)(z-1)(x-\lambda)(y-\kappa)(z-\nu) \rangle$$

is isotropic over every completion $K_v$ associated to a discrete valuation $v$ of $K$, and yet $q$ is anisotropic over $K$.

4. Over general function fields

We have exhibited locally isotropic but globally anisotropic quadratic forms of dimension $2^n$ over the rational function field $\mathbb{C}(x_1,\ldots,x_n)$. In [4, Cor. 6.5], we proved that locally isotropic but anisotropic quadratic forms of dimension 4 exist over any function field of transcendence degree 2 over an algebraically closed field of characteristic zero. Taking these as motivation, we recall Conjecture 2, that over any function field of transcendence degree $n \geq 2$ over an algebraically closed field of characteristic $\neq 2$, there exist locally isotropic but anisotropic quadratic forms of dimension $2^n$. In this section, we provide a possible approach to Conjecture 2, motivated by the geometric realization result in [4, Proposition 6.4].

**Proposition 4.1.** Let $K = k(X)$ be the function field of a smooth projective variety $X$ of dimension $n \geq 2$ over an algebraically closed field $k$ of characteristic $\neq 2$. If either $H^n_{\text{mot}}(K/k, \mu_2^{\otimes n}) \neq 0$ or $H^n_{\text{mot}}(L/k, \mu_2^{\otimes n}) \neq 0$ for some separable quadratic extension $L/K$, then there exists an anisotropic quadratic form of dimension $2^n$ over $K$ that is isotropic over the completion at every discrete valuation.
Proof. First, by a standard application of the Milnor conjectures, every element in $H^n(K, \mu_2^{\otimes n})$ is a symbol since $K$ is a $C_n$-field. If $H^n_{nr}(K/k, \mu_2^{\otimes n}) \neq 0$, then taking a nontrivial element $(a_1) \cdots (a_n)$, the $n$-fold Pfister form $\langle a_1, \ldots, a_n \rangle$ is locally isotropic (by the same argument as in the proof of Theorem 3.1) but is anisotropic, giving an example. So we can assume that $H^n_{nr}(K/k, \mu_2^{\otimes n}) = 0$.

Now assume that $H^n_{nr}(L/k, \mu_2^{\otimes n}) \neq 0$ for some separable quadratic extension $L = K(\sqrt{d})$ of $K$. Then taking a nontrivial element $(a_1) \cdots (a_n)$, the corestriction map $H^n_{nr}(L/k, \mu_2^{\otimes n}) \to H^n_{nr}(K/k, \mu_2^{\otimes n}) = 0$ is trivial, so by the restriction-corestriction sequence for Galois cohomology, we have that $(a_1) \cdots (a_n)$ is in the image of the restriction map $H^n_{nr}(K/k, \mu_2^{\otimes n}) \to H^n_{nr}(L/k, \mu_2^{\otimes n}) = 0$, in which case we can assume that $a_1, \ldots, a_n \in K^\times$. Then the quadratic form $\langle a_1, \ldots, a_n; d \rangle$ is locally isotropic over $K$ (by the same argument as in the proof of Theorem 3.1) but globally anisotropic.

Hence we are naturally led to the following geometric realization conjecture for unramified cohomology classes.

**Conjecture 4.2.** Let $K$ be a finitely generated field of transcendence degree $n$ over an algebraically closed field $k$ of characteristic $\neq 2$. Then either $H^n_{nr}(K/k, \mu_2^{\otimes n}) \neq 0$ or there exists a quadratic extension $L/K$ such that $H^n_{nr}(L/k, \mu_2^{\otimes n}) \neq 0$.

Proposition 4.1 says that the geometric realization Conjecture 4.2 implies Conjecture 2 on the failure of the local-global principle for isotropy of quadratic forms. Proposition 2.2 establishes the conjecture in the case when $K$ is purely transcendental over $k$; in [4, Proposition 6.4], we established the conjecture in dimension 2 and characteristic 0, specifically, that given any smooth projective surface $S$ over an algebraically closed field of characteristic zero, there exists a double cover $T \to S$ with $T$ smooth and $H^n_{nr}(k(T)/k, \mu_2^{\otimes 2}) = \text{Br}(T)[2] \neq 0$. In this latter case, Proposition 4.1 gives a different proof, than the one presented in [4, §6], that there exist locally isotropic but anisotropic quadratic forms of dimension 4 over $K = k(S)$.

**References**


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