AZUMAYA ALGEBRAS WITHOUT INVOLUTION

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Abstract. Generalizing a theorem of Albert, Saltman showed that an Azumaya algebra A over a ring represents a 2-torsion class in the Brauer group if and only if there is an algebra A' in the Brauer class of A admitting an involution of the first kind. Knus, Parimala, and Srinivas later showed that one can choose A' such that $\deg A' = 2 \deg A$. We show that $2 \deg A$ is the lowest degree one can expect in general. Specifically, we construct an Azumaya algebra A of degree 4 and period 2 such that the degree of any algebra A' in the Brauer class of A admitting an involution is divisible by 8.

Separately, we provide examples of split and non-split Azumaya algebras of degree 2 admitting symplectic involutions, but no orthogonal involutions. These stand in contrast to the case of central simple algebras of even degree over fields, where the presence of a symplectic involution implies the existence of an orthogonal involution and vice versa.

Introduction

Let A be a central simple algebra over a field F. It is a classical result of Albert [2, X §9 Thm. 19] (cf. [29, Thm. 3.1(1)]) that A has an involution of the first kind if and only if the Brauer class $[A]$ has period at most 2, i.e., lies in the 2-torsion part of the Brauer group $\mathrm{Br}(F)$. This characterization was later extended and clarified by Scharlau [41] and in unpublished work by Tamagawa.

Albert's theorem does not generally extend to Azumaya algebras A over a commutative ring R. However, Saltman [39, Thm. 3.1(a)] showed that a class $[A] \in \mathrm{Br}(R)$ has period dividing 2 if and only if there is an Azumaya algebra A' in the class $[A]$ admitting an involution of the first kind. Furthermore, one can take $A' = A$ when R is semilocal or when $\deg A = 2$, see [39, Thms. 4.1, 4.4]. A later proof given by Knus, Parimala, and Srinivas [27, Thm. 4.1], which applies in the generality of Azumaya algebras over schemes, shows that in this context A' can be choosen so that $\deg A' = 2 \deg A$.

Suppose $X = \operatorname{Spec} R$ is connected and $[A]$ is a 2-torsion class in $\mathrm{Br}(X)$. It is natural to ask, in the context of the results of [39] and [27], whether in general $2 \deg A$ is the least possible degree of a representative A' in the class $[A]$ admitting an involution of the first kind. We answer this question in the affirmative.

Theorem A. *There exists a commutative domain R and an Azumaya R-algebra A satisfying*

$$\deg A = 4, \qquad \operatorname{per} A = 2,$$

and such that any representative A' in the Brauer class $[A]$ admitting an involution of the first kind satisfies

$$8 \mid \deg A'.$$

This construction is done using approximations of the universal bundle over the classifying stack of SL_4 / μ_2, similarly to the methods of [4].

Involutions of the first kind of central simple algebras may be further classified as being orthogonal or symplectic, see [29, §2]. This classification extends to Azumaya algebras over connected schemes [28, §III.8].

It is well known that a central simple algebra of even degree over a field admits either involutions of both types or of neither type. We show that this is not the case for Azumaya algebras. We construct both split and non-split Azumaya algebras of degree 2, each admitting a symplectic involution but no orthogonal involutions. The split examples are constructed using quadratic spaces and Clifford

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algebras, whereas the non-split example relies on generic methods to construct an Azumaya algebra without zero divisors that specializes to the given split example. In fact, the latter example can made to specialize to any prescribed Azumaya algebra with involution over an affine scheme.

We remark that simply constructing Azumaya R-algebras A without involution and with period 2 is a nontrivial problem. For instance, it is impossible when R is a semilocal ring (as mentioned above). In addition, for many low-dimensional rings, e.g., integer rings of global fields, the bound $\deg A' \leq 2\deg A$ of [27] is not sharp; see [17, Rem. 11.5].

In [39, §4], Saltman provided a criterion for split Azumaya algebras to have an involution, which was used to provide explicit split examples. These methods were extended in [17, §11] to give non-split examples. Alternatively, given a split example, a sufficiently generic algebra of period 2, which specializes to the split example, cannot have any involution. The major difficultly then lies, in trying to fully verify Theorem A, with showing that no Brauer equivalent algebra of the same degree can have an involution. This is where the topological obstruction theory is useful.

Given the above results, we conjecture that:

(a) For any $n \geq 2$, there is an Azumaya algebra A of degree 2^n and period 2 such any $A' \in [A]$ with an involution of the first kind satisfies $2^{n+1} \mid \deg A'$.
(b) For any $n \geq 1$ and $n \geq m \geq 0$, there is an Azumaya algebra A with $\deg A = 2^n$ and $\operatorname{ind} A = 2^m$ which admits a symplectic involution, but not an orthogonal involution.
(c) For any $n \geq 2$ and $n \geq m \geq 0$, there is an Azumaya algebra A with $\deg A = 2^n$ and $\operatorname{ind} A = 2^m$ which admits an orthogonal involution, but not a symplectic involution.

We have settled (a) when $n = 2$ and (b) when $n = 1$ and $m \in \{0, 1\}$.

We would also like to mention the case of involutions of the second kind, which we do not take up in this article. Let K/F be a separable quadratic extension of fields. The Albert–Riehm–Scharlau Theorem [29, Thm. 3.1(2)] asserts that a central simple K-algebra A has an involution of the second kind with fixed field F if and only if the corestriction $\operatorname{cor}_{K/F}[A]$ is trivial in $\operatorname{Br}(F)$. Saltman [39, Thm. 3.1(b)] showed that a class $[A] \in \operatorname{Br}(S)$, where S/R is a quadratic Galois extension of commutative rings, has trivial corestriction if and only if there is an Azumaya algebra A' in the class $[A]$ admitting an involution of the second kind fixing R pointwise. A different proof given by Knus, Parimala, and Srinivas [27, Thm. 4.2], shows that A' can be choosen so that $\deg A' = 2\deg A$. It is then natural to ask, whether in general $2\deg A$ is the least possible degree of a representative A' in the class $[A]$ admitting an involution of the second kind.

Question B. *Do there exist a commutative domain R, a quadratic Galois extension S/R, and an Azumaya S-algebra A of degree n with $\operatorname{cor}_{S/R}[A]$ trivial, such that any representative A' in the Brauer class $[A]$ admitting an involution of the second kind with fixed ring R has $2n \mid \deg A'$?*

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The paper is organized as follows: In the first section, we provide some of the necessary background on Azumaya algebras and their involutions. The second section is topological in content. After a brief primer on the required theory of classifying spaces and cohomology, it is dedicated to giving certain obstructions to the existence of maps between classifying spaces. These obstructions are described in Proposition 10, which is used in Section 3 to construct an example of an Azumaya algebra of period 2 and degree 4 for which any Brauer-equivalent algebra with involution has degree no less than 8. Section 4 exhibits a split Azumaya algebra equipped with a symplectic but not an orthogonal involution. The fifth and final section is concerned with two constructions of non-split Azumaya algebras that specialize to the split example of the previous section, thus consequently giving examples of non-split Azumaya algebras admitting symplectic but not orthogonal involutions.

1. Preliminaries

We assume throughout that 2 is invertible in all rings and schemes.

We begin by recalling some facts about Azumaya algebras and their involutions. For simplicity, we have restricted here to connected rings, schemes, etc., but the extension to disconnected settings requires only some additional bookkeeping.

1.1. **Azumaya algebras.** Let R be a connected commutative ring. Recall that an R-algebra A is called *Azumaya* if A is locally free of finite nonzero rank and the map $\Phi : A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ given by $\Phi(a \otimes b^{\mathrm{op}})(x) = axb$ is an isomorphism. Equivalently, A is Azumaya if there exists a faithfully flat étale R-algebra S such that $A \otimes_R S \cong \mathrm{Mat}_{n\times n}(S)$ as S-algebras, [28, Thm. III.5.1.1]. The number n is called the *degree* of n and is denoted $\deg A$. When R is a field, Azumaya algebras are precisely the central simple algebras.

Two Azumaya algebras A, B over R are *Brauer equivalent* if there are locally free R-modules P and Q of finite rank such that $A \otimes_R \mathrm{End}_R(P) \cong B \otimes_R \mathrm{End}_R(Q)$ as R-algebras. The equivalence class of A is called the *Brauer class* of A and denoted $[A]$. The *index* of A, denoted $\mathrm{ind}\, A$, is $\gcd\{\deg A' \mid A' \in [A]\}$. Unlike in the case where R is a field, it is possible that there might be no representative A' in the class $[A]$ with $\deg A' = \mathrm{ind}\, A$; see [4]. The collection of Brauer classes with the operation $[A] \otimes [B] := [A \otimes_R B]$ forms a group, denoted $\mathrm{Br}(R)$, and called the *Brauer group* of R. See [14], [28, §III.5] for further details.

Let X be a connected scheme. An Azumaya algebra of degree n over X is a sheaf of $\mathcal{O}_X$-modules that is locally isomorphic in the étale topology to $\mathrm{Mat}_{n\times n}(\mathcal{O}_X)$. That is, X has an étale cover (possibly disconnected) $\pi : U \to X$ such that $\pi^* A \cong \mathrm{Mat}_{n\times n}(\mathcal{O}_U)$. If $X = \mathrm{Spec}\, R$ for a commutative ring R, then A is Azumaya over X if and only if $\Gamma(X, A)$ is Azumaya as an R-algebra. The degree of A, the Brauer class $[A]$, the Brauer group of X, etc. are defined as above—replace P and Q by locally free $\mathcal{O}_X$-modules of finite rank and $\mathrm{End}(-)$ by the sheaf $\mathcal{E}nd(-)$. See [32, Chp. IV] for further details.

In the same way, one can define Azumaya algebras, Brauer classes, etc., in any locally ringed topos, [21]. In the case where X is a topological space and $\mathcal{O}_X$ is the sheaf of continuous functions into $\mathbb{C}$, an Azumaya algebra over X is called a *topological Azumaya algebra*. A topological Azumaya algebra A over X is therefore a sheaf of $\mathbb{C}$-algebras over X the fibers of which are isomorphic to $\mathrm{Mat}_{n\times n}(\mathbb{C})$; see [3].

1.2. **Involutions.** Let F be a field and let A be a central simple F-algebra. Recall that an involution on A is an additive map $\sigma : A \to A$ such that $\sigma \circ \sigma = \mathrm{id}_A$ and $\sigma(ab) = \sigma(b) \cdot \sigma(a)$ for all $a, b \in A$. The involution σ is of the *first kind* if it fixes F pointwise and of the *second kind* otherwise. Henceforth, all involutions are assumed to be of the first kind. An Azumaya algebra admitting an involution will be termed *involutary*.

Write $n = \deg A$. Recall from [29, §2] that the F-dimension of $\{\sigma(a) - a \mid a \in A\}$ is either $\frac{1}{2}n(n-1)$ or $\frac{1}{2}n(n+1)$, in which case the type of σ is said to be *orthogonal* or *symplectic* respectively. Symplectic involutions can exist only when n is even.

Let X be a connected scheme (resp. topological space) and let A be an Azuamya algebra (resp. topological Azumaya algebra) over X. An involution on A is an $\mathcal{O}_X$-module isomorphism $\sigma : A \to A$ such that $\sigma \circ \sigma = \mathrm{id}_A$ and the identity $\sigma_*(ab) = \sigma_* b \cdot \sigma_* a$ holds on sections. In this case, we say that the pair (A, σ) is an *Azumaya algebra with involution* (of the first kind). If X is $\mathrm{Spec}\, R$ for a commutative ring R, a map σ is an involution if and only if $\sigma_* : \Gamma(X, A) \to \Gamma(X, A)$ is an involution of Azumaya R-algebras.

Let x be a point of X. In the algebraic case, let $k = k(x)$, and in the topological, let $k = \mathbb{C}$. By pulling back, A induces a central simple k-algebra A_x and σ induces an involution of the first kind $\sigma_x : A_x \to A_x$, which can be either orthogonal or symplectic. Since X is connected, the type of σ_x is independent of x, so we can simply say that σ is orthogonal or symplectic. See [28, §III.8] for an extensive discussion in the case of affine schemes and [34, §1.1] in the case of arbitrary schemes.

Example 1. Denote by t the standard matrix transpose.

(1) The transpose t defines an orthogonal involution of $\mathrm{Mat}_{n\times n}(\mathcal{O}_X)$.

(2) Suppose n is even and let sp denote the standard symplectic involution on $\mathrm{Mat}_{n\times n}(\mathcal{O}_X)$, defined by
$$\mathrm{sp}_*\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right] = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]^{\mathrm{sp}} := \left[\begin{smallmatrix} A^{\mathrm{t}} & -C^{\mathrm{t}} \\ -B^{\mathrm{t}} & D^{\mathrm{t}}\end{smallmatrix}\right]$$
on sections—here A, B, C, D are $\frac{n}{2} \times \frac{n}{2}$ matrices.

(3) Let $b : \mathcal{P} \times \mathcal{P} \to \mathcal{L}$ be nondegenerate symmetric (resp. anti-symmetric) bilinear form, where $\mathcal{P}$ is a locally free $\mathcal{O}_X$-module of finite rank and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module on X. Then b induces an orthogonal (resp. symplectic) involution σ on $\mathcal{E}nd(\mathcal{P})$; on sections, σ_* can be recovered from the equality $b_*(ax, y) = b_*(x, \sigma_*(a)y)$. Moreover any orthogonal (resp. symplectic) involution on $\mathcal{E}nd(\mathcal{P})$ is obtained in this manner, as $\mathcal{L}$ varies, by virtue of [39, Thm. 4.2]. For a generalization see [16, Thm. 5.7].

Example 2. Azumaya algebras of degree 2 may always be equipped with a symplectic involution. To see this, let A be an Azumaya algebra of degree 2 over X. If U is an open affine subscheme of X, define $\sigma_U : \Gamma(U, A) \to \Gamma(U, A)$ by $\sigma_U(a) = \mathrm{Trd}(a) - a$, where $\mathrm{Trd}(a)$ denotes the reduced trace of $a \in \Gamma(U, A)$; see [39, Thm. 4.1].

Example 3. To construct split Azumaya algebras with no involution of the first kind, we simply need to find a locally free $\mathcal{O}_X$-module $\mathcal{P}$ such that $\mathcal{P} \ncong \mathcal{P}^\vee \otimes \mathcal{L}$ for any invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X. Examples over projective schemes are easy to construct, e.g., $X = \mathbb{P}^1$ and $\mathcal{P} = \mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{O}_X(1)$. Similar examples were constructed over appropriate Dedekind domains in [17, §11].

Proposition 4. *Let X be a connected scheme, let A be an Azumaya algebra of degree n over X, and let $\sigma : A \to A$ be an involution of orthogonal or symplectic type. Then (A, σ) is locally isomorphic to $(\mathrm{Mat}_{n\times n}(\mathcal{O}_X), \mathrm{t})$ or $(\mathrm{Mat}_{n\times n}(\mathcal{O}_X), \mathrm{sp})$ in the étale topology, respectively.*

Proof. We can assume X is affine, and in this case we refer to [28, §III.8.5]. See also [34, §1.1]. □

1.3. **Cohomology.** Let X be a scheme and let $X_{ét}$ denote X endowed with the étale topology. Recall that PGL_n is the sheaf of algebra automorphisms of $\mathrm{Mat}_{n\times n}$. There is a standard bijective correspondance between isomorphism classes of Azumaya algebras of degree n and of PGL_n-torsors over $X_{ét}$, and each is classified by $\mathrm{H}^1_{ét}(X, \mathrm{PGL}_n)$; see [28, p. 145] and [32, Chp. III, Cor. 4.7]. The general principle of this correspondence appears as [19, Cor. III.2.2.6].

Suppose now that 2 is invertible on X and let PGO_{2n} and PGSp_{2n} denote the sheaf of automorphisms of $(\mathrm{Mat}_{2n\times 2n}, \mathrm{t})$ and $(\mathrm{Mat}_{2n\times 2n}, \mathrm{sp})$, respectively. In the same manner as the above, there is a bijective correspondence between isomorphism classes of PGO_{2n}-torsors (resp. PGSp_{2n}-torsors) over $X_{ét}$ and isomorphism classes of Azumaya algebras with an orthogonal (resp. symplectic) involution of degree $2n$ over X, see Proposition 4, [28, §III.8.5], or [34, §1.1(iii)].

Similarly, when X is a topological space, isomorphism classes of topological Azumaya algebras with an orthogonal (resp. symplectic) involution of degree $2n$ over X correspond to isomorphism classes of $\mathrm{PGO}_{2n}(\mathbb{C})$-bundles (resp. $\mathrm{PGSp}_{2n}(\mathbb{C})$-bundles) over X. The Skolem–Noether Theorem implies that $\mathrm{PGO}_{2n}(\mathbb{C}) \cong \mathrm{PO}_{2n}(\mathbb{C}) = \mathrm{O}_{2n}(\mathbb{C})/\mu_2$ and $\mathrm{PGSp}_{2n}(\mathbb{C}) \cong \mathrm{PSp}_{2n}(\mathbb{C}) = \mathrm{Sp}_{2n}(\mathbb{C})/\mu_2$ where $\mu_2 = \{\pm 1\}$.

2. Homotopy Theory

In this section, we use algebraic topology to find obstructions to maps between classifying spaces of Lie groups by considering the rational cohomology of maximal tori. This idea appears at least as long ago as [1]. Our application requires that we consider spaces that have the same homotopy and homology groups as classifying spaces in a certain range, but which differ in general; this contrasts with the results of [25] where the classifying space is considered in its entirety.

In this section and the next, the notation $\mathrm{H}^*(X, A)$ will be used for sheaf cohomology groups with values in the sheaf represented by the topological group A. When A is discrete, and X has the homotopy type of a CW complex, this coincides with singular cohomology with coefficients in A.

We make extensive use of classifying spaces of topological groups. A thorough discussion of this material may be found in [30, §8 and 9], but the discussion in [31, Chp. 6] covers most of what we need. That is, if G is a topological group (satisfying a mild topological hypothesis which is satisfied by all Lie groups) there is a functorially defined pointed CW complex $\mathrm{B}G$ and a principal G-bundle

$\mathrm{E}G \to \mathrm{B}G$. If X is any pointed CW complex, then there is a natural bijection of pointed sets between $[X, \mathrm{B}G]$ and $\mathrm{H}^1(X, G)$, where the former denotes the set of homotopy classes of pointed maps between the spaces in question. The correspondence is given by assigning to a map $\xi : X \to \mathrm{B}G$ the pull-back of $\mathrm{E}G$ along ξ. It is a theorem that the isomorphism class of the resulting G-bundle depends on ξ only up to homotopy. One may replace $\mathrm{B}G$ by a homotopy equivalent CW complex without harming this bijection. If G is an abelian group, then $\mathrm{B}G$ is again a topological group, and we may form B^2G, B^3G and so on. If G is discrete, then B^iG is homotopy equivalent to the Eilenberg–MacLane space $K(G, i)$, and $[X, K(A, i)]$ is isomorphic to the reduced cohomology group $\tilde{\mathrm{H}}^i(X, A)$.

2.1. **The 2-torsion of the topological Brauer group.** The relationship between the topological Brauer group, $\mathrm{Br}_{\mathrm{top}}(X)$, and the cohomology groups $\mathrm{H}^2(X, \mathbb{C}^*)_{\mathrm{tors}}$ and $\mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$ shall become important in Section 3, so we discuss it here.

First of all, there is an isomorphism $\mathrm{H}^3(-, \mathbb{Z}) \cong \mathrm{H}^2(-, \mathbb{C}^\times)$. This arises from the exponential short exact sequence of topological groups $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 1$, where $\mathbb{C}$ is a contractible topological group, so $\mathrm{H}^{\geq 1}(-, \mathbb{C})$ vanishes. Often when $\mathrm{H}^*(-, \mathbb{C})$ is written elsewhere, $\mathbb{C}$ is to be understood as a discrete ring, so that $\mathrm{H}^*(-, \mathbb{C}) \cong \mathrm{H}^*(-, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$; this is not the case here.

Secondly, in general $\mathrm{Br}_{\mathrm{top}}(-)$ is a subfunctor of $\mathrm{H}^3(-, \mathbb{Z})$, see [21, Prp. 1.4]. It is the subfunctor consisting of those classes that lie in the image of the coboundary maps $\mathrm{H}^1(-, \mathrm{PGL}_n(\mathbb{C})) \to \mathrm{H}^2(-, \mathbb{C}^*)$ arising from the exact sequence $\mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$. Whenever X has the homotopy type of a finite CW complex, $\mathrm{Br}_{\mathrm{top}}(X)$ may be identified with the torsion subgroup of $\mathrm{H}^3(X, \mathbb{Z})$, this being a theorem of Serre [21, Thm. 1.6]. While there are spaces for which $\mathrm{Br}_{\mathrm{top}}(X) \subsetneq \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$, for example $\mathrm{B}^2\mu_2$ (see [3, Cor. 5.10]), in the sequel we shall deal only with classes $\alpha \in \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$ that arise as the pull-back $f^*(\alpha_0)$ of a canonical class in $\mathrm{H}^2(\mathrm{B}\,\mathrm{PGL}_n, \mathbb{C}^\times)_{\mathrm{tors}}$ corresponding to the universal PGL_n–bundle on $\mathrm{B}\,\mathrm{PGL}_n$. Such a class α necessarily lies in $\mathrm{Br}_{\mathrm{top}}(X)$.

In the sequel, we generally write μ_2 for the group of square roots of 1. We will use $\mathrm{H}^*(-, \mu_2)$ and $\mathrm{H}^*(-, \mathbb{Z}/2)$ interchangeably.

Consider the group μ_2 embedded as scalar matrices $\{\pm I_{2n}\}$ in $\mathrm{GL}_{2n}(\mathbb{C})$. The inclusion $\mu_2 \to \mathrm{GL}_{2n}(\mathbb{C})$ factors through either of the two standard inclusions $\mathrm{O}_{2n}(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$, and it follows that there are natural maps $\mathrm{O}_{2n}(\mathbb{C})/\mu_2 \to \mathrm{GL}_{2n}(\mathbb{C})/\mu_2 \to \mathrm{PGL}_{2n}(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})/\mu_2 \to \mathrm{GL}_{2n}(\mathbb{C})/\mu_2 \to \mathrm{PGL}_{2n}(\mathbb{C})$.

We set about explaining how a principal $\mathrm{O}_{2n}(\mathbb{C})/\mu_2$- or $\mathrm{Sp}_{2n}(\mathbb{C})/\mu_2$-bundle gives rise to a 2-torsion Brauer class, cf. [34, p. 214]. Suppose $G \subset \mathrm{GL}_{2n}(\mathbb{C})$ is a subgroup such that $\mu_2 \subset G$. There is an associated map of short exact sequences

(1)
$$\begin{tikzcd} 1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G/\mu_2 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathrm{GL}_{2n} \arrow[r] & \mathrm{PGL}_{2n} \arrow[r] & 1 \end{tikzcd}$$

and therefore, for any CW complex X, a natural long exact sequences of cohomology groups yielding a commutative diagram:

(2)
$$\begin{tikzcd} \mathrm{H}^1(X, G/\mu_2) \arrow[rr] \arrow[d] & & \mathrm{H}^2(X, \mathbb{Z}/2) \arrow[d] & \\ \mathrm{H}^1(X, \mathrm{PGL}_{2n}(\mathbb{C})) \arrow[r] & \mathrm{Br}_{\mathrm{top}}(X) \arrow[r, hook] & \mathrm{H}^2(X, \mathbb{C}^*) \arrow[r, no head, "\cong"] & \mathrm{H}^3(X, \mathbb{Z}). \end{tikzcd}$$

That is to say, associated to an element in $\mathrm{H}^1(X, G/\mu_2)$, there is an element of $\mathrm{H}^3(X, \mathbb{Z})$ which lies in $\mathrm{Br}_{\mathrm{top}}(X)$ and also in the image of a map from $\mathrm{H}^2(X, \mathbb{Z}/2)$. It is therefore 2-torsion.

There is a diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, "\times i\pi"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "x \mapsto (-1)^x"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r, "\exp"'] & \mathbb{C}^* \arrow[r] & 1 \end{tikzcd}$$

from which it follows that the composite map $\mathrm{H}^2(-,\mathbb{Z}/2) \to \mathrm{H}^2(-,\mathbb{C}^*) \to \mathrm{H}^3(-,\mathbb{Z})$ appearing in Diagram 1 is simply the unreduced Bockstein homomorphism associated to $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2$, and denoted

$$\beta_2 : \mathrm{H}^2(-,\mathbb{Z}/2) \to \mathrm{H}^3(-,\mathbb{Z}) \ .$$

Remark 5. For technical reasons, we find it easier to work with the simply connected group $\mathrm{SL}_{2n}(\mathbb{C})$ rather than $\mathrm{GL}_{2n}(\mathbb{C})$. For comparison, therefore, we should work with $\mathrm{SO}_{2n}(\mathbb{C})$ rather than $\mathrm{O}_{2n}(\mathbb{C})$.

The inclusion of scalar matrices $\mu_2 \to \mathrm{O}_{2n}$ factors as $\mu_2 \to \mathrm{SO}_{2n} \to \mathrm{O}_{2n}$ and there is a diagram

$$\begin{tikzcd}
\mu_2 \arrow[r, equal] \arrow[d] & \mu_2 \arrow[r] \arrow[d] & 1 \arrow[d] \\
\mathrm{SO}_{2n}(\mathbb{C}) \arrow[r] \arrow[d] & \mathrm{O}_{2n}(\mathbb{C}) \arrow[r, "\det"] \arrow[d] & \mu_2 \arrow[d] \\
\mathrm{SO}_{2n}(\mathbb{C})/\mu_2 \arrow[r] & \mathrm{O}_{2n}(\mathbb{C})/\mu_2 \arrow[r, "\det"] & \mu_2
\end{tikzcd}$$

in which every row and column is exact. In particular, the diagram

$$\begin{tikzcd}
\mathrm{H}^1(Y,\mathrm{SO}_{2n}(\mathbb{C})/\mu_2) \arrow[r] \arrow[dr] & \mathrm{H}^1(Y,\mathrm{O}_{2n}(\mathbb{C})/\mu_2) \arrow[r] \arrow[d] & \mathrm{H}^1(Y,\mathbb{Z}/2) \\
 & \mathrm{H}^2(Y,\mathbb{Z}/2) &
\end{tikzcd}$$

commutes and the top row is an exact sequence of pointed sets. If $\mathrm{H}^1(Y,\mathbb{Z}/2)=0$, then the problem of finding a lift of an element of $\mathrm{H}^2(Y,\mathbb{Z}/2)$ to a class in $\mathrm{H}^1(Y,\mathrm{O}_{2n}(\mathbb{C})/\mu_2)$ is equivalent to the problem of lifting to $\mathrm{H}^1(Y,\mathrm{SO}_{2n}(\mathbb{C})/\mu_2)$.

Since $\mathrm{Sp}_{2n}(\mathbb{C}) \subset \mathrm{SL}_{2n}(\mathbb{C})$, no modification of the group is required in the symplectic case.

Restricting our attention to the special-linear setting, we have groups: $\mathrm{SL}_{2n}(\mathbb{C})/\mu_2$, $\mathrm{Sp}_{2n}(\mathbb{C})/\mu_2$ and $\mathrm{SO}_{2n}(\mathbb{C})/\mu_2$. There are induced diagrams

$$\begin{tikzcd}
\mathrm{BSp}_{2n}(\mathbb{C})/\mu_2 \arrow[rr] \arrow[dr, "\xi_s"'] & & \mathrm{B\,SL}_{2n}(\mathbb{C})/\mu_2 \arrow[dl, "\xi"] \\
 & \mathrm{B}^2\mu_2 &
\end{tikzcd} , \quad
\begin{tikzcd}
\mathrm{BSO}_{2n}(\mathbb{C})/\mu_2 \arrow[rr] \arrow[dr, "\xi_o"'] & & \mathrm{B\,SL}_{2n}(\mathbb{C})/\mu_2 \arrow[dl, "\xi"] \\
 & \mathrm{B}^2\mu_2. &
\end{tikzcd} \tag{3}$$

These correspond to diagrams in cohomology for a CW complex X:

$$\begin{tikzcd}
\mathrm{H}^1(X,\mathrm{Sp}_{2n}(\mathbb{C})/\mu_2) \arrow[r] \arrow[d] & \mathrm{H}^2(X,\mu_2) \arrow[d, equal] \\
\mathrm{H}^1(X,\mathrm{SL}_{2n}/\mu_2) \arrow[r] & \mathrm{H}^2(X,\mu_2)
\end{tikzcd} , \quad
\begin{tikzcd}
\mathrm{H}^1(X,\mathrm{SO}_{2n}(\mathbb{C})/\mu_2) \arrow[r] \arrow[d] & \mathrm{H}^2(X,\mu_2) \arrow[d, equal] \\
\mathrm{H}^1(X,\mathrm{SL}_{2n}/\mu_2) \arrow[r] & \mathrm{H}^2(X,\mu_2)
\end{tikzcd} \tag{4}$$

We now summarize the topological argument. In the next section, we will construct a pointed CW complex Y such that $\mathrm{H}^1(Y,\mathbb{Z}/2)=0$ and $\mathrm{H}^2(Y,\mathbb{Z})=0$, and a 2-torsion class $\alpha \in \mathrm{Br}_{\mathrm{top}}(Y) \subset \mathrm{H}^3(Y,\mathbb{Z})$.

The class α admits a lift to a class

$$\zeta \in \mathrm{H}^2(Y,\mathbb{Z}/2)$$

such that the unreduced Bockstein $\beta_2(\zeta)$ is α. The condition $\mathrm{H}^2(Y,\mathbb{Z})=0$ ensures the class ζ is unique. It will be the case that ζ is in the image of a map $\mathrm{H}^1(Y,\mathrm{SL}_4/\mu_2) \to \mathrm{H}^2(Y,\mathbb{Z}/2)$, so that in particular, α is in the image of a map $\mathrm{H}^1(Y,\mathrm{PGL}_4(\mathbb{C})) \to \mathrm{H}^2(Y,\mathbb{C}^*) = \mathrm{H}^3(Y,\mathbb{Z})$; cf. Diagram (2). That is, the index of α divides 4.

The class ζ is represented by a map $\zeta : Y \to \mathrm{B}^2\mu_2$. We will show that there is no factorization in homotopy of ζ as a map $Y \to \mathrm{B}G \to \mathrm{B}^2\mu_2$, where G is either $\mathrm{SO}_{2n}(\mathbb{C})/\mu_2$ or $\mathrm{Sp}_{2n}(\mathbb{C})/\mu_2$ for $n \not\equiv 0 \pmod 4$, thus showing that $\zeta \in \mathrm{H}^2(Y,\mathbb{Z}/2)$ is not in the image of the natural map $\mathrm{H}^1(Y,G) \to \mathrm{H}^2(Y,\mathbb{Z}/2)$. It follows that α is not in the image of the natural map $\mathrm{H}^1(Y,G) \to \mathrm{H}^3(Y,\mathbb{Z})$ for such

G, and this shows that α is not the Brauer class of an Azumaya algebra with involution of degree $2n$ where $n \not\equiv 0 \pmod 4$.

2.2. **The Cohomology of $\mathrm{B}\,\mathrm{SL}_{2n}/\mu_2$.** In order to simplify the notation, we have written in this section SL_{2n} in place of $\mathrm{SL}_{2n}(\mathbb{C})$ and similarly for GL_{2n}.

The Cartan–Iwasawa–Malcev Theorem, see for instance [24], says that the Lie groups $\mathrm{SL}_{2n}(\mathbb{C})$, $\mathrm{O}_{2n}(\mathbb{C})$, and $\mathrm{Sp}_{2n}(\mathbb{C})$ are homotopy equivalent to maximal compact subgroups, which may be taken to be the compact Lie groups SU_{2n}, $\mathrm{O}_{2n}(\mathbb{R})$, and Sp_{2n} respectively.

Our main technical tool is the low-degree part of the Serre spectral sequence in integral cohomology associated to the fiber sequence $\mathrm{B}\,\mathrm{SL}_{2n} \to \mathrm{B}\,\mathrm{SL}_{2n}/\mu_2 \to \mathrm{B}^2\mu_2$. A portion of this is presented in Figure 1.

Necessary background for the Serre spectral sequence *per se* may be found in [31, Chp. 5,6]. The E_2 page of the Serre spectral sequence of a fiber sequence $F \to E \to B$, where B is simply connected, takes the form $\mathrm{E}_2^{p,q} = \mathrm{H}^p(B, \mathrm{H}^q(F,\mathbb{Z}))$, the differentials satisfy $d_r : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$, and it converges strongly to $\mathrm{H}^{p+q}(E,\mathbb{Z})$. In the cases we encounter, $\mathrm{E}_2^{p,q} = \mathrm{H}^p(B,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}^q(F,\mathbb{Z})$, since $\mathrm{H}^*(F,\mathbb{Z})$ will be a free abelian group.

The cohomology of $\mathrm{B}\,\mathrm{SL}_{2n}$ is well-known, and may be deduced from [31, Ex. 5.F, Thm. 6.38], along with the observation that $\mathrm{B}\,\mathrm{SU}_{2n}$ is homotopy equivalent to $\mathrm{B}\,\mathrm{SL}_{2n}$. The homology of $\mathrm{B}^2\mu_2$ can be found in [13], and the cohomology may be deduced from there using the Universal Coefficients Theorem.

$$\begin{tikzcd}
c_2\mathbb{Z} = \mathrm{H}^4(\mathrm{B}\,\mathrm{SL}_{2n},\mathbb{Z}) \arrow[ddddrrrrrr, "d_5"] & & & & & & \\
0 & & & & & & \\
0 & & & & & & \\
0 & & & & & & \\
\mathbb{Z} & & 0 & 0 & q_1\mathbb{Z}/2 & 0 & q_2\mathbb{Z}/4.
\end{tikzcd}$$

FIGURE 1. A portion of the Serre spectral sequence in integral cohomology associated to $\mathrm{B}\,\mathrm{SL}_{2n} \to \mathrm{B}\,\mathrm{SL}_{2n}/\mu_2 \to \mathrm{B}^2\mu_2$.

Lemma 6. *The map $\xi : \mathrm{B}\,\mathrm{SL}_{2n}/\mu_2 \to \mathrm{B}^2\mu_2$, appearing in Diagram (3), represents a generator of*
$$\mathrm{H}^2(\mathrm{B}\,\mathrm{SL}_{2n}/\mu_2,\mathbb{Z}) \cong \mathbb{Z}/2.$$

Proof. By considering Figure 1, we see that ξ induces an isomorphism on $\mathrm{H}^{\le 3}(-,\mathbb{Z})$. It follows that it induces an isomorphism on $\mathrm{H}^2(-,\mathbb{Z}/2)$, but $\mathrm{H}^2(\mathrm{B}^2\mu_2,\mathbb{Z}/2) = \iota\mathbb{Z}/2$, where ι is represented by the identity map $\mathrm{B}^2\mu_2 \to \mathrm{B}^2\mu_2$. The result follows. $\square$

We concern ourselves with the calculation of the d_5 differential by comparison with the case of the (compact) maximal torus of diagonal unitary matrices $ST_{2n} \subset \mathrm{SL}_{2n}$. Write T_{2n} for the (compact) maximal torus of diagonal unitary matrices in GL_{2n}. The following descriptions are well known:

$$\begin{aligned}
\mathrm{H}^*(\mathrm{B}T_{2n},\mathbb{Z}) &\cong \mathbb{Z}[\theta_1,\dots,\theta_{2n}], \quad |\theta_i| = 2,\\
\mathrm{H}^*(\mathrm{B}\,\mathrm{GL}_{2n},\mathbb{Z}) &\cong \mathbb{Z}[c_1,\dots,c_{2n}] \subset \mathbb{Z}[\theta_1,\dots,\theta_{2n}], \quad c_i = \sigma_i(\theta_1,\dots,\theta_{2n}),\\
\mathrm{H}^*(\mathrm{B}ST_{2n},\mathbb{Z}) &\cong \mathbb{Z}[\theta_1,\dots,\theta_{2n}]/(\theta_1+\theta_2+\cdots+\theta_n),\\
\mathrm{H}^*(\mathrm{B}\,\mathrm{SL}_{2n},\mathbb{Z}) &\cong \mathbb{Z}[c_2,\dots,c_{2n}].
\end{aligned}$$

A proof of the relation between $\mathrm{H}^*(\mathrm{B}T_{2n},\mathbb{Z})$ and $\mathrm{H}^*(\mathrm{B}\,\mathrm{GL}_{2n},\mathbb{Z})$ appears as [23, Thm. 18.3.2], provided we recall that the noncompact groups appearing here are weakly equivalent to the compact groups appearing there. The reduction to the special linear case is an easy Serre spectral sequence argument.

The cohomology $\mathrm{H}^*(\mathrm{B}ST_{2n},\mathbb{Z})$ is the polynomial algebra on $2n-1$ generators, which we may as well take to be the images of $\theta_1,\dots,\theta_{2n-1}$, all lying in degree 2. There is a reduced augmentation map
$$\phi : \mathrm{H}^2(\mathrm{B}ST_{2n},\mathbb{Z}) \to \mathbb{Z}/2$$

given by $\theta_i \mapsto 1$ for all i.

There is a comparison of fiber sequences

$$\begin{tikzcd} \mathrm{B}ST_{2n} \arrow[r] \arrow[d] & \mathrm{B}ST_{2n}/\mu_2 \arrow[r] \arrow[d] & \mathrm{B}^2\mu_2 \arrow[d] \\ \mathrm{B}\,\mathrm{SL}_{2n} \arrow[r] & \mathrm{B}\,\mathrm{SL}_{2n}/\mu_2 \arrow[r] & \mathrm{B}^2\mu_2. \end{tikzcd} \tag{5}$$

Lemma 7. *The map $\mathrm{B}ST_{2n} \to \mathrm{B}ST_{2n}/\mu_2$ identifies $\mathrm{H}^*(\mathrm{B}ST_{2n}/\mu_2, \mathbb{Z})$ with the subring of $\mathrm{H}^*(\mathrm{B}ST_{2n}, \mathbb{Z})$ generated by $\ker(\phi : \mathrm{H}^2(\mathrm{B}ST_{2n}, \mathbb{Z}/2) \to \mathbb{Z}/2)$.*

Proof. Since ST_{2n}/μ_2 is a compact connected abelian Lie group of dimension $2n-1$, it is again a torus. Therefore, both $\mathrm{H}^*(\mathrm{B}ST_{2n}, \mathbb{Z})$ and $\mathrm{H}^*(\mathrm{B}ST_{2n}/\mu_2, \mathbb{Z})$ are polynomial rings over $\mathbb{Z}$ on $2n-1$ generators in dimension 2. The assertion of the lemma reduces to the claim that $\mathrm{H}^2(\mathrm{B}ST_{2n}/\mu_2, \mathbb{Z}) \to \mathrm{H}^2(\mathrm{B}ST_{2n}, \mathbb{Z})$ is exactly $\ker \phi$.

Consider the inclusion maps $\mu_2 \xrightarrow{\iota} ST_{2n} \to T_{2n}$. These are compatible with the Σ_{2n} action on T_{2n}, given by permuting factors in T_{2n}. The induced action of Σ_{2n} is transitive on the classes $\theta_i \in \mathrm{H}^2(\mathrm{B}T_{2n}, \mathbb{Z})$, and therefore also on their reductions to $\mathrm{H}^2(\mathrm{B}ST_{2n}, \mathbb{Z})$, while Σ_{2n} acts trivially on $\mathrm{H}^2(\mathrm{B}\mu_2, \mathbb{Z})$. We deduce that $\iota^*(\theta_i) = \iota^*(\theta_j)$ where $1 \leq i, j \leq 2n-1$.

$$\begin{array}{cccc} \mathrm{H}^2(\mathrm{B}\mu_2, \mathbb{Z}) \cong \mathbb{Z}/2 & & & \\ 0 & & & \\ \mathbb{Z} & 0 & \mathrm{H}^2(\mathrm{B}ST_{2n}/\mu_2, \mathbb{Z}) & 0 \end{array}$$

FIGURE 2. A portion of the Serre spectral sequence in integral cohomology associated to $\mathrm{B}\mu_2 \to \mathrm{B}ST_{2n} \to \mathrm{B}ST_{2n}/\mu_2$.

Now we examine the Serre spectral sequence associated to the fiber sequence $\mathrm{B}\mu_2 \to \mathrm{B}ST_{2n} \to \mathrm{B}ST_{2n}/\mu_2$, as given in Figure 2. It collapses in the illustrated range, and we are left with an extension problem

$$0 \to \mathrm{H}^2(\mathrm{B}ST_{2n}/\mu_2, \mathbb{Z}) \to \mathrm{H}^2(\mathrm{B}ST_{2n}, \mathbb{Z}) \xrightarrow{\iota^*} \mathrm{H}^2(\mathrm{B}\mu_2, \mathbb{Z}) \to 0.$$

Since $\iota^* \neq 0$, it follows that $\iota^*(\theta_i) = 1$ for at least one, and therefore for all, values of $i \in \{0, \ldots, 2n-1\}$. Therefore, $\iota^* = \phi$, and the result follows. $\square$

Explicitly, one may present $\mathrm{H}^*(\mathrm{B}ST_{2n}/\mu_2, \mathbb{Z})$ as the subring of $\mathrm{H}^*(\mathrm{B}ST_{2n}, \mathbb{Z})$ generated by the reductions modulo the ideal $(\theta_1 + \theta_2 + \cdots + \theta_{2n})$ of $\{2\theta_1\} \cup \{\theta_i - \theta_1\}_{i=2}^{2n}$.

For convenience, we write $\mathrm{H}^*(\mathrm{B}ST_{2n}, \mathbb{Z}) = \mathbb{Z}[x_1, y_2, \ldots, y_{2n-1}]$, where x_1 is the reduction modulo the ideal $(\theta_1 + \theta_2 + \cdots + \theta_{2n})$ of θ_1, and y_i is the reduction modulo the ideal $(\theta_1 + \theta_2 + \cdots + \theta_{2n})$ of $\theta_i - \theta_1$. Observe that the reduction of θ_{2n} modulo $(\theta_1 + \theta_2 + \cdots + \theta_{2n})$ is $-(2n-1)x_1 - \sum_{i=2}^{2n-1} y_i$. In this notation, $\mathrm{H}^*(\mathrm{B}ST_{2n}/\mu_2, \mathbb{Z}) = \mathbb{Z}[2x_1, y_2, \ldots, y_{2n-1}]$.

Since the spectral sequence of Figure 3 converges to $\mathbb{Z}[2x_1, y_2, \ldots, y_{2n-1}]$, we deduce that $x_1^2\mathbb{Z} \cap \mathrm{E}_\infty^{0,4} = 4x_1^2\mathbb{Z}$, and since $d_3(x_1^2) = 2x_1 d_3(x_1) = 0$, it follows that $d_5(x_1^2) = q_2$ up to a negligeable choice of sign.

We turn to the calculation of the d_5 differential in the spectral sequence of Figure (1).

Lemma 8. *In the spectral sequence of Figure 1, $d_5(c_2) = \gcd(n, 4)q_2$, up to a negligeable choice of sign.*

Proof. There is a comparison map from the spectral sequence of Figure 1 to the spectral sequence of Figure 3.

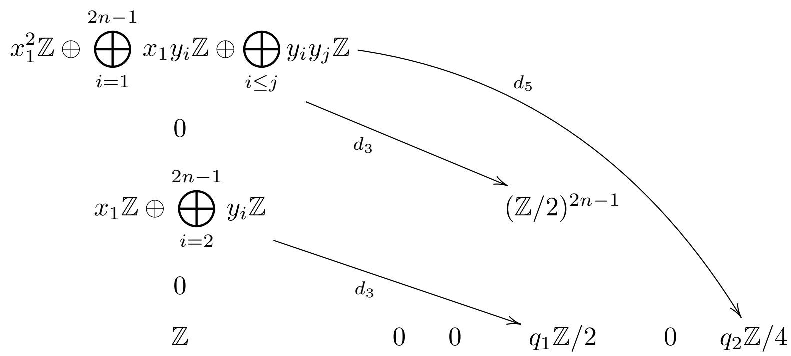

FIGURE 3. A portion of the Serre spectral sequence in integral cohomology associated to $\mathrm{B}ST_{2n} \to \mathrm{B}ST_{2n}/\mu_2 \to \mathrm{B}^2\mu_2$.

One has the following expansion in $\mathrm{H}^*(\mathrm{B}ST_{2n}, \mathbb{Z})$

$$\begin{aligned}
c_2 &= \sigma_2(\theta_1, \dots, \theta_{2n}) \\
&= \sigma_2\Big(x_1, y_2 + x_1, \dots, y_{2n-1} + x_1, -(2n-1)x_1 - \sum_{i=2}^{2n-1} y_i\Big) \\
&= \binom{2n-1}{2} x_1^2 - (2n-1)(2n-1)x_1^2 - 2nx_1(y_2 + \cdots + y_{2n-1}) + p(y_2, \dots, y_{2n-1}) \\
&= -(2n-1)nx_1^2 - 2nx_1(y_2 + \cdots + y_{2n-1}) + p(y_2, \dots, y_{2n-1})
\end{aligned}$$

where p is a homogeneous polynomial of degree 2.

Since $d_5(y_i) = 0$ and $d_5(2x_1) = 0$ in the sequence of Figure 3, it follows that $d_5(c_2) = -(2n-1)nd_5(x_1^2) = u\gcd(n,4)q_2$, for some unit u, in the sequence of Figure 3, and by comparison also in that of Figure 1. □

2.3. **Obstructions to self maps of approximations to $\mathrm{B\,SL}_{2n}/\mu_2$.** We remind the reader that a based space (X, x_0) is *n-connected* if it is connected and the homotopy groups $\pi_i(X, x_0)$ are trivial for $i \le n$. In the sequel we drop the basepoints from the notation and all spaces will be assumed based. A map of spaces $f : X \to Y$ is an n-equivalence if the homotopy fiber, hofib f, is $n-1$-connected. In practice, this means that $f_* : \pi_i(X) \to \pi_i(Y)$ is a bijection for $i < n$ and is surjective for $i = n$. If $f : X \to Y$ is an n-equivalence, then $f^* : \mathrm{H}^i(Y, R) \to \mathrm{H}^i(X, R)$ is an isomorphism when $i < n$; this follows from [43, Thm. IV.7.13] and the Universal Coefficient Theorem.

Proposition 9. *Suppose Y_1, Y_2 are two spaces each equipped with 7-equivalences $Y_1 \to \mathrm{B\,SL}_{2n_1}/\mu_2$, $Y_2 \to \mathrm{B\,SL}_{2n_2}/\mu_2$, where the n_i are not multiples of 4, and $n_1 \ge 2$. Any map $f : Y_1 \to Y_2$ inducing an isomorphism $f^* : \mathrm{H}^2(Y_2, \mathbb{Z}/2) \to \mathrm{H}^2(Y_1, \mathbb{Z}/2) \cong \mathbb{Z}/2$ induces an isomorphism $f^* : \mathrm{H}^6(Y_2, \mathbb{Q}) \to \mathrm{H}^6(Y_1, \mathbb{Q})$.*

Proof. Let Y_i' denote the homotopy fibers of the composite maps $Y_i \to \mathrm{B\,SL}_{2n_i}/\mu_2 \to \mathrm{B}^2\mu_2$. These are 7-equivalent to $\mathrm{B\,SL}_{2n_i}$ by comparison with the case of $\mathrm{B\,SL}_{2n_i} \to \mathrm{B\,SL}_{2n_i}/\mu_2 \to \mathrm{B}^2\mu_2$. The low-degree parts of the Serre spectral sequences in integral cohomology associated to $Y_i' \to Y_i \to \mathrm{B}^2\mu_2$ necessarily take the form shown in Figure 4. This coincides with the Serre spectral sequence associated to $\mathrm{B\,SL}_{2n_i} \to \mathrm{B\,SL}_{2n_i}/\mu_2 \to \mathrm{B}^2\mu_2$, which is presented in Figure 1.

By comparison with the case of $\mathrm{B\,SL}_{2n_i} \to \mathrm{B\,SL}_{2n_i}/\mu_2 \to \mathrm{B}^2\mu_2$ and use of Lemma 8, the differential d_5 takes a generator to $2q_2$.

$$
\begin{tikzcd}
\mathbb{Z} = \mathrm{H}^4(Y_i', \mathbb{Z}) \arrow[ddddrrrrr, "d_5"] & & & & & \\
0 & & & & & \\
0 & & & & & \\
0 & & & & & \\
\mathbb{Z} & 0 & 0 & q_1\mathbb{Z}/2 & 0 & q_2\mathbb{Z}/4
\end{tikzcd}
$$

FIGURE 4. A portion of the Serre spectral sequence in integral cohomology associated to $Y_i' \to Y_i \to \mathrm{B}^2\mu_2$.

The fact that f induces an isomorphism on $\mathrm{H}^2(Y_i, \mathbb{Z}/2)$ for $i \in \{1, 2\}$ implies that there is a homotopy-commutative diagram

$$
\begin{tikzcd}
Y_1' \arrow[r] \arrow[d, "f'"] & Y_1 \arrow[r] \arrow[d, "f"] & \mathrm{B}^2\mu_2 \arrow[d] \\
Y_2' \arrow[r] & Y_2 \arrow[r] & \mathrm{B}^2\mu_2
\end{tikzcd}
\tag{6}
$$

and an attendant map of spectral sequences which induces the identity on $\mathrm{E}_2^{*,0}$. It follows by considering the d_5 differential in Figure 4 that $f'^* : \mathrm{H}^4(Y_2', \mathbb{Z}) \to \mathrm{H}^4(Y_1', \mathbb{Z})$ is multiplication by an odd integer, and so $f'^* : \mathrm{H}^4(Y_2', \mathbb{Z}/2) \to \mathrm{H}^4(Y_1', \mathbb{Z}/2)$ is an isomorphism.

We have $\mathrm{H}^4(\mathrm{B}\,\mathrm{SL}_{2n_i}, \mathbb{Z}/2) = \bar{c}_2\mathbb{Z}/2$, where $\bar{x}$ denotes the reduction modulo 2 of the integral cohomology class x. We now make use of the action of the Steenrod algebra on $\mathbb{Z}/2$ cohomology. The properties of this action may be found in [22, §. 4L]. We compute the action of the Steenrod algebra on $\mathrm{H}^*(\mathrm{B}\,\mathrm{SL}_{2n}, \mathbb{Z}/2)$ by comparison with $\mathrm{H}^*(\mathrm{B}T_{2n}, \mathbb{Z}/2)$. In $\mathrm{H}^*(\mathrm{B}T_{2n}, \mathbb{Z}/2)$, one has $\mathrm{Sq}^2\bar{\theta}_i = \bar{\theta}_i^2$ for all i, and using the axioms, we calculate that

$$\mathrm{Sq}^2\sigma_2(\bar{\theta}_1, \dots, \bar{\theta}_{2n_i}) = \sigma_1(\bar{\theta}_1, \dots, \bar{\theta}_{2n_i})\sigma_2(\bar{\theta}_1, \dots, \bar{\theta}_{2n_i}) + \sigma_3(\bar{\theta}_1, \dots, \bar{\theta}_{2n_i}).$$

Therefore, upon reducing to the case of $\mathrm{B}\,\mathrm{SL}_{2n_i}$, we obtain $\mathrm{Sq}^2\bar{c}_2 = \bar{c}_3$, the reduction of the third Chern class.

By naturality of the Steenrod operations, it follows that $f'^* : \mathrm{H}^6(Y_2', \mathbb{Z}/2) \to \mathrm{H}^6(Y_1', \mathbb{Z}/2) \cong \mathbb{Z}/2$ is an isomorphism. Since $\mathrm{H}^6(Y_i', \mathbb{Z}/2)$ is, in each case, the reduction modulo 2 of the free abelian group $\mathrm{H}^6(Y_i, \mathbb{Z}) \cong \mathbb{Z}$, it follows that the natural maps $\mathrm{H}^6(Y_i, \mathbb{Q}) \to \mathrm{H}^6(Y_i', \mathbb{Q})$ are isomorphisms

Using the Serre spectral sequence, for instance, one deduces that $\mathrm{H}^*(\mathrm{B}\Gamma, \mathbb{Q}) \cong \mathrm{H}^*(\mathrm{B}\Gamma/\mu_2, \mathbb{Q})$ where Γ is a topological group containing μ_2 as a subgroup, so $f'^* : \mathrm{H}^6(Y_2', \mathbb{Q}) \to \mathrm{H}^6(Y_1', \mathbb{Q})$ is also an isomorphism. That $f^* : \mathrm{H}^6(Y_2, \mathbb{Q}) \to \mathrm{H}^6(Y_1, \mathbb{Q})$ is an isomorphism now follows from the previous paragraph and left square of diagram (6). $\square$

Proposition 10. *Suppose $Y \to \mathrm{B}\,\mathrm{SL}_4/\mu_2$ is a 7-equivalence. Let $\eta : Y \to \mathrm{B}^2\mu_2$ denote the composite $\xi \circ (Y \to \mathrm{B}\,\mathrm{SL}_{2n}/\mu_2)$. Let G be one of the groups Sp_{2n}/μ_2 or SO_{2n}/μ_2, where n is not divisible by 4. There is no map $f : Y \to \mathrm{B}G$ making the following diagram commute in homotopy*

$$
\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dr, "\eta"'] & & \mathrm{B}G \arrow[dl, "\xi_?"] \\
& \mathrm{B}^2\mu_2 &
\end{tikzcd}
.
$$

Proof. The cohomology $\mathrm{H}^*(\mathrm{BSp}_{2n}, \mathbb{Z})$ is a polynomial ring on classes in dimensions $\{4, 8, \dots, 4n\}$; this calculation is classical and is to be found in [11] following Proposition 29.2. The cohomology ring $\mathrm{H}^*(\mathrm{BSO}_{2n}, \mathbb{Z})$ is calculated in [12]. In each case, $\mathrm{H}^6(\mathrm{B}G, \mathbb{Z})$ is torsion, so it follows in our situation that $\mathrm{H}^6(\mathrm{B}G, \mathbb{Q}) = 0$.

Now we turn to disproving the existence of a map f. If there were such a map, then by composing $Y \to \mathrm{B}G$ with the appropriate map in Diagram (3), we could construct a map $g : Y \to \mathrm{B}G \to$

$\mathrm{B}\,\mathrm{SL}_4/\mu_2$ over $\mathrm{B}^2\mu_2$. By Lemma 6, we know that both $\eta : Y \to \mathrm{B}^2\mu_2$ and $\mathrm{B}\,\mathrm{SL}_4/\mu_2 \to \mathrm{B}^2\mu_2$ represent a generator of $\mathrm{H}^2(-, \mathbb{Z}/2)$, and so $g^* : \mathrm{H}^2(\mathrm{B}\,\mathrm{SL}_4/\mu_2, \mathbb{Z}/2) \to \mathrm{H}^2(Y, \mathbb{Z}/2)$ is an isomorphism. By Proposition 9, the map g induces an isomorphism $\mathrm{H}^6(\mathrm{B}\,\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}) \to \mathrm{H}^6(Y, \mathbb{Q}) \cong \mathbb{Q}$, factoring through $\mathrm{H}^6(\mathrm{B}G, \mathbb{Q}) \cong 0$, a contradiction. $\square$

3. Back to Algebra

In this section, the notation reverts to $\mathrm{SL}_{2n}, \mathrm{GL}_{2n}$ for a group scheme and $\mathrm{SL}_{2n}(\mathbb{C}), \mathrm{GL}_{2n}(\mathbb{C})$ for the Lie group of complex points. We can now prove the main Theorem A, which we rephrase.

Theorem 11. *There exists a nonsingular affine variety* $\operatorname{Spec} R$ *over* $\mathbb{C}$ *with an Azumaya algebra* A *of period* 2 *and degree* 4 *such that if* B *is any Azumaya algebra with involution and* $[B] = [A]$ *then* $8 | \deg(B)$.

The construction is similar to that of [4, Thm. 1.1].

Proof. Let V be an algebraic linear representation of SL_4/μ_2 over $\mathbb{C}$ such that SL_4/μ_2 acts freely outside an invariant closed subscheme S of codimension at least 5, and such that $(V - S)/(\mathrm{SL}_4/\mu_2)$ exists as a smooth quasi-projective complex variety. Such representations exist by [42, Rem. 1.4]. As the codimension of S is at least 5, the space $(V - S)(\mathbb{C})$ is $2(5) - 2 = 8$–connected.

Since we would like to have an affine example in particular, we use the Jouanolou device, [26], to replace $(V - S)/(\mathrm{SL}_4/\mu_2)$ by an affine vector bundle torsor $p : \operatorname{Spec} R \to (V - S)/(\mathrm{SL}_4/\mu_2)$. In order to simplify the notation, we write $Y = \operatorname{Spec} R$. The map $Y(\mathbb{C}) \to (V - S)/(\mathrm{SL}_4/\mu_2)(\mathbb{C})$ is a homotopy equivalence.

We pull the evident (SL_4/μ_2)-torsor on $(V - S)/(\mathrm{SL}_4/\mu_2)$ back along p, giving an SL_4/μ_2-torsor, T, on Y. There is a map $Y(\mathbb{C}) \to \mathrm{B}(\mathrm{SL}_4(\mathbb{C})/\mu_2)$, classifying the complex realization $T(\mathbb{C})$, and the map $Y(\mathbb{C}) \to \mathrm{B}(\mathrm{SL}_4(\mathbb{C})/\mu_2)$ is an 8–equivalence of topological spaces.

As explained in Section 2.1, the algebraic (SL_4/μ_2)-torsor T on Y induces an Azumaya algebra A over Y of degree 4 whose image in $\mathrm{Br}(Y)$ is 2-torsion. Consider the topological Azumaya algebra $A(\mathbb{C})$ over $Y(\mathbb{C})$. Its image in $\mathrm{H}^2(Y(\mathbb{C}), \mathbb{Z}/2)$ is classified by a map $\eta : Y(\mathbb{C}) \to \mathrm{B}^2\mu_2$, which factors through the 8-equivalence $Y(\mathbb{C}) \to \mathrm{B}\,\mathrm{SL}_4(\mathbb{C})/\mu_2$. Therefore, by Lemma 6, the map η represents a generator of $\mathrm{H}^2(Y(\mathbb{C}), \mathbb{Z}/2) \cong \mathbb{Z}/2$. Since $\mathrm{H}^2(Y(\mathbb{C}), \mathbb{Z}) = 0$, the map $\beta_2 : \mathrm{H}^2(Y(\mathbb{C}), \mathbb{Z}/2) \to \mathrm{H}^3(Y(\mathbb{C}), \mathbb{Z})_{\mathrm{tors}} = \mathrm{Br}(Y(\mathbb{C}))$ is injective, and therefore $A(\mathbb{C})$ is not split.

Suppose B is some Azumaya algebra on Y equivalent to A, and B is equipped with an involution of the first kind. Since A is not split, the degree of B is an even integer, $2n$. Then the topological realization $B(\mathbb{C})$ is equivalent to $A(\mathbb{C})$, and $B(\mathbb{C})$ is classified by a map $\beta : Y \to \mathrm{B}G$ where $G = \mathrm{SO}_{2n}(\mathbb{C})/\mu_2$ or $\mathrm{Sp}_{2n}(\mathbb{C})/\mu_2$, depending on whether the involution is orthogonal or symplectic; here we use Remark 5 to replace $\mathrm{O}_{2n}(\mathbb{C})$ by $\mathrm{SO}_{2n}(\mathbb{C})$. Since the Brauer classes of $A(\mathbb{C})$ and $B(\mathbb{C})$ are the same, there is a homotopy commutative diagram

$$\begin{tikzcd} Y(\mathbb{C}) \arrow[rr, "\beta"] \arrow[dr, "\eta"'] & & \mathrm{B}G \arrow[dl, "\xi_?"] \\ & \mathrm{B}^2\mu_2 & \end{tikzcd}$$

from which Proposition 10 implies that $8|2n$. $\square$

We remark that once $\operatorname{Spec} R$ has been found of some, possibly large, dimension, the affine Lefschetz theorem (see [20, Introduction, §2.2]), ensures we can replace it by a 7-dimensional smooth affine variety.

We further note that any algebra A as in Theorem 11 has index 4 and no zero divisors. Indeed, if $A' \in [A]$, then by the proof of Saltman's result given by Knus, Parimala, and Srinivas [27, §4], there is $A'' \in [A]$ admitting an involution with $\deg A'' = 2 \deg A'$. By assumption, $8 \mid \deg A''$, and hence $4 \mid \deg A'$. It follows that $\operatorname{ind} A = \gcd\{\deg A' : A' \in [A]\}$ is at least 4. Next, let K be the fraction field of R. Since $\operatorname{Spec} R$ is nonsingular, $\operatorname{ind} A \otimes_R K = \operatorname{ind} A$ ([3, Prp. 6.1]). This implies $\deg A \otimes_R K = 4 = \operatorname{ind} A \otimes_R K$, and therefore $A \otimes_R K$ is a division algebra.

4. Split Azumaya Algebras With Only Symplectic Involutions

Let X be a scheme, let $\mathcal{P}$ be a locally free $\mathcal{O}_X$-module of finite rank, and let $\mathcal{L}$ be a line bundle over X. Given a symmetric bilinear form $b : \mathcal{P} \times \mathcal{P} \to \mathcal{L}$, there is an associated even Clifford algebra $C_0(b)$ and a Clifford bimodule $C_1(b)$ over the even Clifford algebra constructed by Bichsel [10], [9]. For a summary of its main properties, see [5, §1.8], [6, §1.2], or [7, §1.5]. In particular, if $\mathcal{P}$ has rank n, then $C_0(b)$ and $C_1(b)$ are locally free $\mathcal{O}_X$-modules of rank 2^{n-1}. There is a natural vector bundle embedding $\mathcal{P} \to C_1(b)$. Assume that b is regular, i.e., the associated map $\mathcal{P} \to \mathcal{H}om(\mathcal{P}, \mathcal{L})$ is an isomorphism. If moreover n is even, then the center $Z(b)$ of $C_0(b)$ is an étale quadratic $\mathcal{O}_X$-algebra, and the left action of $Z(b)$ on $C_1(b)$ is a twist of the right action by the unique nontrivial $\mathcal{O}_X$-algebra automorphism of $Z(b)$. The class of the associated étale quadratic cover $Z \to X$ defines the discriminant class $d(b) \in \mathrm{H}^1_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z})$.

Proposition 12. *Let $b : \mathcal{P} \times \mathcal{P} \to \mathcal{L}$ be a regular symmetric bilinear form of rank 2 over a scheme X. If b has trivial discriminant then $\mathcal{P}$ is a direct sum of line bundles.*

Proof. This is a consequence of [5, Cor. 5.6], but we will give a direct proof here for completeness. Since $\mathcal{P}$ has rank 2, the vector bundle embedding $\mathcal{P} \to C_1(b)$ is an $\mathcal{O}_X$-module isomorphism, and similarly, the embedding $Z(b) \to C_0(b)$ is an $\mathcal{O}_X$-algebra isomorphism. As b is regular, another property is that $C_1(b)$ is an invertible right $C_0(b)$-module, i.e., $\mathcal{P}$ has a canonical structure of an invertible $Z(b)$-module. If b has trivial discriminant, then $Z(b)$ is the split étale quadratic algebra $\mathcal{O}_X \times \mathcal{O}_X$, and therefore there is a decomposition of $\mathcal{P}$ into a direct sum of two invertible $\mathcal{O}_X$-modules. □

This gives a nontrivial necessary condition for the existence of a regular symmetric bilinear form on a rank 2 vector bundle $\mathcal{P}$. On the other hand, any rank 2 vector bundle $\mathcal{P}$ has a canonical regular skew-symmetric form $\mathcal{P} \times \mathcal{P} \to \bigwedge^2 \mathcal{P}$ defined by wedging.

Corollary 13. *Let X be a scheme such that $\mathrm{H}^1_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z}) = 0$. If $A = \mathcal{E}nd(\mathcal{P})$ is a split Azumaya algebra of degree 2 with orthogonal involution over X, then $\mathcal{P}$ is a direct sum of line bundles. In particular, if $\mathcal{P}$ is an indecomposable vector bundle of rank 2 on X, then $\mathcal{E}nd(\mathcal{P})$ carries a symplectic involution but no orthogonal involution.*

Proof. By a result of Saltman [39, Thm. 4.2a], any orthogonal (resp. symplectic) involution on $\mathcal{E}nd(\mathcal{P})$ is adjoint to a regular symmetric (resp. skew-symmetric) bilinear form $b : \mathcal{P} \times \mathcal{P} \to \mathcal{L}$ in the sense of Example 1. If A admits an orthogonal involution and b is the corresponding form, then the discriminant $d(b) \in \mathrm{H}^1_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z})$ is trivial by assumption, hence $\mathcal{P}$ is a direct sum of two line bundles by Proposition 12. Finally, the algebra A always has a symplectic involution by Example 2. □

It is easy to provide a projective scheme and an indecomposable rank 2 vector bundle, e.g., $X = \mathbb{P}^2$ and $\mathcal{P} = \Omega^1_{\mathbb{P}^2}$. We now give an example of an integral affine scheme X and a locally free sheaf $\mathcal{P}$ satisfying the conditions of Corollary 13, thus giving rise to a split Azumaya algebra of degree 2 admitting a symplectic involution but no orthogonal involution over a domain.

Example 14. Let $R = \mathbb{C}[x, y, z, s, t, u]/(xs+yt+zu-1)$. The vector $v = (x, y, z) \in R^3$ is unimodular, hence $\mathcal{P} = R^3/Rv$ is a projective R-module of rank 2. Indeed, the ring R is $(A_{3,1})_{\mathbb{C}}$ and the module $\mathcal{P}$ is $P_{3,1}$ in the notation of [35]. There is an affine vector bundle torsor $\operatorname{Spec} R \to \mathbb{A}^3 \setminus \{0\}$, which on the level of coordinates is given by $(x, y, z, s, t, u) \mapsto (x, y, z)$, and it follows that $\operatorname{Pic}(R) = \mathrm{CH}^1(\operatorname{Spec} R) = 0$, see [18, Thm. 1.9]. By [35, Cor. 6.3], the R-module $\mathcal{P}$ is not free, and as a consequence is indecomposable.

By Artin's Comparison theorem, $\mathrm{H}^1_{\acute{e}t}(R, \mathbb{Z}/2\mathbb{Z}) = 0$, and thus $A = \operatorname{End}(\mathcal{P})$ has a symplectic involution, but no orthogonal involution by Corollary 13.

5. Non-Split Azumaya Algebras With Only Symplectic Involutions

We now show that any Azumaya algebra with involution over an affine scheme is a specialization of an involutary Azumaya algebra without zero divisors. Applying this to Example 14, we obtain non-split Azumaya algebras of degree 2 admitting only symplectic involutions.

We shall give two different constructions, both arising from orders in certain generic division algebras. The first construction is in fact a sequence of involutary Azumaya algebras that together specialize to any involutary Azumaya algebra. The centers of these algebras are regular, but their

Krull dimension is very large. The second construction is not universal in the previous sense, but its center has smaller Krull dimension.

Henceforth, we shall restrict to Azumaya algebras over commutative rings. We write A/R to denote that A is an R-algebra. We remind the reader that if A/R and A'/R' are two Azumaya algebras of degree n, then any ring homomorphism $\phi : A \to A'$ is in fact a *specialization*, meaning that $\phi(R) \subseteq R'$, and $A \otimes_\phi R' \cong A'$ via $a \otimes r' \mapsto \phi(a)r'$ (see [40, Cor. 2.9b]).

We will also need Rowen's version of the Artin–Procesi Theorem. If $g(x_1, \dots, x_n)$ is a polynomial in non-commuting variables over $\mathbb{Z}$ and A is a ring, then let $g(A)$ denote $\{g(a_1, \dots, a_n) \,|\, a_1, \dots, a_n \in A\}$. We further let $\mathrm{Id}(A)$ denote the *multilinear polynomial identities* of A over $\mathbb{Z}$ (see [36, Chp. 6]). The center of A is denoted $Z(A)$.

Theorem 15 (Artin, Procesi, Rowen)**.** *Let A be a ring and let $g_n(x_1, \dots, x_{4n^2+1})$ denote the Formanek central polynomial (see* [36, Def. 6.1.21]*). The following conditions are equivalent:*

(a) *A is Azumaya of degree n over $Z(A)$.*
(b) $\mathrm{Id}(\mathrm{Mat}_{n\times n}(\mathbb{Z})) \subseteq \mathrm{Id}(A)$*, and there exists a multilinear polynomial identity of* $\mathrm{Mat}_{(n-1)\times(n-1)}(\mathbb{Z})$ *that is not a polynomial identity of any nonzero image of A.*
(c) $\mathrm{Id}(\mathrm{Mat}_{n\times n}(\mathbb{Z})) \subseteq \mathrm{Id}(A)$*, the polynomial g_n is a central polynomial of A, and 1_A is in the additive group spanned by $g_n(A)$.*

Proof. See [36, Thm. 6.1.35] or [37, §6], for instance. □

5.1. **First Construction.** This construction is inspired by Rowen's generic division algebras with involution [38].

Fix an integral domain Ω with $2 \in \Omega^\times$ and fix $n > 1$. Recall that g_n denotes the Formanek central polynomial, and let $N = 4n^2 + 1$ denote the number of variables of g_n. Theorem 15 implies that any Azumaya algebra A/R of degree n admits vectors $v_1, \dots, v_m \in A^N$ such that $\sum_{i=1}^m g_n(v_i) = 1_A$. We call the minimal possible such m the *Formanek number* of A and denote it by $\mathrm{For}(A)$.

For every $k \in \mathbb{N}$, let $T = T(n,k)$ be the free commutative Ω-algebra spanned by $\{x_{ij}^{(r)}\}_{i,j,r}$ where $1 \le i,j \le n$ and $1 \le r \le k$. We let F denote the fraction field of T. Let X_r denote the generic matrix $(x_{ij}^{(r)}) \in \mathrm{Mat}_{n\times n}(T)$. Let $A_0(n,k)$ be the Ω-subalgebra of $\mathrm{Mat}_{n\times n}(T)$ generated by $X_1, X_1^{\mathrm{t}}, \dots, X_k, X_k^{\mathrm{t}}$. If n is even, let $B_0(n,k)$ be the Ω-subalgebra of $\mathrm{Mat}_{n\times n}(T)$ generated by $X_1, X_1^{\mathrm{sp}}, \dots, X_k, X_k^{\mathrm{sp}}$ (see Example 1). We alert the reader that the algebras $A_0(n,k)$ and $B_0(n,k)$ are not Azumaya in general.

Lemma 16. *Let (A,σ) be an Azumaya algebra of degree n with an orthogonal (resp. symplectic) involution and such that $R := Z(A)$ is an Ω-algebra. Let $a_1, \dots, a_k \in A$. Then there is a homomorphism of Ω-algebras with involution $\phi : (A_0(n,k), \mathrm{t}) \to (A,\sigma)$ (resp. $\phi : (B_0(n,k), \mathrm{sp}) \to (A,\sigma)$) such that $\phi(X_i) = a_i$ for all $1 \le i \le k$.*

Proof (compare with [38, Thm. 27(i)]*).* We will treat the orthogonal case; the symplectic case is similar.

The lemma is clear when $A = \mathrm{Mat}_{n\times n}(R)$ and $\sigma = \mathrm{t}$, indeed, simply specialize X_i to a_i for all i. For general A, choose an étale R-algebra S such that $(A',\sigma') := (A \otimes_R S, \sigma \otimes_R \mathrm{id}_S)$ is isomorphic to $(\mathrm{Mat}_{n\times n}(S), \mathrm{t})$ (see Proposition 4), and view (A,σ) as an involutary subring of (A',σ'). Then there is $\phi : A_0(n,k) \to A'$ with $\phi(X_i) = a_i$ and $\phi(X_i^{\mathrm{t}}) = \sigma'(a_i) = \sigma(a_i)$. Since $A_0(n,k)$ is generated as an Ω-algebra by $X_1, \dots, X_k, X_1^{\mathrm{t}}, \dots, X_k^{\mathrm{t}}$, we have $\mathrm{im}(\phi) \subseteq \Omega[a_1, \dots, a_k, \sigma(a_1), \dots, \sigma(a_k)] \subseteq A$. □

Suppose now that $k = Nm$ for $m \in \mathbb{N}$ and let

$$\omega = \sum_{i=1}^m g_n(X_{N(i-1)+1}, \dots, X_{N(i-1)+N}) \in A_0(n,k) \cap B_0(n,k)$$

Since g_n is a central polynomial of $\mathrm{Mat}_{n\times n}(T)$, the matrix ω is diagonal, and hence $\omega^{\mathrm{t}} = \omega^{\mathrm{sp}} = \omega$. Furthermore, since g_n is not a polynomial identity of $\mathrm{Mat}_{n\times n}(\Omega)$, we have $\omega \ne 0$ (because the matrices $X_1, \dots, X_{Nm}$ can be specialized to any $n \times n$ matrix over Ω). We define

$$A(n,m) := A_0(n,Nm)[\omega^{-1}] \subseteq \mathrm{Mat}_{n\times n}(F) \ .$$

The involution t on $\mathrm{Mat}_{n\times n}(F)$ restricts to an involution on $A(n,m)$, which we also denote by t. In the same way, we define $B(n,m)$ for n even by replacing "A_0" with "B_0" and "t" with "sp".

Theorem 17. *Suppose $n > 1$ is a power of 2. Then $A(n,m)$ (resp. $B(n,m)$) is an Azumaya algebra of degree n without zero divisors. Furthermore, $(A(n,m),\mathrm{t})$ (resp. $(B(n,m),\mathrm{sp})$) specializes to any Azumaya algebra with an orthogonal (resp. symplectic) involution (E,σ) satisfying $\deg E = n$ and $\mathrm{For}(E) \le m$, and such that $Z(E)$ is an Ω-algebra.*

Proof. Again, we prove only the orthogonal case.

We use condition (c) of Theorem 15 to check that $A(n,m)$ is Azumaya. That $\mathrm{Id}(\mathrm{Mat}_{n\times n}(\mathbb{Z})) \subseteq \mathrm{Id}(A(n,m))$ and g_n is a central polynomial of $A(n,m)$ follow from the fact that $A(n,m) \subseteq \mathrm{Mat}_{n\times n}(F)$. Next, we have $\omega = \sum_{i=1}^m g_n(X_{N(i-1)+1},\dots,X_{N(i-1)+N})$, and hence

$$1 = \sum_{i=1}^m g_n(\omega^{-1}X_{N(i-1)+1}, X_{N(i-1)+2},\dots,X_{N(i-1)+N})$$

because g_n is multilinear, so 1 is in the additive group generated by $g_n(A(n,m))$, as required.

That $A(n,m)$ has no zero divisors follows from [38, Thm. 29].

To finish, let (E,σ) be as in the theorem. Since $\mathrm{For}(E) \le m$, there are vectors $v_1,\dots,v_m \in E^N$ such that $\sum_i g_n(v_i) = 1_E$. Define $a_1,\dots,a_{Nm} \in E$ via $v_i = (a_{N(i-1)+1},\dots,a_{N(i-1)+N})$. By Lemma 16, there is a homomorphism $\phi : (A_0(n,Nm),\mathrm{t}) \to (E,\sigma)$ such that $\phi(X_i) = a_i$ and $\phi(X_i^{\mathrm{t}}) = \sigma(a_i)$ for all $1 \le i \le Nm$. It follows that the element $\omega \in A_0(n,Nm)$ is mapped by ϕ to 1_E. Thus, ϕ extends to a homomorphism of rings with involution $(A(n,m),\mathrm{t}) \to (E,\sigma)$ by setting $\phi(\omega^{-1}) = 1$. □

Write $Z(n,m) = Z(A(n,m))$ and $W(n,m) = Z(B(n,m))$. We now show that the morphisms $\operatorname{Spec} Z(n,m) \to \operatorname{Spec}\Omega$ and $\operatorname{Spec} W(n,m) \to \operatorname{Spec}\Omega$ are smooth when Ω is noetherian. As a result, $Z(n,m)$ and $W(n,m)$ are regular when Ω is a field.

Lemma 18. *Let A/R be an Azumaya algebra. Suppose that there is a noetherian subring $R_0 \subseteq R$ such that A is finitely generated as an R_0-algebra. Then R is finitely generated as an R_0-algebra. In particular, A and R are noetherian.*

Proof. Since A is Azumaya, it is finitely generated as a module over R. The proposition therefore follows from a variant of the Artin–Tate Lemma (see [33, Lem. 1], for instance). □

Lemma 19. *Let R be a commutative ring and let I be an ideal in R satisfying $I^2 = 0$. Let G be a smooth affine group scheme over R. Then the map $\mathrm{H}^1_{\acute{e}t}(R,G) \to \mathrm{H}^1_{\acute{e}t}(R/I,G)$ is bijective.*

Proof. Write $R' = R/I$.

By [SGA4, Exp. viii, Thm. 1.1], the base change $\operatorname{Spec} R' \to \operatorname{Spec} R$ induces an equivalence between the étale site of $\operatorname{Spec} R$ and the étale site of $\operatorname{Spec} R'$. It is therefore enough to prove that for any étale cover $\operatorname{Spec} S \to \operatorname{Spec} R$, the map $\mathrm{H}^1_{\acute{e}t}(S/R,G) \to \mathrm{H}^1_{\acute{e}t}(S'/R',G)$ is an isomorphism, where $S' = S\otimes_R R'$. By [SGA3, Exp. xxiv, Lem. 8.1.8], the map $\mathrm{H}^1_{fpqc}(S/R,G) \to \mathrm{H}^1_{fpqc}(S'/R',G)$ is an isomorphism (here we need G to be smooth), and since $\operatorname{Spec} S \to \operatorname{Spec} R$ is étale and $G \to \operatorname{Spec} R$ is smooth, we may replace the fpqc topology by the étale topology, see the introduction to [SGA3, Exp. xxiv]. □

Proposition 20. *When Ω is noetherian, the morphisms $\operatorname{Spec} Z(n,m) \to \operatorname{Spec}\Omega$ and $\operatorname{Spec} W(n,m) \to \operatorname{Spec}\Omega$ are smooth.*

Proof. We prove only that $\operatorname{Spec} Z(n,m) \to \operatorname{Spec}\Omega$ is smooth. By Lemma 18, this morphism is finitely presented, and hence it is enough to show that $\operatorname{Spec} Z(n,m) \to \operatorname{Spec}\Omega$ is formally smooth. Let S be a commutative Ω-algebra, let I be an ideal of S with $I^2 = 0$, and let $\phi : Z(n,m) \to S/I$ be a homomorphism of Ω-algebras. We show that ϕ can be lifted to a homomorphism $\phi' : Z(n,m) \to S$.

Let $A = A(n,m) \otimes_\phi (S/I)$ and let σ be the involution induced by t on A. Then (A,σ) is an Azumaya algebra with an orthogonal involution over S/I. The smoothness of $\mathrm{PGO}_n \to \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ and Lemma 19 (see also Section 1.3) imply that there is an Azumaya S-algebra with involution (A',σ') such that $(A,\sigma) \cong (A' \otimes_S (S/I), \sigma' \otimes_S \mathrm{id}_{S/I})$. We identify A with $A'/A'I$. For all $1 \le i \le Nm$, write $a_i = \phi(X_i)$ and choose some $a'_i \in A'$ whose image in A is a_i. By Lemma 16, there is a morphism of Ω-algebras with involution $\phi' : (A_0(n,Nm),\mathrm{t}) \to (A',\sigma')$ such that $\phi'(X_i) = a'_i$. The morphism

ϕ' extends uniquely to $A(n,m)$ provided $\phi'(\omega) \in A'^{\times}$. This holds because $\phi'(\omega) + A'I = \phi(\omega) \in A^{\times}$ (since $\phi(\omega)^{-1} = \phi(\omega^{-1})$) and $A'I$ is nilpotent. We have shown $\phi' : A(n,m) \to A'$ is a homomorphism of Azumaya algebras, and therefore it restricts to a homomorphism on the centers $\phi' : Z(n,m) \to S$, which is the lift we required. □

Corollary 21. *There exists an affine regular $\mathbb{C}$-algebra R and an Azumaya R-algebra A of degree 2 without zero divisors such that A has no orthogonal involutions.*

Proof. Let m_0 to be the Formanek number of the Azumaya algebra constructed in Example 14. Take $\Omega = \mathbb{C}$ and let $R = W(n, m_0)$ and $A = B(n, m_0)$. □

5.2. **Second Construction.** This construction uses versal torsors in the sense of [15]. It lacks the universal character of the first construction and it may have a non-regular center, but the Krull dimension of the center is much smaller and can be effectively estimated.

We start by showing that weakly versal PGO_{2n}-torsors and PGSp_{2n}-torsors satisfy a slightly stronger version of weak versality.

Let K be a field and let G be a smooth affine group scheme over K. Recall that a weakly versal G-torsor is a K-scheme X together with a G-torsor $T \to X$ such for every field extension K'/K with K' infinite and every G-torsor $T' \to \operatorname{Spec} K'$ there exists a K-morphism $i : \operatorname{Spec} K' \to X$ such that $T' \cong i^*T$. The torsor $T \to X$ is called versal if, for every open immersion $j : U \to X$, the restriction $j^*T \to U$ is weakly versal. The minimal possible dimension of a base scheme of a versal torsor is equal to the essential dimension of G, denoted $\mathrm{ed}_K(G)$. These versal torsors of minimal dimension can be chosen to be integral and affine over K. See [8], [15] for further details and proofs.

Lemma 22. *Let L be a field and let R be a commutative ring. Let E/L and A/R be Azumaya algebras of degree n. Suppose that there is a subring $E_0 \subseteq E$ and a surjective ring homomorphism $\phi : E_0 \to A$. Then there exist elements $s \in E_0 \cap K^{\times}$ and $x_1, \dots, x_m \in E_0$ such that:*

(i) *ϕ extends uniquely to a ring homomorphism $E_0[s^{-1}] \to A$.*
(ii) *Any subring $E_1 \subseteq E$ containing s^{-1} and $x_1, \dots, x_m$ is Azumaya of degree n over its center.*

In particular, $E_0[s^{-1}]$ is an Azumaya algebra that specializes to A via ϕ.

Proof. Let $N = 4n^2 + 1$ denote the number of variables of the Formanek central polynomial g_n. Since A is Azumaya of degree n, Theorem 15 implies that there are vectors $v_1, \dots, v_t \in A^N$ such that $\sum_{i=1}^t g_n(v_i) = 1$. Choose $u_1, \dots, u_t \in E_0^N$ such that $\phi_0^N(u_i) = v_i$ for all $1 \le i \le t$. Write $m = tN$ and define $x_1, \dots, x_m$ via $u_i = (x_{(i-1)N+1}, \dots, x_{(i-1)N+N})$. In addition, let $s = \sum_{i=1}^t g_n(u_i)$. Observe that $s \in Z(E) = L$ because g_n is a central polynomial, and $s \neq 0$ because $\phi(s) = \phi(\sum_i g_n(u_i)) = \sum_i g_n(v_i) = 1_A$. We claim that s and $x_1, \dots, x_m$ satisfy (i) and (ii). Indeed, extend ϕ to $E_0[s^{-1}]$ by defining $\phi(s^{-1}) = 1_A$. That any E_1 as in (ii) is Azumaya is similar to the proof of Theorem 17. □

Proposition 23. *Let K'/K be a field extension with K' infinite, and let G be PGO_n or PGSp_n. Let $T \to X$ be a weakly versal G-torsor over K such that X is integral and affine over $\operatorname{Spec} K$, and let $X' = X \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ and $T' = T \times_{\operatorname{Spec} K} \operatorname{Spec} K'$. Let R be a subring of K' and let $U \to \operatorname{Spec} R$ be a G-torsor. Then there are G-torsors $T'_0 \to X'_0$, $U' \to Y$ and morphisms as illustrated*

$$\begin{tikzcd} U \arrow[d] & U' \arrow[d] & T'_0 \arrow[d] & T' \arrow[d] \\ \operatorname{Spec} R \arrow[r, "f"] & Y & X'_0 \arrow[l, "j"'] \arrow[r, "i"] & X' \end{tikzcd}$$

such that i is an open immersion, j is dominant, and there are isomorphisms of G-torsors

$$U \cong f^*U', \qquad j^*U' \cong T'_0 \cong i^*T' .$$

If R contains a noetherian subring R_0, then Y may be chosen to be of finite type over $\operatorname{Spec} R_0$ and f may be chosen to be an R_0-morphism.

Proof. We treat only the case $G = \mathrm{PGO}_n$. The other case is similar. If R_0 is not specified, take it to be the image of $\mathbb{Z}[\frac{1}{2}]$ in R. Since weak versality is preserved under base change, we may replace $T \to X$ by $T' \to X'$ and assume that $K' = K$.

Let (B,τ) be the involutary Azumaya algebra corresponding to $T \to X$ and let (A,σ) be the involutary Azumaya algebra corresponding to $U \to \operatorname{Spec} R$. Since X is weakly versal, there is a K-section $g : \operatorname{Spec} K \to X$ specializing (B,τ) to $(A_K, \sigma_K) := (A \otimes_R K, \sigma \otimes_R \mathrm{id}_K)$. Denote the induced ring homomorphism by $\phi : B \to A_K$. The map ϕ is onto because it is K-linear and any epimorphic image of B is an Azumaya algebra of degree n. View A as a subring of A_K and let $E_0 = \phi^{-1}(A)$. Since ϕ is a homomorphism of rings with involution, we have $\tau(E_0) \subseteq E_0$. Let ξ be the generic point of X and let E/L be the Azumaya algebra induced by the PGO_n-torsor T_ξ. The involution $\tau : B \to B$ extends to an involution on E. We now apply Lemma 22 to get elements $x_1, \dots, x_m \in E_0$ and $s \in Z(E_0) \cap L^\times$.

Let $E_1 = R_0[s^{-1}, x_1, \dots, x_m, \tau(x_1), \dots, \tau(x_m)]$. Then E_1 is Azumaya of degree n over its center, and ϕ induces a specialization of Azumaya algebras from E_1 to A. This specialization is compatible with the involutions τ and σ, hence $(E_1, \tau|_{E_1})$ specializes to (A,σ). We therefore take $Y = \operatorname{Spec} Z(E_1)$ and $X'_0 = \operatorname{Spec} Z(B)[s^{-1}]$, and let U and T'_0 be the PGO_n-torsors corresponding to (E_1,τ) and $(B[s^{-1}],\tau)$, respectively. The morphism i corresponds to the inclusion $Z(B) \subseteq Z(B)[s^{-1}]$, the morphism j corresponds to the inclusion $Z(E_1) \subseteq Z(E_0)[s^{-1}] \subseteq Z(B)[s^{-1}]$, and the morphism f corresponds to $\phi : Z(E_1) \to Z(A) = R$. When R_0 is noetherian, $Z(E_1)$ is affine over R_0 by Lemma 18. □

Proposition 23 also holds for PGL_n-torsors; the proof is similar. We do not know whether the proposition holds for torsors of general smooth affine group schemes.

We now use Proposition 23 to show that any Azumaya algebra with involution (A,σ) over a domain R is a specialization of an involutary Azumaya algebra without zero divisors.

Lemma 24. *Let K be a field and let $T \to X$ be a versal PGO_n-torsor (resp. PGSp_n-torsor). Suppose that X is integral and affine, $X = \operatorname{Spec} R$, and let (A,σ) denote the Azumaya algebra with involution corresponding to $T \to X$. If n is a power of 2, then A has no zero divisors.*

Proof. It is well-known that there exists a field extension K'/K such that K' is infinite and a central simple K'-algebra with an orthogonal (resp. symplectic) involution (D,τ) of degree n such that D is a division ring; see [38, Thm. 29], for instance.

Suppose that $a, b \in A$ satisfy $ab = 0$. View A as a sheaf of algebras over X and let Z_a and Z_b be the vanishing loci of a and b, respectively. Let $U = X \setminus (Z_a \cup Z_b)$ and assume for the sake of contradiction that U is nonempty. Shrinking U if necessary, we may assume $U = \operatorname{Spec} R[r^{-1}]$ for some $0 \neq r \in R$. By the versality of $T \to X$, the algebra $(A[r^{-1}],\sigma)$ specializes to (D,τ) over K, and by the construction of U, the images of a and b in D are nonzero. This means D has zero divisors, a contradiction. Consequently U must be empty and since X is irreducible, either $Z_a = X$ or $Z_b = X$. This implies that either $a = 0$ or $b = 0$, since X is reduced. □

Theorem 25. *Let (A,σ) be an Azumaya algebra of degree 2^n with an orthogonal (resp. symplectic) involution over a domain R. Then there exists an Azumaya algebra of degree 2^n with an orthogonal (resp. symplectic) involution (B,τ) such that B has no zero divisors and (B,τ) specializes to (A,σ). If R is an affine algebra over a field k, then (B,τ) can be chosen such that $Z(B)$ is affine over k and*

$$\dim Z(B) \le \dim R + \mathrm{ed}_K(G)$$

where $G = \mathrm{PGO}_{2^n}$ (resp. $G = \mathrm{PGSp}_{2^n}$) and K is the fraction field of R.

Proof. We prove only the orthogonal case.

Let $U \to \operatorname{Spec} R$ be the torsor corresponding to (A,σ). Choose a versal PGO_{2^n}-torsor $T \to X$ over K such that X is an affine, integral K-scheme and $\dim X = \mathrm{ed}_K(\mathrm{PGO}_{2^n})$. We now apply Proposition 23 with $K' = K$ (and $R_0 = k$, if necessary) to obtain $U' \to Y$, $T' \to X'_0$, f, i, j as in the proposition.

Let (B,τ) and (E,θ) be the Azumaya algebras with involution corresponding to $U' \to Y$, and $T \to X$, respectively. By Lemma 24, E has no zero divisors. Since i is an open immersion and j is dominant, this means B has no zero divisors. Finally, if R is affine over k, then $\dim Z(B) = \mathrm{trdeg}_k K(Y) \le \mathrm{trdeg}_k K(X) = \mathrm{trdeg}_k K + \mathrm{trdeg}_K K(X) = \dim R + \mathrm{ed}_K(\mathrm{PGO}_{2^n})$. □

We note that Theorem 25 gives an Azumaya algebra whose center is *a priori* not regular.

Corollary 26. *There exists an Azumaya algebra B/S of degree 2 without zero divisors that admits only symplectic involutions. The ring S can be taken to be an affine $\mathbb{C}$-algebra with $\dim S \leq 7$.*

Proof. This follows from Example 14 and Theorem 25 since $\mathrm{ed}_{\mathbb{C}}(\mathrm{PGSp}_2) \leq 2$. □

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