SEMIORTHOGONAL DECOMPOSITIONS AND BIRATIONAL GEOMETRY OF DEL PEZZO SURFACES OVER ARBITRARY FIELDS

ASHER AUEL AND MARCELLO BERNARDARA

Abstract. We study the birational properties of geometrically rational surfaces from a derived categorical perspective. In particular, we give a criterion for the rationality of a del Pezzo surface S over an arbitrary field, namely, that its derived category decomposes into zero-dimensional components. For $\deg(S) \geq 5$, we construct explicit semiorthogonal decompositions by subcategories of modules over semisimple algebras arising as endomorphism algebras of vector bundles and we show how to retrieve information about the index of S from Brauer classes and Chern classes associated to these vector bundles.

Introduction

In their address to the 2002 International Congress of Mathematicians in Beijing, Bondal and Orlov [31] suggest that the bounded derived category $\mathsf{D}^{\mathrm{b}}(S)$ of coherent sheaves on a smooth projective variety S could provide new tools to explore the birational geometry of S, in particular via semiorthogonal decompositions. The work of many authors [72], [75], [20], [11], [1] now provides evidence for the usefulness of semiorthogonal decompositions in the birational study of complex projective varieties of dimension at most 4. A survey can be found in [77]. At the same time, the relevance of semiorthogonal decompositions to other areas of algebraic and noncommutative geometry has been growing rapidly, as Kuznetsov [76] points out in his address to the 2014 ICM in Seoul.

In this context, based on the classical notion of representability for Chow groups and motives, Bolognesi and the second author [21] proposed the notion of *categorical representability* (see Definition 1.14), and formulated the following question.

Question 1. *Is a rational variety always categorically representable in codimension 2?*

When the base field k is not algebraically closed, the existence of k-rational points on S (being a necessary condition for rationality) is a major open question in arithmetic geometry. We are indebted to H. Esnault, who posed the following question to us in 2009.

Question 2. *Let S be a smooth projective variety over a field k. Can the bounded derived category $\mathsf{D}^{\mathrm{b}}(S)$ of coherent sheaves detect the existence of a k-rational point on S?*

This question, which was the original motivation for the current work, is now central to a growing body of research into arithmetic aspects of the theory of derived categories, see [3], [8], [11], [12], [79], [55], [109], [10]. As an example, if S is a smooth projective surface, Hassett and Tschinkel [55, Lemma 8] prove that the index of S can be recovered from $\mathsf{D}^{\mathrm{b}}(S)$ as the greatest common divisor of the second Chern classes of objects. Recall that the *index* $\mathrm{ind}(S)$ of a variety S over k is the greatest common divisor of the degrees of closed points of S.

These questions also intertwine with concurrent developments in equivariant and descent aspects of triangulated and dg categories in the context of noncommutative geometry, see [7], [47], [74], [97], [105].

In the present text, we study these two questions for geometrically rational surfaces over an arbitrary field k, with special attention paid to the case of del Pezzo surfaces of Picard rank 1. The k-birational classification of such surfaces was achieved by Enriques, Manin [82], and Iskovskikh [57]; the study of arithmetic properties (e.g., existence of rational points, stable rationality, Hasse principle, Chow groups) has been an active area of research since the 1970s, see [38] for a survey.

One of the most important invariants of a geometrically rational surface S over k is the Néron–Severi lattice $\mathrm{NS}(S_{k^s}) = \mathrm{Pic}(S_{k^s})$ of $S_{k^s} = S \times_k k^s$ as a module over the absolute Galois group $G_k = \mathrm{Gal}(k^s/k)$, where k^s is a fixed separable closure of k. The structure of this Galois lattice controls the exceptional lines on S, and hence the possible birational contractions. As the canonical bundle ω_S is always defined over k, one often considers the orthogonal complement $\omega_S^\perp$. For del Pezzo surfaces, this lattice is a twist of a semisimple root lattice with the Galois action factoring through the Weyl group, see [70, Thm. 2.12]. C. Vial [109] has recently studied how one can, given a k-exceptional collection on a surface S, obtain information on the Néron–Severi lattice. In particular, he shows that a geometrically rational surface with an exceptional collection of maximal length is rational. In this paper, we deal with the opposite extreme, of surfaces of small Picard rank.

Even when S has Picard rank 1, and there are no exceptional curves on S defined over k, our perspective is that the missing information concerning the birational geometry of S can be filled in, to some extent, by considering higher rank vector bundles on S that are generators of canonical components of $\mathsf{D}^{\mathrm{b}}(S)$.

For example, let S be a quadric surface with $\mathrm{Pic}(S)$ generated by the hyperplane section $\mathscr{O}_S(1)$ of degree 2. It is well known that the rationality of S is equivalent to the existence of a rational point. By a result in the theory of quadratic forms, a quadric surface having a rational point is equivalent to the corresponding even Clifford algebra C_0 (a quaternion algebra over the quadratic extension defining the rulings of the quadric) being split over its center. Given a rational point $x \in S(k)$, the Serre construction yields a rank 2 vector bundle W as an extension of $\mathscr{I}_x$ by $\mathscr{O}_S(-1)$. On the other hand, the surface $S_{k^s} = \mathbb{P}^1_{k^s} \times \mathbb{P}^1_{k^s}$ carries two rank one *spinor bundles*, corresponding to $\mathscr{O}(1,0)$ and $\mathscr{O}(0,1)$ (see, e.g., [87]). The vector bundle W_{k^s} is isomorphic to $\mathscr{O}(1,0) \oplus \mathscr{O}(0,1)$, as the point x is the complete intersection of lines in two different rulings. Not assuming the existence of a rational point, such rank 2 vector bundles are only defined étale locally; the obstruction to their existence is the Brauer class of C_0. There do, however, exist naturally defined vector bundles V of rank 4 on S, via descent of $W \oplus W$, and satisfying $\mathrm{End}(V) = C_0$. One could say that V "controls" the birational geometry of S in a noncommutative way.

Derived categories and their semiorthogonal decompositions provide a natural setting for a *noncommutative* counterpart of the Néron–Severi lattice with its Galois action. One can consider the Euler form $\chi(A,B) = \sum_{i=0}^{2} \dim \mathrm{Ext}^i(A,B)$ on the derived category, and the χ-semiorthogonal systems of simple generators, the so-called *exceptional collections*. Over k^s, these are known to exist and have interesting properties for rational surfaces. In this paper, we consider the descent properties of such collections and show how they indeed give a complete way to interpret the link between vector bundles, semisimple algebras, rationality, and rational points. In particular, on a geometrically rational surface S, the canonical line bundle ω_S is naturally an element of such a system. Then one can consider the subcategory $\langle \omega_S \rangle^\perp$, whose object are χ-semiorthogonal to ω_S. In practice, it can be more natural to consider the category $\mathsf{A}_S = \langle \mathscr{O}_S \rangle^\perp$, which is equivalent to $\langle \omega_S \rangle^\perp$. We describe decompositions (or indecomposability) of this subcategory by explicit descent methods for vector bundles. In this context, certain semisimple k-algebras naturally arise and control the birational geometry of S. These algebras also provide a presentation of the K-theory of S.

A sample result of this kind, that does not seem to be contained in the literature, is as follows. Let S be a del Pezzo surface of degree 5 over a field k. It is known that S is uniquely determined by an étale algebra l of degree 5 over k, see [96, Thm. 3.1.3]. We prove a k-equivalence $\mathsf{D}^{\mathrm{b}}(S) = \mathsf{D}^{\mathrm{b}}(A)$, where A is a finite dimensional algebra whose semisimplification is $k \times k \times l$. In particular, there is an isomorphism

$$K_i(S) \cong K_i(k) \times K_i(k) \times K_i(l)$$

in algebraic K-theory, for all $i \geq 0$. In the context of computing the algebraic K-theory of geometrically rational surfaces, this adds to the work of Quillen [91] for Severi–Brauer surfaces (i.e., del Pezzo surfaces of degree 9), Swan [98] and Panin [88] for quadric surfaces and involution surfaces (i.e., minimal del Pezzo surfaces of degree 8), and Blunk [25] for del Pezzo surfaces of

degree 6. Our method provides a uniform way to compute the algebraic K-theory of all del Pezzo surfaces degree ≥ 5.

S. Gille [49] recently studied the Chow motive with integer coefficients of a geometrically rational surface, showing that they are zero dimensional if and only if the surface has a zero cycle of degree one and the Chow group of zero-cycles is torsion free after base changing to the function field. Tabuada (cf. [102]) has shown how the Morita equivalence class of the derived category of a smooth projective variety (with its canonical dg-enhancement) gives the *noncommutative motive* of the variety. We refrain here from defining the additive, monoidal, and idempotent complete category of noncommutative motives, whose objects are smooth and proper dg categories up to Morita equivalence; the interested reader can refer to [102]. We recall only that any semiorthogonal decomposition gives a splitting of the noncommutative motive (see [101]) and that, as well as Chow motives, noncommutative motives can be defined with coefficients in any ring R. Working with $\mathbb{Q}$-coefficients, there is a well established correspondence between noncommutative and Chow motives, see [103]. Roughly speaking, the category of Chow motives embeds fully and faithfully into the category of noncommutative motives, if one "forgets" the Tate motive. As pointed out in [22], this relation does not hold in general for integer coefficients. In particular, for surfaces, one has to invert 2 and 3 in the coefficient ring to have such a direct comparison (see [22, Cor. 1.6]). We think that a comparison between Gille's results and those of the present paper could lead to a deep understanding of the different types of geometric information encoded by Chow and noncommutative motives.

By a del Pezzo surface S over a field k, we mean a smooth, projective, and geometrically integral surface over k with ample anticanonical class. The degree of S is the self-intersection number of the canonical class ω_S. Our main result is the following.

Theorem 1. *A del Pezzo surface S of Picard rank one over a field k is k-rational if and only if S is categorically representable in dimension 0.*

In particular, if S has degree $d \geq 5$ and any Picard rank, the following are equivalent:

(i) S is k-rational.
(ii) S has a k-rational point.
(iii) S is categorically representable in dimension 0.
(iv) $\mathsf{A}_S \simeq \langle \omega_S \rangle^{\perp}$ is representable in dimension 0.

The equivalence of *(i)* and *(ii)* for del Pezzo surfaces of degree ≥ 5 is a result of Manin [82, Thm. 29.4]. Our work is a detailed analysis of how birational properties interact with certain semiorthogonal decompositions. In particular, we study collections (so-called *blocks*) of completely orthogonal exceptional vector bundles on S_{k^s} and their descent to S. The theory of 3-block decompositions due to Karpov and Nogin [64] is indispensable for our work.

In our results for del Pezzo surfaces of higher Picard rank, the bound on the degree is quite important. This is not surprising, as del Pezzo surfaces of degree $d \geq 5$ have a much simpler geometry and arithmetic than those of lower degree. For example, in smaller degree the existence of a rational point only implies unirationality [82, Thm. 29.4]. On the other hand, a succinct corollary of Theorem 1 gives a positive answer to Question 1 for all del Pezzo surfaces.

Corollary 1. *Any rational del Pezzo surface S is categorically representable in dimension 0.*

This relies on the fact that there is no minimal k-rational del Pezzo surface of degree $d \leq 4$ (see [83, Thm. 3.3.1]). So Question 2 in smaller degrees, where the existence of a k-point is weaker than categorical representability in dimension 0, remains open.

One of the ingredients in the proof of Theorem 1 is the description of semiorthogonal decompositions of minimal del Pezzo surfaces of degree ≥ 5. This relies on, and extends, the special cases established by Kuznetsov [71], Blunk [26], Blunk, Sierra, and Smith [27], and the second author [19]. In these cases, it always turns out that there are semisimple k-algebras A_1 and A_2, Azumaya over their centers, and a semiorthogonal decomposition

$$\mathsf{A}_S \simeq \langle \omega_S \rangle^{\perp} = \langle \mathsf{D}^{\mathrm{b}}(k, A_1), \mathsf{D}^{\mathrm{b}}(k, A_2) \rangle \tag{0.1}$$

over k. The algebras A_i arise as endomorphism algebras of vector bundles with special homological features. In particular, over k^s, these bundles split into a sum of completely orthogonal exceptional bundles, whose existence was proved by Rudakov and Gorodentsev [52], [92], and by Karpov and Nogin [64]. This generalizes the example of quadric surfaces explained above.

Theorem 2. *Let S be a del Pezzo surface of degree $d \geq 5$. There exist vector bundles V_1 and V_2 and a semiorthogonal decomposition*

$$\mathsf{A}_S = \langle V_1, V_2 \rangle = \langle \mathsf{D}^{\mathrm{b}}(k, A_1), \mathsf{D}^{\mathrm{b}}(k, A_2) \rangle = \langle \mathsf{D}^{\mathrm{b}}(l_1/k, \alpha_1), \mathsf{D}^{\mathrm{b}}(l_2/k, \alpha_2) \rangle \tag{0.2}$$

where $A_i = \mathrm{End}(V_i)$ are semisimple k-algebras with centers l_i, and α_i is the class of A_i in $\mathrm{Br}(l_i)$. Finally, S has Picard rank 1 if and only if the vector bundles V_i are indecomposable or, equivalently, the algebras A_i are simple.

One of our main technical results is that such a decomposition of $\mathsf{A}_S \simeq \langle \omega_S \rangle^{\perp}$ can never occur if S has Picard rank 1 and degree ≤ 4, see Theorem 5.1. In fact, in this case one should expect A_S to have a very complicated algebraic description, but not much is known (see Theorem 5.7 and Theorem 5.8).

A feature of these semiorthogonal decompositions is that the k-birational class of S is determined by the unordered pair of semisimple algebras (A_1, A_2) up to pairwise Morita equivalence. As pointed out by Kuznetsov [77, §3], one of most tempting ideas in the theory of semiorthogonal decompositions with a view toward rationality is to define, in any dimension and independently of the base field, a categorical analog of the Griffiths component of the intermediate Jacobian of a complex threefold. Such an analog would be the best candidate for a birational invariant. It turns out that such a component is not well defined in general. However, this is not the case for del Pezzo surfaces. Indeed, we can define the *Griffiths–Kuznetsov component* GK_S of a del Pezzo surface S in the case when S has degree $d \geq 5$ or Picard rank 1 as follows.

Definition 1. *Let S be a minimal del Pezzo surface of degree d over k. We define the Griffiths–Kuznetsov component GK_S of S as follows: if A_S is representable in dimension 0, set $\mathsf{GK}_S = 0$. If not, GK_S is either the product of all indecomposable components of A_S of the form $\mathsf{D}^{\mathrm{b}}(l, \alpha)$ with l/k a field extension and $\alpha \in \mathrm{Br}(l)$ nontrivial or else $\mathsf{GK}_S = \mathsf{A}_S$.*

If S is not minimal, then we set $\mathsf{GK}_S = \mathsf{GK}_{S'}$ for a minimal model $S \to S'$.

Definition 1 may appear ad hoc, but it can roughly be rephrased by saying that the Griffiths–Kuznetsov component is the product of all components of A_S which are not representable in dimension 0. If $\deg(S) \geq 5$, then GK_S is determined up to equivalence by Brauer classes related to the algebras A_1 and A_2. If $\deg(S) \leq 4$ and S has Picard rank 1, we conjecture its indecomposability (see Conjecture 5.6).

Theorem 3. *Let S be a del Pezzo surface of degree d over k. The Griffiths-Kuznetsov component GK_S is well-defined, unless $\deg(S) = 4$ and $\mathrm{ind}(S) = 1$, or $\deg(S) = 8$ and $\mathrm{ind}(S) = 2$. Moreover, if $S' \dashrightarrow S$ is a birational map, then $\mathsf{GK}_S = \mathsf{GK}_{S'}$.*

We notice that the two "bad" cases of degree 4 and index 1 or degree 8 and index 2 are birational to conic bundles which are not forms of Hirzebruch surfaces. In the Appendix, we will show how A_S can be interpreted as categories naturally attached to conic bundles.

Theorem 3 could be considered as an analogue of Amitsur's theorem that k-birational Severi–Brauer varieties have Brauer classes that generate the same subgroup of $\mathrm{Br}(k)$.

In the case where $d \geq 5$, we push this further to analyze how the algebras A_i obstruct the existence of closed points of given degree. The *index* $\mathrm{ind}(A)$ of a central simple k-algebra A is the greatest common divisor of the degrees of all central simple k-algebras Morita equivalent to A. We extend this definition to finite dimensional semisimple algebras by taking the least common multiple of the indices of simple components considered over their centers.

Theorem 4. *Let S be a non-k-rational del Pezzo surface of degree $d \geq 5$, and A_1, A_2 the associated semisimple algebras. If A_i has index d_i then $\mathrm{ind}(S) = d_1 d_2 / \gcd(d_1, d_2)$. In particular:*

- *If $d = 9$, then $d_1 = d_2$ is either 1 if $S \simeq \mathbb{P}^2_k$ or 3 if $S(k) \neq \varnothing$.*

- *If $d = 8$ and S is an involution surface, then (up to renumbering) A_1 is a central simple k-algebra, $d_1|4$ and $d_2|2$, and $d_1 = 1$ if and only if S is a quadric in $\mathbb{P}^3_k$ while $d_2 = 1$ if and only S is rational.*
- *If $d = 6$, then (up to renumbering) $d_1|3$ and $d_2|2$, with S birationally rigid if $d_i > 1$.*
- *In all other cases, $d_1 = d_2 = 1$ and S is rational.*

The key of the proof of Theorem 4 is the explicit description of the vector bundles generating the exceptional blocks of S, together with an analysis of all possible Sarkisov links, cf. [58].

For del Pezzo surfaces of Picard rank one and degree at least 5, a consequence of our results is that the index of S can be calculated only in terms of the second Chern classes of generators of the 3 blocks, improving upon, in this case, a result of Hassett and Tschinkel [55, Lemma 8].

Theorem 5. *Let S be a del Pezzo surface of degree $d \geq 5$. There exist generators V_i for the three blocks such that*

$$\operatorname{ind}(S) = \gcd\{c_2(V_0), c_2(V_1), c_2(V_2)\}.$$

Moreover, unless $d = 5$, $d = \operatorname{ind}(S) = 6$, or S is an anisotropic quadric surface in $\mathbb{P}^3$ then

$$\operatorname{ind}(S) = \gcd\{c_2(V_1), c_2(V_2)\}$$

for indecomposable generators of the blocks of A_S.

Notice that more precise and detailed statements can be given in each specific case. Consult Sections 6–10 and Tables 2–5 for details.

In summary, our results establish a complete understanding of the relationship between birational geometry, derived categories, and vector bundles on del Pezzo surfaces of degree ≥ 5.

Structure. The paper is organized as follows. Part 1 organizes the necessary algebraic and geometric background. In Section 1 we treat semiorthogonal decompositions and a generalized notion of exceptional objects, whose descent is treated in Section 2. Section 3 collects useful results on geometrically rational surfaces. Part 2 is dedicated to a detailed statement and proof of the main results. Before proceeding to the proofs, we recall the exceptional block theory in Section 4. Section 5 proves the main results for surfaces of low degree, while the higher degree cases are treated separately in Sections 6–10. Part 3 consists of three appendices containing useful calculations related to elementary links.

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Notations. Fix an arbitrary field k. If X is a k-scheme, we will denote by $\mathsf{D}^{\mathrm{b}}(X)$ the bounded derived category of complexes of coherent sheaves on X, considered as a k-linear category. If B is an $\mathscr{O}_X$-algebra, we will denote by $\mathsf{D}^{\mathrm{b}}(X, B)$ the bounded derived category of complexes of B-modules, considered as a k-linear category. If $X = \operatorname{Spec}(K)$ is an affine k-scheme and B is associated to a K-algebra, we will write $\mathsf{D}^{\mathrm{b}}(K/k, B)$ as shorthand. Also $\mathsf{D}^{\mathrm{b}}(K, A)$ is shorthand for $\mathsf{D}^{\mathrm{b}}(K/K, A)$. If A is the restriction of scalars of B down to k, we remark that $\mathsf{D}^{\mathrm{b}}(K/k, B)$ is k-equivalent to $\mathsf{D}^{\mathrm{b}}(k, A)$. Finally, if B is Azumaya with class β in $\operatorname{Br}(X)$, by abuse of notation, we will write $\mathsf{D}^{\mathrm{b}}(X, \beta)$ for $\mathsf{D}^{\mathrm{b}}(X, B)$.

Triangulated categories will be commonly denoted with sans-serif font; dg-categories with calligraphic font; finite dimensional k-algebras with upper case Roman letters near the beginning of the alphabet; finite products of field extensions of k with l; vector bundles with upper case Roman letters; and Brauer classes as lower case Greek letters.

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Part 1. Background on derived categories and geometrically rational surfaces

1. Semiorthogonal decompositions and categorical representability

We start by recalling the categorical notions that play the main role in this paper. Let k be an arbitrary field. The theory of exceptional objects and semiorthogonal decompositions in the case where k is algebraically closed and of characteristic zero was studied in the Rudakov seminar at the end of the 80s, and developed by Rudakov, Gorodentsev, Bondal, Kapranov, and Orlov among others, see [52], [28], [29], [32], and [93]. As noted in [11], most fundamental properties persist over any base field k.

1.1. **Semiorthogonal decompositions and their mutations.** Let T be a k-linear triangulated category. A full triangulated subcategory A of T is called *admissible* if the embedding functor admits a left and a right adjoint.

Definition 1.1 ([29])**.** A *semiorthogonal decomposition* of T is a sequence of admissible subcategories $\mathsf{A}_1, \dots, \mathsf{A}_n$ of T such that

- $\mathrm{Hom}_{\mathsf{T}}(A_i, A_j) = 0$ for all $i > j$ and any A_i in A_i and A_j in A_j;
- for all objects A_i in A_i and A_j in A_j, and for every object T of T, there is a chain of morphisms $0 = T_n \to T_{n-1} \to \dots \to T_1 \to T_0 = T$ such that the cone of $T_k \to T_{k-1}$ is an object of A_k for all $k = 1, \dots, n$.

Such a decomposition will be written

$$\mathsf{T} = \langle \mathsf{A}_1, \dots, \mathsf{A}_n \rangle.$$

If $\mathsf{A} \subset \mathsf{T}$ is admissible, we have two semi-orthogonal decompositions

$$\mathsf{T} = \langle \mathsf{A}^{\perp}, \mathsf{A} \rangle = \langle \mathsf{A}, {}^{\perp}\mathsf{A} \rangle,$$

where $\mathsf{A}^{\perp}$ and ${}^{\perp}\mathsf{A}$ are, respectively, the left and right orthogonal of A in T (see [29, §3]).

Given a semiorthogonal decomposition $\mathsf{T} = \langle \mathsf{A}, \mathsf{B} \rangle$, Bondal [28, §3] defines left and right mutations $L_{\mathsf{A}}(\mathsf{B})$ and $R_{\mathsf{B}}(\mathsf{A})$ of this pair. In particular, there are equivalences $L_{\mathsf{A}}(\mathsf{B}) \simeq \mathsf{B}$ and

$R_{\mathsf{B}}(\mathsf{A}) \simeq \mathsf{A}$, and semiorthogonal decompositions

$$\mathsf{T} = \langle L_{\mathsf{A}}(\mathsf{B}), \mathsf{A}\rangle, \qquad \mathsf{T} = \langle \mathsf{B}, R_{\mathsf{B}}(\mathsf{A})\rangle.$$

We refrain from giving an explicit definition for the mutation functors in general, which can be found in [28, §3]. In §1.2 we will give an explicit formula in the case where A and B are generated by exceptional objects.

1.2. **Exceptional objects, blocks, and mutations.** Very special examples of admissible subcategories, semiorthogonal decompositions, and their mutations are provided by the theory of exceptional objects and blocks. These constructions appear naturally on Fano varieties, and especially on geometrically rational surfaces, and they play a central role in our study. We provide here a detailed treatment, generalizing to any field notions usually studied over algebraically closed fields.

Let T be a k-linear triangulated category. The triangulated category $\langle\{V_i\}_{i\in I}\rangle$ *generated* by a class of objects $\{V_i\}_{i\in I}$ of T is the smallest thick (that is, closed under direct summands) full triangulated subcategory of T containing the class. We will write $\mathrm{Ext}^r_{\mathsf{T}}(V,W) = \mathrm{Hom}_{\mathsf{T}}(V, W[r])$.

Definition 1.2. Let A be a division (not necessarily central) k-algebra (e.g., A could be a field extension of k). An object V of T is called *A-exceptional* if

$$\mathrm{Hom}_{\mathsf{T}}(V,V) = A \quad \text{and} \quad \mathrm{Ext}^r_{\mathsf{T}}(V,V) = 0 \quad \text{for} \quad r \neq 0.$$

An exceptional object in the classical sense [51, Def. 3.2] of the term is a k-exceptional object. By *exceptional* object, we mean A-exceptional for some division k-algebra A.

A totally ordered set $\{V_1, \dots, V_n\}$ of exceptional objects is called an *exceptional collection* if $\mathrm{Ext}^r_{\mathsf{T}}(V_j, V_i) = 0$ for all integers r whenever $j > i$. An exceptional collection is *full* if it generates T, equivalently, if for an object W of T, the vanishing $\mathrm{Hom}_{\mathsf{T}}(W, V_i) = 0$ for all i implies $W = 0$. An exceptional collection is *strong* if $\mathrm{Ext}^r_{\mathsf{T}}(V_i, V_j) = 0$ whenever $r \neq 0$.

Remark 1.3. The extension of scalars of an exceptional object (in this generalized sense) need not be exceptional, see Remark 2.1 for more details. However, if k is algebraically closed, then all exceptional objects are automatically k-exceptional, hence remain exceptional under any extension of scalars.

Exceptional collections provide examples of semiorthogonal decompositions when T is the bounded derived category of a smooth projective scheme.

Proposition 1.4 ([28, Thm. 3.2])**.** *Let $\{V_1, \dots, V_n\}$ be an exceptional collection on the bounded derived category $\mathsf{D}^{\mathrm{b}}(X)$ of a smooth projective k-scheme X. Then there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(X) = \langle V_1, \dots, V_n, \mathsf{A}\rangle,$$

where A is the full subcategory of objects W such that $\mathrm{Hom}_{\mathsf{T}}(W, V_i) = 0$. In particular, the sequence if full if and only if $\mathsf{A} = 0$.

Given an exceptional pair $\{V_1, V_2\}$ with V_i being A_i-exceptional, consider the admissible subcategories $\langle V_i\rangle$, forming a semiorthogonal pair. We can hence perform right and left mutations, which provide equivalent admissible subcategories.

Recall that mutations provide equivalent admissible subcategories and flip the semiorthogonality condition. It easily follows that the object $R_{V_2}(V_1)$ is A_1-exceptional, the object $L_{V_1}(V_2)$ is A_2-exceptional, and the pairs $\{L_{V_1}(V_2), V_1\}$ and $\{V_2, R_{V_2}(V_1)\}$ are exceptional. We call $R_{V_2}(V_1)$ the *right mutation* of V_1 through V_2 and $L_{V_1}(V_2)$ the *left mutation* of V_2 through V_1.

In the case of k-exceptional objects, mutations can be explicitly computed.

Definition 1.5 ([51, §3.4])**.** Given a k-exceptional pair $\{V_1, V_2\}$ in T, the *left mutation* of V_2 with respect to V_1 is the object $L_{V_1}(V_2)$ defined by the distinguished triangle:

$$\mathrm{Hom}_{\mathsf{T}}(V_1, V_2) \otimes V_1 \xrightarrow{ev} V_2 \longrightarrow L_{V_1}(V_2), \tag{1.1}$$

where ev is the canonical evaluation morphism. The *right mutation* of V_1 with respect to V_2 is the object $R_{V_2}(V_1)$ defined by the distinguished triangle:

$$R_{V_2}(V_1) \longrightarrow V_1 \xrightarrow{coev} \mathrm{Hom}_{\mathsf{T}}(V_1, V_2) \otimes V_2,$$

where $coev$ is the canonical coevaluation morphism.

Given an exceptional collection $\{V_1, \dots, V_n\}$, one can consider any exceptional pair $\{V_i, V_{i+1}\}$ and perform either right or left mutation to get a new exceptional collection.

Exceptional collections provide an algebraic description of admissible subcategories of T. Indeed, if V is an A-exceptional object in T, the triangulated subcategory $\langle V \rangle \subset \mathsf{T}$ is equivalent to $\mathsf{D}^{\mathrm{b}}(k, A)$. The equivalence $\mathsf{D}^{\mathrm{b}}(k, A) \to \langle V \rangle$ is obtained by sending the complex A concentrated in degree 0 to V. Moreover, as shown by Bondal [28] and Bondal–Kapranov [30], full exceptional collections give an algebraic description of a triangulated category.

Proposition 1.6 ([28, Thm. 6.2])**.** *Suppose that T is the bounded derived category of either a smooth projective k-scheme or a k-linear abelian category with enough injective objects. Let $\{V_1, \dots, V_n\}$ be a full strong exceptional collection on T, and consider the object $V = \bigoplus_{i=1}^n V_i$ and the k-algebra $A = \mathrm{End}_{\mathsf{T}}(V)$. Then $\mathbf{R}\mathrm{Hom}_{\mathsf{T}}(V, -) : \mathsf{T} \to \mathsf{D}^{\mathrm{b}}(k, A)$ is a k-linear equivalence.*

Remark 1.7. The assumption on the category T and on the strongness of the exceptional sequence may seem rather restrictive, and both find a natural solution when triangulated categories are enriched with a dg-category structure. The first assumption can be indeed replaced by considering a dg-enhancement of T (see [30, Thm. 1]). When the exceptional collection is not strong, dg-algebras are required. See §1.3 for details.

Example 1.8. The full strong k-exceptional collection $\{\mathscr{O}, \mathscr{O}(1), \dots, \mathscr{O}(n)\}$ on $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^n_k)$ is due to Beilinson [17], [18] and Bernšteĭn–Gelfand–Gelfand [24]. In this case $A = \mathrm{End}\big(\bigoplus_{i=0}^n \mathscr{O}(i)\big)$ is isomorphic to the path algebra of the Beilinson quiver with $n+1$ vertices, see [28, Ex. 6.4]. We remark that the semisimplification of A is the algebra k^{n+1}.

Proposition 1.6 provides an alternative approach to exceptional collections and semiorthogonal decompositions by considering *tilting objects*. In this paper, we will not use this approach, but we find the language convenient at times, especially in relation to issues of Galois descent.

Definition 1.9. Given a strong exceptional collection $\{V_1, \dots, V_m\}$ of a k-linear triangulated category T, the object $V = \bigoplus_{i=1}^m V_i$ is called a *tilting object* for the subcategory $\langle V_1, \dots, V_m \rangle$. Recall that Proposition 1.6 implies that $\langle V_1, \dots, V_m \rangle \simeq \mathsf{D}^{\mathrm{b}}(k, \mathrm{End}_{\mathsf{T}}(V))$. If the V_i are vector bundles on a smooth projective variety X, we call V a *tilting bundle*.

The existence of a tilting bundle also yields, analogously to Proposition 1.6, a presentation in algebraic K-theory.

Proposition 1.10. *Let $\{V_1, \dots, V_n\}$ be a full strong exceptional collection of vector bundles on a smooth projective k-variety X and let $V = \bigoplus_{i=1}^n V_i$ be the associated tilting bundle. Suppose that V_i is A_i-exceptional. Then $\mathrm{Hom}_X(V, -) : K_i(X) \to K_i(A_1 \times \cdots \times A_n)$ is an isomorphism for all $i \geq 0$.*

Proof. Letting $A = \mathrm{End}_{\mathsf{D}^{\mathrm{b}}(X)}(V)$. Since the exceptional objects are vector bundles, $\mathrm{Hom}_X(V, -)$ defines a morphism from the category of vector bundles on X to the category of A-modules. Proposition 1.6 implies that $\mathbf{R}\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(X)}(V, -) : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(k, A)$ is an equivalence. Hence we can apply [107, Thm. 1.9.8, §3], implying that $\mathrm{Hom}_X(V, -) : K_i(X) \to K_i(A)$ is an isomorphism for all $i \geq 0$. But A has a nilpotent ideal I so that $A/I \cong A_1 \times \cdots \times A_n$, so the result follows from the fact that $K_i(A) \cong K_i(A/I)$. □

Remark 1.11. One can prove the same result for any additive invariant of $\mathsf{D}^{\mathrm{b}}(X)$ via noncommutative motives and Tabuada's universal additive functor (see [101]).

A special case of an exceptional pair is a *completely orthogonal* pair $\{V_1, V_2\}$, i.e., an exceptional pair such that $\{V_2, V_1\}$ is also exceptional. In this case, $R_{V_2}(V_1) = V_1$ and $L_{V_1}(V_2) = V_2$.

Definition 1.12 ([64], 1.5)**.** An *exceptional block* in a k-linear triangulated category T is an exceptional collection $\{V_1, \dots, V_n\}$ such that $\mathrm{Ext}^r_{\mathsf{T}}(V_i, V_j) = 0$ for every r whenever $i \neq j$. Equivalently, every pair of objects in the collection is completely orthogonal. By abuse of notation, we denote by E the exceptional block as well as the subcategory that it generates.

If E is an exceptional block, then $\mathrm{End}_{\mathsf{T}}(\bigoplus_{i=1}^n V_i)$ is isomorphic to the k-algebra $A_1 \times \cdots \times A_n$, where V_i is A_i-exceptional. Proposition 1.6 then yields a k-equivalence $\mathsf{E} \simeq \mathsf{D}^{\mathrm{b}}(A_1 \times \cdots \times A_n)$.

Moreover, given an exceptional block, any internal mutation acts by simply permuting the exceptional objects. Given an exceptional collection $\{V_1, \dots, V_n, W_1, \dots, W_m\}$ consisting of two blocks E and F, the left mutation $L_{\mathsf{E}}(\mathsf{F})$ and the right mutation $R_{\mathsf{F}}(\mathsf{E})$ are obtained by mutating all the objects of one block to the other side of all the objects of the other block, or, equivalently, as mutations of semiorthogonal admissible subcategories.

1.3. **dg-enhancements.** Recall that a k-linear *dg-category* $\mathscr{A}$ is a category enriched over dg-complexes of k-vector spaces, that is, for any pair of objects x, y in $\mathscr{A}$, the morphism set $\mathrm{Hom}_{\mathscr{A}}(x, y)$ has a functorial structure of a differential graded complex of vector spaces. The category $H^0(\mathscr{A})$ has the same objects as $\mathscr{A}$, and $\mathrm{Hom}_{H^0(\mathscr{A})}(x, y) = H^0(\mathrm{Hom}_{\mathscr{A}}(x, y))$, in particular $H^0(\mathscr{A})$ is triangulated. We will only consider *pretriangulated* dg-categories (see [66, §4.5]). A semiorthogonal decomposition of $\mathscr{A} = \langle \mathscr{B}_1, \dots, \mathscr{B}_n \rangle$ is a set of pretriangulated full subcategories such that $\langle H^0(\mathscr{B}_1), \dots, H^0(\mathscr{B}_n) \rangle$ is a semiorthogonal decomposition of $H^0(\mathscr{A})$. Consult [66] for an introduction to dg-categories.

Let X and Y be a smooth proper k-schemes, and recall that a *Fourier–Mukai functor* of kernel $P \in \mathsf{D}^{\mathrm{b}}(X \times Y)$ is the exact functor $\Phi : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Y)$ given by the formula $\Phi_P(-) = (\mathbf{R}p_Y)_*(p_X^*(-) \otimes^{\mathbf{L}} P)$ (see, e.g., [56, §5]).

Considering the bounded derived category $\mathsf{D}^{\mathrm{b}}(X)$ as a triangulated k-linear category gives rise to some functorial problems. One of them has a geometric nature: given an exact functor $\Phi : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Y)$ it is not known whether it is isomorphic to a Fourier–Mukai functor, except in some special cases, see [37]. For example, suppose that A_X and A_Y are admissible triangulated categories of $\mathsf{D}^{\mathrm{b}}(X)$ and $\mathsf{D}^{\mathrm{b}}(Y)$, respectively, and suppose that there is a triangulated equivalence $\mathsf{A}_X \simeq \mathsf{A}_Y$. The composition functor

$$\Phi : \mathsf{D}^{\mathrm{b}}(X) \twoheadrightarrow \mathsf{A}_X \simeq \mathsf{A}_Y \hookrightarrow \mathsf{D}^{\mathrm{b}}(Y)$$

is conjectured to be a Fourier–Mukai functor ([73, Conj. 3.7]).

As proved by Lunts and Orlov [80], there is a unique functorial enhancement $\mathscr{D}^{\mathrm{b}}(X)$ for $\mathsf{D}^{\mathrm{b}}(X)$, that is a dg-category such that $H^0(\mathscr{D}^{\mathrm{b}}(X)) = \mathsf{D}^{\mathrm{b}}(X)$. Given X and Y as before, a functor $\Phi : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Y)$ is of Fourier–Mukai type if and only if there is a functor $\varPhi : \mathscr{D}^{\mathrm{b}}(X) \to \mathscr{D}^{\mathrm{b}}(Y)$ such that $H^0(\varPhi) = \Phi$, see [23, Prop. 9.4] for example, or [81] for a more general statement.

On the other hand, Kuznetsov [74] has shown that, for any admissible subcategory $\mathsf{A}_X \subset \mathsf{D}^{\mathrm{b}}(X)$, the projection functor (i.e., the right adjoint to the embedding functor) is a Fourier–Mukai functor. Hence, the choice of such a functor endows A_X with a dg-structure, which is in principle not unique, but depends on the choice of projection. Hence, when dealing with a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{A}_1, \dots, \mathsf{A}_n \rangle,$$

one always has a decomposition

$$\mathscr{D}^{\mathrm{b}}(X) = \langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle,$$

where the dg-category $\mathscr{D}^{\mathrm{b}}(X)$ is unique, though the dg-enhancements $\mathscr{A}_i$ of A_i may not be.

Finally, we recall that if there exists a smooth projective scheme Z and a Brauer class α in $\mathrm{Br}(Z)$ such that $\mathsf{A}_i \simeq \mathsf{D}^{\mathrm{b}}(Z, \alpha)$, then the embedding functor $\Phi : \mathsf{D}^{\mathrm{b}}(Z, \alpha) \to \mathsf{D}^{\mathrm{b}}(X)$ is a Fourier–Mukai functor [36], and hence the dg-structure $\mathscr{D}^{\mathrm{b}}(Z, \alpha)$ is unique. Other specific cases are provided by Homological Projective Duality [73].

1.4. **Categorical representability.** Using semiorthogonal decompositions, one can define a notion of *categorical representability* for a triangulated category. In the case of smooth projective varieties, this is inspired by the classical notions of representability of cycles, see [21].

Definition 1.13 ([21])**.** A k-linear triangulated category T is *representable in dimension* m if it admits a semiorthogonal decomposition

$$\mathsf{T} = \langle \mathsf{A}_1, \dots, \mathsf{A}_r \rangle,$$

and for each $i = 1, \dots, r$ there exists a smooth projective connected k-variety Y_i with $\dim Y_i \leq m$, such that A_i is equivalent to an admissible subcategory of $\mathsf{D}^{\mathrm{b}}(Y_i)$.

Definition 1.14 ([21])**.** Let X be a projective k-variety of dimension n. We say that X is *categorically representable* in dimension m (or equivalently in codimension $n-m$) if there exists a categorical resolution of singularities of $\mathsf{D}^{\mathrm{b}}(X)$ that is representable in dimension m.

The following explains why categorical representability in codimension 2 is conjecturally related to birational geometry in the spirit of Question 1.

Lemma 1.15. *Let $X \to Y$ be the blow-up of a smooth projective k-variety along a smooth center. If Y is categorically representable in codimension 2 then so is X.*

Proof. Denoting by $Z \subset Y$ the center of the blow-up $f : X \to Y$, which has codimension at least 2 in Y, Orlov's blow-up formula [84] gives a semiorthogonal decomposition $\mathsf{D}^{\mathrm{b}}(X) = \langle f^*\mathsf{D}^{\mathrm{b}}(Y), \mathsf{D}^{\mathrm{b}}(Z) \rangle$. As X and Y have the same dimension, if Y is categorically representable in codimension 2, then so is X. □

An additive category T is *indecomposable* if for any product decomposition $\mathsf{T} \simeq \mathsf{T}_1 \times \mathsf{T}_2$ into additive categories, we have that $\mathsf{T} \simeq \mathsf{T}_1$ or $\mathsf{T} \simeq \mathsf{T}_2$. Equivalently, T has no nontrivial completely orthogonal decomposition. Remark that if X is a k-scheme then $\mathsf{D}^{\mathrm{b}}(X)$ is indecomposable if and only if X is connected (see [33, Ex. 3.2]). More is known if X is the spectrum of a field or a product of fields.

Lemma 1.16. *Let K be a k-algebra.*

- *(i) If K is a field and A is a nonzero admissible k-linear triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(k, K)$, then $\mathsf{A} = \mathsf{D}^{\mathrm{b}}(k, K)$.*
- *(ii) If $K \cong K_1 \times \cdots \times K_n$ is a product of field extensions of k and A is a nonzero admissible indecomposable k-linear triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(k, K)$, then $\mathsf{A} \simeq \mathsf{D}^{\mathrm{b}}(k, K_i)$ for some $i = 1, \dots, n$.*
- *(iii) If $K \cong K_1 \times \cdots \times K_n$ is a product of field extensions of k and A is a nonzero admissible k-linear triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(k, K)$, then $\mathsf{A} \simeq \prod_{j \in I} \mathsf{D}^{\mathrm{b}}(k, K_j)$ for some subset $I \subset \{1, \dots, n\}$.*

Proof. For *(i)*, it suffices to note that any nonzero object of $\mathsf{D}^{\mathrm{b}}(k, K)$ is a generator. For *(ii)*, with respect to the product decomposition $\mathsf{D}^{\mathrm{b}}(k, K) \simeq \mathsf{D}^{\mathrm{b}}(k, K_1) \times \cdots \times \mathsf{D}^{\mathrm{b}}(k, K_n)$, consider the projections x_i to $\mathsf{D}^{\mathrm{b}}(k, K_i)$ of an object x of A. Any nonzero x_i will generate the respective subcategory $\mathsf{D}^{\mathrm{b}}(k, K_i)$.

Given a nonzero object x in A, there exist $1 \leq i \leq n$ such that $x_i \neq 0$. If there exists another object y in A with $y_j \neq 0$ for $j \neq i$, then the objects x_i and y_j will generate completely orthogonal nontrivial subcategories of A. Since A is indecomposable, this is impossible. Hence for every object x in A, we have that $x_j = 0$ for every $j \neq i$. In particular, $\mathsf{A} \subset \mathsf{D}^{\mathrm{b}}(k, K_i)$, hence they are equal by *(i)*. For *(iii)*, we apply *(ii)* to the indecomposable components of A. □

Next, we need the following affine case of a conjecture of Căldăraru [35, Conj. 4.1]. The simple proof below is taken from the unpublished manuscript [13] as part of a proof of Căldăraru's conjecture in the relative case. A proof of the conjecture (using different techniques) was recently obtained by Antieau [5], with the case of arbitrary (possibly non-torsion) classes over general spaces handled by [34], after progress by [36] and [90].

Recall that a *central simple k-algebra* A is a simple k-algebra whose center is k. More generally, if X is a scheme, an *Azumaya algebra* A over X is a locally free $\mathscr{O}_X$-algebra of

finite rank such that $A \otimes_R A^{\mathrm{op}}$ is isomorphic to the matrix algebra $\mathrm{End}(A)$. Azumaya algebras A and B are *Brauer equivalent* if there exist locally free $\mathscr{O}_X$-modules P and Q such that $A \otimes \mathrm{End}(P) \cong B \otimes \mathrm{End}(Q)$, and they are *Morita equivalent* over X if their stacks of coherent modules over X are equivalent. Then Brauer equivalence coincides with Morita equivalence over X, and the group of equivalence classes under tensor product is the *Brauer group* $\mathrm{Br}(X)$. For an Azumaya algebra A over X, we write A^n for $A^{\otimes n}$ for $n \geq 0$ and $(A^{\mathrm{op}})^{\otimes -n}$ for $n \leq 0$.

Theorem 1.17. *Let R be a noetherian commutative ring, U and V be R-algebras, and A and B be Azumaya algebras over U and V, respectively. Then A and B are Morita R-equivalent if and only if there exists an R-algebra isomorphism $\sigma : U \to V$ such that A and $\sigma^* B$ are Brauer equivalent over U.*

Proof. First suppose that A and $\sigma^* B$ are Brauer equivalent, where $\sigma : U \to V$ is an R-algebra isomorphism. Then by Morita theory for Azumaya algebras (cf. [65, §19.5]), there is a Morita U-equivalence $\mathsf{Coh}(U, A) \to \mathsf{Coh}(U, \sigma^* B)$, which by restriction of scalars, is a Morita R-equivalence. Also, $(\sigma^{-1})^* : \mathsf{Coh}(U, \sigma^* B) \to \mathsf{Coh}(V, B)$ is an R-equivalence. Composing these yields the required Morita R-equivalence.

Now suppose that $F : \mathsf{Coh}(U, A) \to \mathsf{Coh}(V, B)$ is a Morita R-equivalence. Since F is essentially surjective, we can choose $P \in \mathsf{Coh}(U, A)$ such that $F(P) \cong B^{\mathrm{op}}$ as B-modules. Since F is fully faithful, any choice of isomorphism $\theta : F(P) \cong B^{\mathrm{op}}$ defines a left B-module structure on P. In this way, P has a B-A-module structure with commuting R-structure. In particular, for any A-module P', $\mathrm{Hom}_A(P, P')$ has a natural right B-module structure via precomposition.

The equivalence F being R-linear yields an induced R-algebra isomorphism

$$\psi : \mathrm{End}_A(P) = \mathrm{Hom}_A(P, P) \cong \mathrm{Hom}_B(F(P), F(P)) \cong \mathrm{End}_B(B^{\mathrm{op}}) = B.$$

In particular, ψ restricts to an R-algebra isomorphism $\sigma : U \to V$ of the centers, hence becomes a U-module isomorphism $\mathrm{End}_A(P) \to \sigma^* B = (\sigma^* B)^{\mathrm{op}}$, so that A is Brauer equivalent to $\sigma^* B$. □

Corollary 1.18. *Let A and B be simple k-algebras with respective centers K and L. Then $\mathsf{D}^{\mathrm{b}}(K/k, A)$ and $\mathsf{D}^{\mathrm{b}}(L/k, B)$ are (k-linearly) equivalent if and only if there exists some k-automorphism $\sigma : K \to L$ such that A and $\sigma^* B$ are Brauer equivalent over K.*

Proof. By [110, Cor. 2.7], if A or B is either commutative or local artinian (not necessarily commutative), then they are derived k-equivalent if and only if they are Morita k-equivalence. Then we apply Theorem 1.17. □

We remark that Corollary 1.18 is also a consequence of the results of [6].

Lemma 1.19. *A k-linear triangulated category T is representable in dimension 0 if and only if there exists a semiorthogonal decomposition*

$$\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_r \rangle,$$

such that for each i, there is a k-linear equivalence $\mathsf{A}_i \simeq \mathsf{D}^{\mathrm{b}}(K_i/k)$ for an étale k-algebra K_i.

Proof. The smooth k-varieties of dimension 0 are precisely the spectra of étale k-algebras. Hence the semiorthogonal decomposition condition is certainly sufficient for the representability of T in dimension 0. On the other hand, if T is representable in dimension 0, we have that each A_i an admissible subcategory of the derived category of an étale k-algebra. By Lemma 1.16*(iii)*, we have that A_i is thus itself such a category. □

2. Descent for semiorthogonal decompositions

Given a smooth projective variety X over a field k, and fixing a separable closure k^s of k, we are interested in comparing k-linear semiorthogonal decompositions of $\mathsf{D}^{\mathrm{b}}(X)$ and k^s-linear semiorthogonal decompositions of $\mathsf{D}^{\mathrm{b}}(X_{k^s})$. The general question of how derived categories and semiorthogonal decompositions behave under base field extension has started to be addressed by several authors [3, §2], [4], [7], [47], [74], [97], [105]. Galois descent does not generally hold for objects in a triangulated category, due to the fact that cones are only defined up to quasi-isomorphism. However, for a linearly reductive algebraic group G acting on a variety X,

a general theory of descent of semiorthogonal decompositions has been developed by Elagin [47, 46, 45]. In the case where K/k is a finite G-Galois extension and we consider G acting on X_K via k-automorphisms, this theory works as long as the characteristic of k does not divide the order of G.

We are then faced with two main questions. First, given a k^s-linear triangulated category $\overline{\mathsf{T}}$, what are all the k-linear triangulated categories T such that T_{k^s} is equivalent to T? This first question is very challenging (see [4] for some examples), and we usually restrict ourselves to consider T_{k^s} in a restricted class of categories (e.g., those generated by a strong exceptional collection). Notice that we are actually dealing with triangulated categories which admit a canonical dg-enhancement, and this should be the natural structure to consider. Second, given a set of admissible subcategories in $\mathsf{D}^{\mathrm{b}}(X)$, determine how this can be related to the semiorthogonal decomposition of $\mathsf{D}^{\mathrm{b}}(X_{k^s})$.

In our geometric applications, a central rôle is played by descent of exceptional blocks and vector bundles.

2.1. **Scalar extension of triangulated categories.** Let X be a k-scheme and V an $\mathscr{O}_X$-module. For a ring extension K/k we write $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ and $f_K : X_K \to X$ for the projection and $V_K = f_K^* V$ for the *extension* of V to K. If K is a finite k-algebra and F is an $\mathscr{O}_{X_K}$-module, we denote by $\operatorname{tr}_{K/k} F = (f_K)_* F$ the *trace* of F from K down to k. We similarly define the trace of an object of $\mathsf{D}^{\mathrm{b}}(X_K)$. If F is locally free of rank r on X_K then $\operatorname{tr}_{K/k} F$ is locally free of rank $r[K:k]$ on X. We denote by $G_k = \operatorname{Gal}(k^s/k)$ the absolute Galois group of k.

If T is a k-linear triangulated category with a dg-enhancement and K/k is a field extension, then denote by T_K the K-linear extension of scalars category defined in [97]. This category is a thick dg-enhanced K-linear triangulated category. As expected, there is a K-linear equivalence $\mathsf{D}^{\mathrm{b}}(X)_K \simeq \mathsf{D}^{\mathrm{b}}(X_K)$. If K/k is a Galois extension, then the Galois group acts by k-linear equivalences on T_K. If K/k is a finite G-Galois extension then G acts on X_K as a k-scheme and there is a k-linear equivalence $\mathsf{D}^{\mathrm{b}}(X) \simeq \mathsf{D}^{\mathrm{b}}_G(X_K)$ with the bounded derived category of G-equivariant coherent sheaves on X_K considered as a k-scheme.

Given an orthogonal pair of admissible subcategories $\{\mathsf{A}, \mathsf{B}\}$ in $\mathsf{D}^{\mathrm{b}}(X)$, we can perform right and left mutations. These operations commute with base change, i.e., $L_{\mathsf{A}}(\mathsf{B})_K = L_{\mathsf{A}_K}(\mathsf{B}_K)$, for any finite extension K of k, and similarly for the right mutation.

Remark 2.1. Assume that $[K:k]$ is finite and that X is a smooth projective k-variety. For an object V in $\mathsf{D}^{\mathrm{b}}(X)$, the scalar extension V_K in $\mathsf{D}^{\mathrm{b}}(X)_K = \mathsf{D}^{\mathrm{b}}(X_K)$ satisfies $\operatorname{End}_{\mathsf{T}_K}(V_K) = \operatorname{End}_{\mathsf{T}}(V) \otimes_k K$, see [53, §0.5.4.9, 1.9.3.3]. If A is a division k-algebra, then $A \otimes_k K$ may not be. Hence if V is an exceptional object in $\mathsf{D}^{\mathrm{b}}(X)$, then V_K may fail to be an exceptional object in $\mathsf{D}^{\mathrm{b}}(X_K)$. However, if V is k-exceptional, then V_K is always K-exceptional.

Lemma 2.2. *Let T be an admissible k-linear subcategory of $\mathsf{D}^{\mathrm{b}}(X)$ for a smooth projective k-variety X. Let K be any extension of k. Then T_K is admissible in $\mathsf{D}^{\mathrm{b}}(X_K)$.*

Proof. By Kuznetsov [74], the inclusion functor of the admissible subcategory $\mathsf{T} \to \mathsf{D}^{\mathrm{b}}(X)$ has an adjoint $\rho : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{T}$ that is of Fourier–Mukai type with kernel P an object in $\mathsf{D}^{\mathrm{b}}(X \times X)$. Taking $P \otimes_k K$ in $\mathsf{D}^{\mathrm{b}}(X_K \times X_K)$ as the kernel of a Fourier–Mukai functor, we obtain an adjoint for $\mathsf{T}_K \to \mathsf{D}^{\mathrm{b}}(X_K)$. □

Based on base change results for Fourier–Mukai functor due to Orlov (see [85]), the following statement was proved in [11, Lemma 2.9] (see also [3, Prop. 2.1]).

Lemma 2.3. *Let X be a smooth projective variety over k, and K a finite extension of k. Suppose that $\mathsf{T}_1, \dots, \mathsf{T}_n$ are admissible subcategories of $\mathsf{D}^{\mathrm{b}}(X)$ such that $\mathsf{D}^{\mathrm{b}}(X_K) = \langle \mathsf{T}_{1K}, \dots, \mathsf{T}_{nK} \rangle$. Then $\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{T}_1, \dots, \mathsf{T}_n \rangle$.*

2.2. **Classical descent theory for vector bundles.** For a vector bundle V on a scheme X, denote by $A(V) = \operatorname{End}(V)/\operatorname{rad}(\operatorname{End}(V))$ and $Z(V)$ the center of $A(V)$. Assuming that X is a proper k-scheme, then vector bundles on X enjoy a Krull–Schmidt decomposition, hence $A(V)$

is a semisimple k-algebra. If $V = \bigoplus_{i=1}^m V_i^{\oplus d_i}$ is the Krull–Schmidt decomposition of V, then $A(V)$ is the product of the algebras $M_{d_i}(A(V_i))$. In particular, V is indecomposable on X if and only if $A(V)$ is a division algebra over k.

If K/k is a field extension and E is a vector bundle, then we have $\mathrm{End}(V_K) = \mathrm{End}(V)_K$ (cf. Remark 2.1), and hence $A(V_K) = A(V)_K$ if K/k is separable.

Lemma 2.4 ([9, Lemma 1.1])**.** *Let V be an indecomposable vector bundle on X. Let K/k be a normal extension containing $Z(V)$ and splitting $A(V)$. Write $m = [Z(V) : k]_{\mathrm{sep}}$ and $d = \deg_{Z(V)}(A(V))$. Then V_K has a Krull–Schmidt decomposition of the form $V_K = \bigoplus_{i=1}^m V_i^{\oplus d}$ where V_i are indecomposable over X_K and $A(V_i) = K$.*

Let W be an indecomposable vector bundle on X_{k^s}. A vector bundle V on X is *pure (of type W)* if V_{k^s} is a direct sum of vector bundles isomorphic to W.

Proposition 2.5 ([9, Prop. 3.4])**.** *Let X be a proper variety over k. If W be a G_k-invariant indecomposable vector bundle on X_{k^s} then there exists a pure indecomposable vector bundle V on X, unique up to isomorphism, of type W.*

Definition 2.6. Given a G_k-invariant indecomposable vector bundle W on X_{k^s}, let V be the vector bundle on X given in Proposition 2.5. Define $\alpha(W) \in \mathrm{Br}(k)$ to be the Brauer class of $A(V)$. In fact, $\alpha(W)$ is the Brauer class of $A(V')$ for any pure vector bundle V' on X of type W.

Remark 2.7. Recall the Brauer obstruction for Galois invariant line bundles on a smooth proper geometrically integral variety X over k, see [100, §2]. The sequence of low degree terms of the Leray spectral sequence for $\mathbb{G}_{\mathrm{m}}$ associated to the structural morphism $X \to \operatorname{Spec} k$ is

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{k^s})^{G_k} \xrightarrow{d} \mathrm{Br}(k) \to \mathrm{Br}(X). \tag{2.1}$$

The differential d is called the Brauer obstruction for a Galois invariant line bundle; a class in $\mathrm{Pic}(X_{k^s})^{G_k}$ descends to a line bundle on X if and only if it has trivial Brauer obstruction. The cokernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k^s})^{G_k}$ is torsion since the Brauer group is. We also recall that if A is a central simple algebra with class $d(L)$, for a Galois invariant line bundle L, and Y is the Severi–Brauer variety associated to A, then up to tensoring by elements of $\mathrm{Pic}(X)$, we can assume that L corresponds to a morphism $X \to Y$, see [100, Prop. 8].

Corollary 2.8 ([9, Prop. 3.15])**.** *Let r be the index of the Brauer class $\alpha(W) \in \mathrm{Br}(k)$. Then $V_{k^s} \cong W^{\oplus r}$, and hence $c_1(V_{k^s}) = rc_1(W)$ is in the image of $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k^s})^{G_k}$. In particular, W descends to X if and only if $\alpha(W)$ is trivial*

Theorem 2.9 ([9, Thm. 1.8])**.** *Let X be a proper variety over k and V an indecomposable vector bundle on X. If K is a maximal subfield of $A(V)$ then there is an absolutely indecomposable vector bundle W on X_K such that $V \cong \mathrm{tr}_{K/k} W$.*

Definition 2.10. Let V be a vector bundle on a proper smooth scheme X over k. We write $V = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$ the Krull–Schmidt decomposition, with $V_1, \ldots, V_r$ indecomposable. The *reduced part* of V is defined to be $V^{\min} = V_1 \oplus \cdots \oplus V_r$. We remark that V and $V^{\min}$ generate the same thick subcategory of $\mathsf{D}^{\mathrm{b}}(X)$.

2.3. **Descent of exceptional blocks.** If T_{k^s} admits a full exceptional block, we wish to classify all the k-linear categories T base-changing to T_{k^s}. In particular, if T_{k^s} is generated by an exceptional block of vector bundles (i.e., it admits a tilting bundle) in $\mathsf{D}^{\mathrm{b}}(X_{k^s})$ for some smooth proper scheme X, the descent of T_{k^s} to T inside $\mathsf{D}^{\mathrm{b}}(X)$ is described by descent of the tilting bundle.

When T_{k^s} is generated by a k^s-exceptional object, the descent question has been studied by Toën [105]. Recall (see Corollary 1.18) that Brauer equivalence between Azumaya algebras over a ring R is the same as Morita R-equivalence, and that Morita k-equivalence between local (possibly noncommutative) k-algebras is the same as derived k-equivalence by [110, Cor. 2.7].

Theorem 2.11 (Toën [105, Cor. 2.15])**.** *If T is a k-linear triangulated category such that $\mathsf{T}_{k^s} = \mathsf{D}^{\mathrm{b}}(k^s)$, then there exists a central simple algebra A over k such that $\mathsf{T} = \mathsf{D}^{\mathrm{b}}(k, A)$. In particular, T is generated by a k-exceptional object if and only if A is trivial in the Brauer group.*

Proof. Since $\mathsf{D}^{\mathrm{b}}(k^s)$ has compact generator, then so does T by [105, Prop. 4.12]. Then the dg-category T is equivalent to $\mathsf{D}^{\mathrm{b}}(k, A)$ for some dg-algebra A over k. As A_{k^s} is Morita equivalent to k^s and k^s/k is faithfully flat, then by [105, Prop. 2.3], A is a derived Azumaya algebra over k. By [105, Prop. 2.12], A is Morita equivalent to an Azumaya algebra over k. □

Recall that K is a finite étale k-algebra if and only if $K \otimes_k k^s \cong k^s \times \cdots \times k^s$. Equivalently, $K \cong l_1 \times \cdots \times l_m$ where each l_i/k is a finite separable field extension. The k-dimension of K is called the *degree* of K over k. An Azumaya algebra A over $K \cong l_1 \times \cdots \times l_m$ is simply a product $A \cong A_1 \times \cdots \times A_m$ where each A_i is a central simple l_i-algebra. A *separable algebra* of dimension nr^2 over k is an Azumaya algebra of degree r over an étale algebra K over k of dimension n.

The set of k-isomorphism classes of étale k-algebras of degree n is in bijection with the Galois cohomology set $H^1(k, S_n)$. The set of k-isomorphism classes of degree n Azumaya algebras over an étale k-algebra of degree r is in bijection with the Galois cohomology set $H^1(k, G)$, where G is the group scheme defined by the functor $G(R) = \mathrm{Aut}_R(M_r(R^n))$. Note that G is isomorphic to the wreath product $\mathrm{PGL}_r \wr S_n = \mathrm{PGL}_r^n \rtimes S_n$ with S_n acting via permutation of the factors. Then there is a natural map $H^1(k, G) \to H^1(k, S_n)$ given by taking the isomorphism class of the center (this map is clearly surjective), whose fiber over the class of an étale algebra K of dimension n over k is given by the set of isomorphism classes of Azumaya algebras over K of degree r.

Finally, by the classical theory of central simple algebras over a field, each central simple k-algebra A of degree n contains a maximal étale k-subalgebra K of degree n splitting A, i.e., such that $A \otimes_k K \cong M_n(K)$.

Proposition 2.12. *Let T be a k-linear triangulated category such that T_{k^s} is k^s-equivalent to $\mathsf{D}^{\mathrm{b}}(k^s, (k^s)^n)$. Then there exists an étale algebra K of degree n over k, an Azumaya algebra A over K, and a k-linear equivalence $\mathsf{T} \simeq \mathsf{D}^{\mathrm{b}}(K/k, A)$. In this case, T is an indecomposable category if and only if K is a field extension of k.*

Proof. Toën [105, Cor. 3.12] shows that the derived group stack of autoequivalences $\mathbf{Aut}_{dAz/k}(k)$ of the trivial derived Azumaya k-algebra k is equivalent to $\mathbb{Z} \times K(\mathbb{G}_{\mathrm{m}}, 1)$, as any dg-autoequivalence of an étale k-algebra R is given by tensoring with some invertible perfect dg-R-module, which by [106, Thm. 8.15], are all of the form $L[n]$ for an invertible R-module L and $n \in \mathbb{Z}$.

As a generalization, we claim that the derived group stack of autoequivalences $\mathbf{Aut}_{dg/k}(k^n)$ of the étale k-algebra k^n (thought of as a dg-algebra over k), is equivalent to the wreath product $(\mathbb{Z} \times K(\mathbb{G}_{\mathrm{m}}, 1)) \wr S_n$, thought of as $n \times n$ generalized permutation matrices filled with shifts of invertible modules. Indeed, for an étale k-algebra R, any dg-endofunctor of R^n can be viewed as an $n \times n$ matrix M of perfect dg-R-modules by [106, Thm. 8.15]. Considering the $n \times n$ matrix $\overline{M}$ of ranks of the entries of M, we see that M is invertible implies that $\overline{M}$ is invertible and its inverse also consists of nonnegative integers, which is well-known to imply that $\overline{M}$ is a permutation matrix (i.e., $\mathrm{GL}_n(\mathbb{N}) = S_n$). Hence M must have exactly a single nonzero module in each row and column, and this entry must have rank 1, giving the desired form.

We are now interested in the stack $\mathbf{F} = \mathbf{Forms}_{dg/k}(k^n)$ associated to the prestack of dg-algebras étale locally Morita equivalent to k^n, which is simply $K(\mathbf{Aut}_{dg/k}(k^n), 1)$, as in [105, Cor. 3.12]. The exact sequence of group stacks

$$1 \to (\mathbb{Z} \times K(\mathbb{G}_{\mathrm{m}}, 1))^n \to \mathbf{Aut}_{dg/k}(k^n) \to S_n \to 1$$

gives rise to a fibration of stacks

$$(K(\mathbb{Z}, 1) \times K(\mathbb{G}_{\mathrm{m}}, 2))^n \to \mathbf{F} \to K(S_n, 1).$$

Hence to any dg-algebra A étale locally Morita equivalent to k^n, we have a class $\phi(A)$ in $\mathbf{F}(k)$ (the sections over k of the stack $\mathbf{F}$) and the association $A \mapsto \phi(A)$ is injective. Hence $\mathbf{F}(k)$ fits

into an exact sequence

$$\mathrm{Br}(k)^n \to \mathsf{F}(k) \to H^1(k, S_n) \to 1,$$

(using that $H^1(k, \mathbb{Z}) = 0$) where the map $z : \mathsf{F}(k) \to H^1(k, S_n)$ sends $\phi(A)$ to the isomorphism class of the center $Z(A)$. By a standard cohomological twisting argument, the fiber of z over an étale algebra K of degree n over k, is the image of $\mathrm{Br}(K)$. This gives the desired description. The final claim follows since for any field extension K over k, the k-linear category $\mathsf{D}^{\mathrm{b}}(K/k, A)$ is indecomposable; conversely, given any decomposition $K = K_1 \times K_2$, there is an induced decomposition $A = A_1 \times A_2$, with A_i Azumaya over K_i, and then $\mathsf{D}^{\mathrm{b}}(K/k, A) \simeq \mathsf{D}^{\mathrm{b}}(K_1/k, A_1) \times \mathsf{D}^{\mathrm{b}}(K_2/k, A_2)$. □

Let us state explicitly the following corollary which will be extensively used in our applications.

Corollary 2.13. *Let X be a smooth projective k-variety, K/k a Galois extension with group G. Suppose that there is an admissible subcategory T of $\mathsf{D}^{\mathrm{b}}(X)$ such that $\mathsf{T}_K = \langle V_1, \dots, V_n \rangle$ is an exceptional block of vector bundles on X_K. Then for any object A in T the Chern class $c_1(A)$ is a $\mathbb{Z}$-linear combination of the $c_1(V_i)$. In particular, there exists a nonzero G-invariant $\mathbb{Z}$-linear combination of the $c_1(V_i)$.*

Proof. Proposition 2.12 implies that T has cohomological dimension 0, so that every object is the direct sum of its cohomologies, and this decomposition is clearly G-invariant. By the thickness of T, we can then suppose that A is a sheaf. Then A_K is a direct sum of the vector bundles V_i, and is nontrivial if $A \neq 0$. □

Let us end this section by illustrating Lemma 2.3 by examples known in the literature of descent of vector bundles.

Example 2.14. Let X be the Severi–Brauer variety associated to a central simple algebra A of degree $n+1$ over k, see Example 3.2 for definitions. Let K/k be a field extension such that $X_K \simeq \mathbb{P}^n_K$. Thanks to Beilinson [17], we have the following semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(\mathbb{P}^n_K) = \langle \mathscr{O}_{\mathbb{P}^n_K}, \mathscr{O}_{\mathbb{P}^n_K}(1), \dots, \mathscr{O}_{\mathbb{P}^n_K}(n) \rangle \tag{2.2}$$

On the other hand, $\mathsf{D}^{\mathrm{b}}(X)$ admits the following semiorthogonal decomposition[1] [19, Cor. 4.7]

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k, A), \dots, \mathsf{D}^{\mathrm{b}}(k, A^n) \rangle. \tag{2.3}$$

We can see the semiorthogonal decomposition (2.2) as the base change of (2.3) using descent of vector bundles as follows. The exceptional line bundle $\mathscr{O}_{\mathbb{P}^n_K}$ clearly descends to X, and hence generates $\mathsf{D}^{\mathrm{b}}(k)$ inside $\mathsf{D}^{\mathrm{b}}(X)$. Consider now the exceptional line bundle $\mathscr{O}_{\mathbb{P}^n_K}(1)$, and consider it as a stand-alone block. It is well known (cf. [91, §8.4]), that $V = \mathscr{O}_{\mathbb{P}^n_K}(1)^{\oplus n+1}$ descends to X and that $\mathrm{End}(V) = A$. Similar arguments apply to $\mathscr{O}_{\mathbb{P}^n_K}(i)$. Hence the decomposition (2.3) of $\mathsf{D}^{\mathrm{b}}(X)$ can be obtained by descending the exceptional collection (2.2) of $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^n_K)$.

Other known (similar) examples include: the decomposition of a relative Severi–Brauer variety [19], whose base change is the decomposition of a projective bundle given by Orlov [84]; the decomposition of a generalized Severi–Brauer variety given by Blunk [26], whose base change is the decomposition of a Grassmannian variety given by Kapranov [60]; and the decomposition of a quadric hypersurface given in [11] (generalizing the work of Kuznetsov [71]), whose base change is the decomposition of a quadric given by Kapranov [60].

Remark 2.15. One might wonder, motivated by the previous examples, whether the descent of an exceptional block $\mathsf{E} \subset \mathsf{D}^{\mathrm{b}}(X_{k^s})$, generated by vector bundles V_i can be realized by the descent of the tilting bundle $V = \bigoplus_i V_i$. If V were Galois invariant, the results of §2.2 would produce a vector bundle W on X pure of type V. Moreover, this would explicitly let us consider the

[1] In [19], the full and faithful embedding of the subcategory of A^{-1}-modules into $\mathsf{D}^{\mathrm{b}}(X)$ is given by the choice of an "étale local form" $\mathscr{F}$ of $\mathscr{O}_X(1)$; an A-module M is sent to $M \otimes \mathscr{F}$, which can be endowed with the structure of an $\mathscr{O}_X$-module. Notice that in *loc.cit.* A^{-1} is the algebra obstructing the descent of $\mathscr{O}(1)$, so that our notations are opposite to the ones used there.

endomorphism algebra $\mathrm{End}(W)$ to obtain an algebraic description of the descended category. We wonder whether a converse statement holds: if the block E descends, then is the tilting bundle V Galois-invariant?

3. Geometrically rational surfaces

In this section we collect together some known results on geometrically rational surfaces, including their classification, Chow groups, and derived categories.

3.1. **Classification and first properties.** Let k be a field, k^s a separably closure, and $\bar{k}$ an algebraic closure. A smooth projective geometrically integral surface S over k such that $\overline{S} = S \times_k \bar{k}$ is $\bar{k}$-rational is called a *geometrically rational surface*. Recall that S is a *del Pezzo* surface if $\omega_S^\vee$ is ample. The *degree* of a geometrically rational surface is the self-intersection number $d = \omega_S \cdot \omega_S$.

We say that a field extension l of k is a *splitting field* for S if $S \times_k l$ is birational to $\mathbb{P}^2_l$ via a sequence of monoidal transformations centered at closed l-points. An important fact is that geometrically rational surfaces are separably split. The following result allows one to consider the separable closure k^s instead of the algebraic closure.

Proposition 3.1 ([42, Thm. 1], [108, Thm. 1.6])**.** *If S is a geometrically rational surface over a field k, then S is split over k^s.*

A surface S is *minimal* over k, or *k-minimal*, if any birational morphism $f : S \to S'$, defined over k, is an isomorphism. Over a separably closed field, the only minimal rational surfaces are the projective plane and projective bundles over the projective line. Over a general field, this is no longer true. Minimal geometrically rational surfaces have been completely classified, and we have the following list (see [82], and the recent [54]):

(i) $S = \mathbb{P}^2_k$ is a projective plane, so $\mathrm{Pic}(S) = \mathbb{Z}$, generated by the hyperplane $\mathscr{O}(1)$;
(ii) $S \subset \mathbb{P}^3_k$ is a smooth quadric and $\mathrm{Pic}(S) = \mathbb{Z}$, generated by the hyperplane section $\mathscr{O}(1)$;
(iii) S is a del Pezzo surface with $\mathrm{Pic}(S) = \mathbb{Z}$, generated by the canonical class ω_S;
(iv) S is a conic bundle $f : S \to C$ over a geometrically rational curve, with $\mathrm{Pic}(S) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Example 3.2. A *Severi–Brauer variety* is a variety X over k such that $\overline{X} \cong \mathbb{P}^{n-1}_{\bar{k}}$ for some $n \geq 2$. The set of isomorphism classes of Severi–Brauer varieties X of dimension $n-1$ over k is in bijection with the set of k-isomorphism classes of central simple algebras A of degree n over k, and we will write $X = \mathrm{SB}(A)$ accordingly. By a theorem of Châtelet, a Severi–Brauer variety X is k-rational if and only if X is k-isomorphic to projective space if and only if $X(k) \neq \varnothing$, cf. [50, Thm. 5.1.3]. In this case, we say that X *splits* and remark that X always splits after a finite separable field extension. The Galois action on $\mathrm{Pic}(X_{k^s}) = \mathrm{Pic}(\mathbb{P}^{n-1}_{k^s}) = \mathbb{Z}$ is trivial since it preserves dimensions of spaces of global sections. It is a theorem of Lichtenbaum that the Brauer obstruction (see Remark 2.7) for the Galois invariant class of $\mathscr{O}_{\mathbb{P}^2_{k^s}}(1)$ to descend to X is precisely the Brauer class of A, cf. [50, Thm. 5.4.10]. Thus for a nonsplit Severi–Brauer variety X, the low terms of the Leray spectral sequence (see (2.1)) show that ω_X generates $\mathrm{Pic}(X)$.

A *Severi–Brauer surface* S is a Severi–Brauer variety of dimension 2, hence is a minimal del Pezzo surface. As intersection numbers do not change under scalar extension, S has degree 9. By the above analysis of the Picard group, a nonsplit Severi–Brauer surface belongs to the case *(iii)*, while the split Severi–Brauer surface $\mathbb{P}^2_k$ belongs to case *(i)*.

Example 3.3. An *involution variety* is a variety X over k such that $\overline{X}$ is $\bar{k}$-isomorphic to a smooth quadric in $\mathbb{P}^n_{\bar{k}}$ for some $n \geq 2$. The set of isomorphism classes of involution varieties over k is in bijection with the set of k-isomorphism classes of central simple algebras (A, σ) of degree $n+1$ over k together with a *quadratic pair* σ, a generalization to arbitrary characteristic of the notion of an orthogonal involution, see [67, §5.B] for definitions. In this case, X is a degree 2 hypersurface in the Severi–Brauer variety $\mathrm{SB}(A)$. Attached to an involution variety X is the even Clifford algebra $C_0(A, \sigma)$, defined by Jacobson [59] via Galois descent in characteristic not 2 and by [67, §8] in general. If A is split, then the quadratic pair on A is adjoint to a

quadratic form, uniquely defined up to scaling and $X \subset \mathrm{SB}(A) \cong \mathbb{P}^n_k$ is the associated quadric hypersurface. We say that X is an *anisotropic quadric* if A is split yet $X(k) = \varnothing$.

From now on, we assume that X has even dimension. In this case, $C_0(A, \sigma)$ is an Azumaya algebra over its center l, which is an étale quadratic extension of k, called the *discriminant extension* of X. The Galois action on $\mathrm{Pic}(X_{k^s}) \cong \mathbb{Z}^2$, since it preserves dimensions of spaces of global sections factors through a permutation of the divisors defining the two rulings of the quadric over k^s. This action coincides with the Galois action on the embeddings $l \to k^s$ of the discriminant extension. By a comparison of the low degree terms of the Leray spectral sequences for $X \to \operatorname{Spec} k$ and $\mathrm{SB}(A) \to \operatorname{Spec} k$, we see that the Brauer obstruction to the Galois invariant class of $\mathscr{O}_{X_{k^s}}(1)$ (which is the sum of the two classes of rulings) is precisely the Brauer class of A. If the discriminant extension is trivial, then both ruling classes are Galois invariant, and each one could have its own Brauer obstruction. Thus for an involution variety X, the low terms of the Leray spectral sequence (see (2.1)) show that ω_X generates $\mathrm{Pic}(X)$ if and only if the discriminant extension is nontrivial.

An *involution surface* S is an involution variety of dimension 2, i.e., $S_{k^s} \cong \mathbb{P}^1_{k^s} \times \mathbb{P}^1_{k^s}$. In particular, S is a minimal geometrically rational del Pezzo surface. As intersection numbers do not change under scalar extension, S has degree 8. The set of k-isomorphism classes of involution surfaces is also in bijection with the set of isomorphism classes of pairs (l, B), where l is an étale quadratic extension of k and B is a quaternion algebra over l, see [67, §15.B]. Given an involution variety S corresponding to a central simple algebra (A, σ) of degree 4 with quadratic pair, the even Clifford algebra $C_0(A, \sigma)$ is a quaternion algebra over the discriminant extension. Conversely, given a pair (l, B), the associated involution variety is the Weil restriction $S = R_{l/k}\mathrm{SB}(B)$. As far as placing involution surfaces into the classification of minimal geometrically rational surfaces, there are several cases: If the discriminant extension is trivial, then S belongs to case *(iv)*. In this case $S \cong C_1 \times C_2$ is a product of Severi–Brauer curves. Writing $C_i = \mathrm{SB}(B_i)$ for quaternion algebras B_i over k, then $A \cong B_1 \otimes B_2$. In this case, the Brauer class of A is trivial if and only if $C_1 \cong C_2$. If the discriminant extension is nontrivial, then S belongs to case *(ii)* or *(iii)* depending on whether the Brauer class of A is trivial or not, respectively.

Example 3.4. Let S be a geometrically rational del Pezzo surface of degree 8 not isomorphic to an involution variety. The unique exceptional curve on S_{k^s} is a Galois invariant subvariety, hence can be contracted to arrive at a del Pezzo surface of degree 9 with a rational point, which is thus $\mathbb{P}^2_k$. Thus $S \to \mathbb{P}^2_k$ is the blow up of $\mathbb{P}^2_k$ at a single k-rational point. In particular, S is a rational del Pezzo surface and is never minimal.

Let X be a geometrically rational del Pezzo surface of degree 7. As S_{k^s} is the blow-up of $\mathbb{P}^2_{k^s}$ at two points, the exceptional divisor consists of a Galois invariant pair of (-1)-curves, hence can be contracted to arrive at a del Pezzo surface of degree 9 with a point of degree 2, which is thus $\mathbb{P}^2_k$. Thus $S \to \mathbb{P}^2_k$ is the blow up of $\mathbb{P}^2_k$ at a closed point of degree 2. In particular, S is a rational del Pezzo surface and is never minimal.

Remark 3.5. If S is a minimal del Pezzo surface of degree ≤ 6, then $\mathrm{Pic}(S) \to \mathrm{Pic}(S_{k^s})^{G_k}$ is an isomorphism, cf. [41, Lemma 2.5, Prop. 5.3].

We continue our general discussion of geometrically rational surfaces. Denote by $\rho(S)$ the Picard rank of S.

Proposition 3.6. *If S is a geometrically rational surface over a field k, then $K_0(S)_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space of dimension $2 + \rho(S)$. In particular, if S is minimal then $K_0(S)_{\mathbb{Q}}$ has dimension 3 or 4, and in the latter case S has a conic bundle structure.*

Proof. The Chern character

$$ch : K_0(S) \otimes_{\mathbb{Z}} \mathbb{Q} \to \bigoplus_i \mathrm{CH}^i(S) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. We have $\mathrm{CH}^0(S) \cong \mathbb{Z}$ and $\mathrm{CH}^1(S) \cong \mathrm{Pic}(S)$. Finally, to describe $\mathrm{CH}^2(S) = \mathrm{CH}_0(S)$, we have an exact sequence

$$0 \to A_0(S) \to \mathrm{CH}_0(S) \xrightarrow{\deg} \mathbb{Z} \to \mathbb{Z}/i(S)\mathbb{Z} \to 0$$

where $A_0(S)$ is defined to be the kernel of the degree map and $i(S)$ is the *index*, i.e., the greatest common divisor of degrees of closed points on S. Since S is rational after a finite separable extension (by Proposition 3.1), a restriction-corestriction argument shows that $A_0(S)$ is torsion, cf. [40, Prop. 6.4]. This implies that the degree map becomes an isomorphism $\mathrm{CH}_0(S)\otimes_{\mathbb{Z}}\mathbb{Q} \xrightarrow{\simeq} \mathbb{Q}$ after tensoring with $\mathbb{Q}$. This completes the calculation of the dimension. The final statement is a result of the classification of minimal geometrically rational surfaces. □

Corollary 3.7. *Let S be a geometrically rational surface over k. If there exist field extensions $l_1, \dots, l_n$ of k and Azumaya algebras A_i over l_i, such that there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(l_1/k, A_1), \dots, \mathsf{D}^{\mathrm{b}}(l_n/k, A_n)\rangle,$$

then $n = 2 + \rho(S)$. In particular, S is categorically representable in dimension 0 if and only if there exist field extensions $l_1, \dots, l_n$, with $n = \rho(S) + 2$, and a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(l_1/k), \dots, \mathsf{D}^{\mathrm{b}}(l_n/k)\rangle. \tag{3.1}$$

Proof. The first statement is a corollary of Proposition 3.6, since the semiorthogonal decomposition gives a splitting

$$K_0(S) = \bigoplus_{i=1}^{n} K_0(l_i, A_i)$$

and $K_0(l, A) \cong \mathbb{Z}$ for any field l and any Azumaya algebra A over l. To prove the second statement notice that S is categorically representable in dimension zero if and only if a semiorthogonal decomposition like (3.1) exists by Lemma 1.19, and the number of component is given by the first statement. □

Let us recall a straightforward consequence of Orlov's result on blow-ups, reducing the question of being categorically representable in dimension 0 to minimal surfaces.

Lemma 3.8. *Let S be a smooth projective non-minimal surface over k. Then there is a smooth projective minimal surface S', and a fully faithful functor $\Phi : \mathsf{D}^{\mathrm{b}}(S') \to \mathsf{D}^{\mathrm{b}}(S)$ such that the orthogonal complement of $\Phi(\mathsf{D}^{\mathrm{b}}(S'))$ is representable in dimension 0.*

Proof. Since S is not minimal, there exists a k-birational morphism $\pi : S \to S'$ to a minimal surface. Then π is the blow-up of a closed zero-dimensional subvariety $Z \subset S'$. The proof follows from Orlov blow-up formula [84]. □

Manin has proved that, given a (non necessarily minimal) del Pezzo surface of degree $d \geq 2$, the existence of a k-rational point (not lying on any exceptional curve if $d \leq 4$) implies the existence of a unirational parametrization, i.e., a map $\mathbb{P}^2_k \to S$ of finite degree.

Theorem 3.9 (Manin [82, Thm. 29.4])**.** *Let S be a del Pezzo surface of degree $d \geq 2$ over k with $S(k) \neq \varnothing$. If $d \leq 4$ suppose moreover that the point does not lie on any exceptional curve. Then there exists a rational map $\phi : \mathbb{P}^2_k \to S$ whose degree δ_d is given by the following table*

d	≥ 5	4	3	2
δ_d	1	2	6	24

In particular, if $d \geq 5$, the surface S has a k-rational point if and only if it is k-rational.

Minimal del Pezzo surfaces of degree ≤ 7 can be characterized by the Galois action on exceptional curves. We say that a del Pezzo surface S over k is *totally split* if S is k-rational and all exceptional curves are defined over k. Any field extension of k over which a del Pezzo surface becomes totally split will be called a *total splitting field* for S. We can always choose a finite Galois total splitting field for a del Pezzo surface. We remark that S can be split, but not necessarily totally split, over a given field.

We end this section by recalling the following classification of birational maps between non-rational minimal del Pezzo surfaces, which can be proved by classifying all the possible links in the Sarkisov program (see [58]).

Proposition 3.10 (Iskovskikh [58, Thms. 1.6, 4.5, 4.6])**.** *Let S be a non rational del Pezzo surface of Picard rank 1, S' a minimal surface, and $\phi : S \dashrightarrow S'$ a k-birational map.*

(i) If $\deg(S) = 1$ or if S has no closed point x of degree $< \deg(S)$, then ϕ is an isomorphism (i.e., S is birationally rigid*).*

(ii) If $\deg(S) = 2$ and $S(k) \neq \varnothing$, or if $\deg(S) = 3$ and $S(k) \neq \varnothing$ or if $\deg(S) = 4$ and S has a point of degree 2 and no point of degree 1 or 3, or if S has degree 8 and a point of degree 4 (but no point of lower degree), then ϕ can be composed with a birational map $S \dashrightarrow S$ to give an isomorphism (i.e. S is birationally semirigid*).*

(iii) If $\deg(S) = 6$ or $\deg(S) = 9$, then $\deg(S) = \deg(S')$ (i.e. S is deg-rigid*).*

(iv) If $\deg(S) = 4$ and S has a k-rational point or if $\deg(S) = 8$ and S has a degree 2 point (but no k-rational point), then the blow-up of S along such a point is a conic bundle of degree 3 or 6, respectively. In particular, S is not deg-rigid.

All non rational del Pezzo surfaces of Picard rank one are covered by one of these cases.

Proof. A proof of all of these results can be found in [58] (some of them were previously known). Item *(i)* is [58, Thm. 1.6], while items *(ii)*, *(iii)*, and *(iv)* are summarized in [58, Comment 5]. Notice that if S has degree 8 and a point of degree 4 but no point of lower degree, then [58, Thm. 4.4] only says that S is deg-rigid. In this case, S is an involution variety, for which the birational semirigidity can be proved directly via the theory of quadratic forms, see Proposition C.3.

Finally, the list is exhaustive since if $\deg(S) \leq 6$ and S has a closed point of degree $< \deg(S)$ and coprime with $\deg(S)$, then S is rational (for an argument, see [58, p. 624]). Similarly if S has degree 5 or 7 then it is rational. □

Part 2. Del Pezzo surfaces

The plan of this part is as follows. In Section 4 we recall the necessary facts concerning 3-block decompositions. In Section 5 we consider the case of degree $d \leq 4$ and prove Theorem 1 and 3. The rest of the proofs will be divided into the cases of Severi–Brauer surfaces (Section 6), involution surfaces (Section 7), del Pezzo surfaces of degree 7 (Section 8), 6 (Section 9), and 5 (Section 10).

4. Exceptional objects and blocks on del Pezzo surfaces

In this section, we assume that k is separably closed. The derived category of a totally split del Pezzo surface (over an algebraically closed field) has been extensively studied by the Moscow school in the 1990's (see, e.g., [51], [52], [92], [69, 64]), with a particular attention to the structure of exceptional collections.

While these authors restrict to working over algebraically closed fields of characteristic zero, their proofs are based on properties of vector bundles and on the description of a del Pezzo surface as a blow-up of $\mathbb{P}^2$, hence they actually hold for any totally split del Pezzo surface, in particular, over any separably closed field.

Let S be a del Pezzo surface over k. Then S is totally split and S is either a quadric surface and has degree 8, or S is a blow-up of $\mathbb{P}^2$ in $9 - d$ points, for $1 \leq d \leq 9$, and has degree d. A *3-block decomposition* of $\mathsf{D}^{\mathrm{b}}(S)$ is a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{E}, \mathsf{F}, \mathsf{G} \rangle$$

consisting of exceptional blocks. Gorodentsev–Rudakov, Rudakov, and Karpov–Nogin (see [52], [92], [64]) have proved the existence of 3-block decompositions by exceptional vector bundles with some unicity property (up to mutations). Indeed, the ranks and degrees of vector bundles are constant in a block, and there is an algorithm to compute how slopes change under mutations, as explained in [64]. A 3-block decomposition is *minimal* if any mutation increases the rank of one of the blocks. The existence of minimal decomposition is related to solutions of Markov-type equations.

deg(S)	E			F			G		
	n	r	exc. collection	n	r	exc. collection	n	r	exc. collection
9	1	1	$\mathscr{O}$	1	1	$\mathscr{O}(H)$	1	1	$\mathscr{O}(2H)$
8 (inv)	1	1	$\mathscr{O}$	2	1	$\mathscr{O}(1,0)$ $\mathscr{O}(0,1)$	1	1	$\mathscr{O}(1,1)$
8 (dP) 7	No 3-block decomposition								
6	1	1	$\mathscr{O}$	2	1	$\mathscr{O}(H)$ $\mathscr{O}(2H-L_{\leq 3})$	3	1	$\mathscr{O}(2H-L_1-L_2)$ $\mathscr{O}(2H-L_1-L_3)$ $\mathscr{O}(2H-L_2-L_3)$
5	1	1	$\mathscr{O}$	1	2	F	5	1	$\mathscr{O}(H)$ $\mathscr{O}(L_1-\omega-H)$ $\mathscr{O}(L_2-\omega-H)$ $\mathscr{O}(L_3-\omega-H)$ $\mathscr{O}(L_4-\omega-H)$
4	2	1	$\mathscr{O}(L_4)$ $\mathscr{O}(L_5)$	2	1	$\mathscr{O}(H)$ $\mathscr{O}(2H-L_{\leq 3})$	4	1	$-\omega$ $\mathscr{O}(2H-L_1-L_2)$ $\mathscr{O}(2H-L_1-L_3)$ $\mathscr{O}(2H-L_2-L_3)$
3 (i)	3	1	$\mathscr{O}(L_4)$ $\mathscr{O}(L_5)$ $\mathscr{O}(L_6)$	3	1	$\mathscr{O}(H-L_1)$ $\mathscr{O}(H-L_2)$ $\mathscr{O}(H-L_3)$	3	1	$-\omega$ $\mathscr{O}(H)$ $\mathscr{O}(2H-L_{\leq 3})$
3 (ii)	1	2	T_6	2	1	$\mathscr{O}(H)$ $-\omega$	6	1	$\mathscr{O}(L_1-\omega)$ $\mathscr{O}(L_2-\omega)$ $\mathscr{O}(L_3-\omega)$ $\mathscr{O}(L_4-\omega)$ $\mathscr{O}(L_5-\omega)$ $\mathscr{O}(L_6-\omega)$
2 (i)	1	2	E_7	1	2	T_7	8	1	$-\omega$ $\mathscr{O}(H-L_1)$ … $\mathscr{O}(H-L_7)$
2 (ii)	2	2	E_7 E_7'	4	1	$\mathscr{O}(L_4)$ $\mathscr{O}(L_5)$ $\mathscr{O}(L_6)$ $\mathscr{O}(L_7)$	4	1	$-\omega$ $\mathscr{O}(H-L_1)$ $\mathscr{O}(H-L_2)$ $\mathscr{O}(H-L_3)$
2 (iii)	1	3	E_7''	3	1	$\mathscr{O}(H-L_1)$ $\mathscr{O}(H-L_2)$ $\mathscr{O}(H-L_3)$	6	1	$\mathscr{O}(H)$ $\mathscr{O}(2H-L_{\leq 3})$ $\mathscr{O}(L_4-\omega)$ … $\mathscr{O}(L_7-\omega)$
1 (i)	1	3	E_8	1	3	F_8	9	1	$-\omega$ $\mathscr{O}(-L_1)$ … $\mathscr{O}(-L_8)$
1 (ii)	1	4	E_8'	2	2	T_8 T_8'	8	1	$\mathscr{O}(L_4-\omega)$ … $\mathscr{O}(L_8-\omega)$ $\mathscr{O}(H-L_1)$ $\mathscr{O}(H-L_2)$ $\mathscr{O}(H-L_3)$
1 (iii)	2	4	E_7'' E_8''	3	2	T_8 T_8' T_8''	6	1	$\mathscr{O}(L_4-\omega)$ $\mathscr{O}(L_5-\omega)$ $\mathscr{O}(L_6-\omega)$ $\mathscr{O}(H-L_1)$ $\mathscr{O}(H-L_2)$ $\mathscr{O}(H-L_3)$
1 (iv)	1	5	E_8'''	5	2	$F_{4,8}$ … $F_{8,8}$	5	1	$\mathscr{O}(H)$ $\mathscr{O}(2H-L_{\leq 3})$ $\mathscr{O}(H-L_1-\omega)$ … $\mathscr{O}(H-L_3-\omega)$

TABLE 1. Representative sets of exceptional objects generating the various 3-block decompositions of $\mathsf{D}^{\mathrm{b}}(S_{k^s})$, taken up to mutation, tensoring through by a line bundle, and the Weyl group action. For each block n = number of bundles in the block and r = rank of these bundles. Here: $S \to \mathbb{P}^2_{k^s}$ is the blow-up at $9-\deg(S)$ points, L_i the exceptional divisors, and H the pull back of the hyperplane class, $L_{\geq 4} := \sum_{i\geq 4} L_i$, and $L_{\leq 3} := \sum_{i\leq 3} L_i$. For the higher rank vector bundles, we follow the notation in Karpov–Nogin [64, §4].

Proposition 4.1 ([52], [92], [64]). *Let S be either a quadric surface or a del Pezzo surface of degree $d \neq 7, 8$ over a separably closed field k. Set $s = \max\{1, 5-d\}$. Then $\mathsf{D}^{\mathrm{b}}(S)$ has s (up to tensoring by line bundles and the action of the Weyl group) minimal 3-block decompositions such that any other 3-block decomposition of $\mathsf{D}^{\mathrm{b}}(S)$ can obtained from one of these by a finite number of mutations.*

In all cases, the blocks are generated by a completely orthogonal set of vector bundles, so that each block has a tilting bundle.

In the cases of degree $d = 7, 8$ (see Example 3.4), there is always a 4-block decomposition, but not a 3-block one. On the other hand, del Pezzo surfaces of these degrees are never minimal (see Example 3.4).

We summarize the possible minimal 3-block decompositions in Table 1. Recall that these decompositions hold over a separably closed field, or in general for totally split del Pezzo surfaces, so that S is either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or the blow up of $\mathbb{P}^2$ in $9-d$ rational points, and exceptional means k-exceptional. In the first case we use the standard notation $\mathscr{O}(a, b)$ for line bundles of bidegree (a, b). In the latter, we denote by L_i (for $i = 1, \dots r$) the exceptional divisors of $S \to \mathbb{P}^2$, and by H the pull-back of the hyperplane class in $\mathbb{P}^2$. For each block, the table lists the number n of exceptional bundles, their rank r (which is constant within the block), and the 1st Chern class of the tilting bundle of the block (i.e., the sum of the 1st Chern classes of the bundles in the block).

5. Del Pezzo surfaces of Picard rank 1 and low degree

From now on, let k be an arbitrary field and S be a del Pezzo surface of degree d and Picard rank 1 over k. Recall that $K_0(S)_{\mathbb{Q}} \simeq \mathbb{Q}^{\oplus 3}$, so that we can wonder, according to Proposition 3.6, whether there is a semiorthogonal decomposition given by three simple k-algebras. If such a decomposition exists, it must base change to a 3-block decomposition of $\mathsf{D}^{\mathrm{b}}(S_{k^s})$ by Proposition 2.12. Conversely, if we suppose that there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{E}, \mathsf{F}, \mathsf{G} \rangle$$

whose base change is a 3-block decomposition, then Proposition 2.12 guarantees that the three components are equivalent to derived categories of simple k-algebras.

Combining the explicit description of the vector bundles forming exceptional blocks on S_{k^s}, together with the previous observations, we gain control over semiorthogonal decompositions of derived categories of del Pezzo surfaces of Picard rank 1.

Theorem 5.1. *Let S be a del Pezzo surface of degree $d \leq 4$. Then there is no semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(l_1/k, \alpha_1), \mathsf{D}^{\mathrm{b}}(l_2/k, \alpha_2), \mathsf{D}^{\mathrm{b}}(l_3/k, \alpha_3) \rangle, \tag{5.1}$$

with l_i field extensions of k and α_i in $\mathrm{Br}(l_i)$.

Proof. If a decomposition (5.1) exists, then $\rho(S) = 1$ by Corollary 3.7, hence $\mathrm{Pic}(S) = \mathbb{Z}[\omega]$ by the classification. Moreover, Proposition 2.12 ensures us that its base change to k^s is a 3-block decomposition. Up to mutating the three blocks, tensoring with line bundles, and the action of the Weyl group, we can appeal to the Karpov–Nogin classification (cf. Proposition 4.1) and proceed in a case-by-case analysis. Such exceptional blocks are generated by vector bundles, so Corollary 2.13 guarantees that a nontrivial $\mathbb{Z}$-linear combination of the first Chern classes of these vector bundles is a multiple of ω.

First, the Galois action on $\langle \omega \rangle^{\perp} \subset \mathrm{Pic}(S_{k^s})$ factors through the Weyl group of the associated root system, see [44, Thm. 2]. Hence over k^s, there is a choice of $9-d$ pairwise disjoint exceptional lines $L_1, \dots, L_d$ so that the three blocks are described as in Table 1, up to tensoring all exceptional bundles by the same line bundle $\mathscr{O}(M)$ in $\mathrm{Pic}(S_{k^s})$.

Our argument will follow two different paths, depending on the degree and subcase:

(1) Either we show that one of the blocks contains a proper admissible subcategory generated by ω, and then Lemma 1.16 shows that this contradicts $\rho(S) = 1$.

(2) Or we show the impossibility of descending a non-trivial generator of one of the blocks, by proving that its first Chern class could never be a multiple of ω.

In what follows, we will consider a line bundle M on S_{k^s} written as $M = nH + \sum_{i=1}^{d} a_i L_i$. Tensoring by powers of $\omega = -3H + \sum_{i=1}^{d} L_i$, we can choose to fix one of the coefficients a_i or a representative of n modulo 3.

Degree 4. In degree 4, E is generated by $\mathscr{O}(L_4)$ and $\mathscr{O}(L_5)$ over k^s. Let $\mathscr{O}(M)$ be a line bundle on S_{k^s}. If there exists a pair of integers a and b such that

$$a(L_4 + M) + b(L_5 + M) = r\omega, \tag{5.2}$$

then $r = 0$. Indeed, we make this first calculation explicit: let us assume that $M = nH + \sum_{i=1}^{5} a_i L_i$ with $a_1 = 0$. Then it follows that $r = 0$, in which case, we must also have $n = 0$ and $a_2 = a_3 = 0$. Of course, we can assume that both a and b are not both zero, otherwise no nontrivial generator of the block E descends. In fact, since $\mathrm{Pic}(S_{k^s})$ is torsion-free, we can further take a and b to be coprime.

With this in mind, equation (5.2) yields

$$a(L_4 + a_4 L_4 + a_5 L_5) + b(L_5 + a_4 L_4 + a_5 L_5) = 0.$$

It follows, looking at the coefficients of L_4 and L_5, respectively, that

$$\begin{cases} a + (a+b)a_4 = 0 \\ b + (a+b)a_5 = 0 \end{cases} \tag{5.3}$$

Since a and b are coprime, $a + b \neq 0$ and hence the only integer solutions to this system of equations has $a_4 = a_5 = 0$ (which implies $a = b = 0$, a contradiction) or $a + b = \pm 1$.

Suppose that $a + b = 1$. From (5.3), we get that $a_4 = -a$ and $a_5 = a - 1$, so that the possibilities to descend the block E are obtained by tensoring the Karpov–Nogin 3-block decomposition over S_{k^s} by $M = -aL_4 + (a-1)L_5$, for some integer a. Now we consider the block F. After tensoring with $\mathscr{O}(M)$, the block F is generated, over k^s, by $\mathscr{O}(H + M)$ and $\mathscr{O}(2H - L_1 - L_2 - L_3 + M)$. If the block descends to k, we have integers α, β, and ρ such that

$$\alpha(H + M) + \beta(2H - L_1 - L_2 - L_3 + M) = \rho\omega. \tag{5.4}$$

Looking at coefficients of L_1 in (5.4), we get $\beta = -\rho$. Then considering the coefficient of H in (5.4) gives $\alpha = -\rho$. Finally, the coefficients of L_4 and L_5 give $\alpha\rho = \rho$ and $(1-\alpha)\rho = \rho$, respectively. From this, it follows that $\rho = 0$, so that $\alpha = \beta = 0$, hence no nontrivial generator of the block F can descend.

If we suppose that $a + b = -1$, then $M = aL_4 - (a+1)L_5$, and similar arguments show that no nontrivial generator of F can descend. Using Corollary 2.13, this means that there is no way to descend both the blocks E and F to k at the same time.

Degree 3, case (i). The block E is generated by $\mathscr{O}(L_1)$, $\mathscr{O}(L_2)$, and $\mathscr{O}(L_3)$. We consider the equation

$$a(L_1 + M) + b(L_2 + M) + c(L_3 + M) = r\omega.$$

As before, up to tensoring by multiples of ω, we can assume that $M = nH + \sum_{i=1}^{6} a_i L_i$ with $a_4 = 0$, from which we similarly conclude that $r = 0$, hence $c_1(\mathsf{E}) = 0$, and $n = a_4 = a_5 = a_6 = 0$.

The block F is generated by $\mathscr{O}(H - L_4)$, $\mathscr{O}(H - L_5)$, $\mathscr{O}(H - L_6)$, so that we are looking for nontrivial integers α, β, and γ such that

$$\alpha(H - L_4 + M) + \beta(H - L_5 + M) + \gamma(H - L_6 + M) = \rho\omega$$

where $M = a_1 L_1 + a_2 L_2 + a_3 L_3$, as just shown. From this, we get $\alpha + \beta + \gamma = -3\rho$ (considering the coefficients of H) and $\alpha + \beta + \gamma = -\rho$ (considering the coefficients of L_4). It follows that $\rho = 0$, and hence $c_1(\mathsf{F}) = 0$. Notice that the latter does not mean that the coefficients α, β, and γ are trivial, so we can't appeal to Corollary 2.13 here. However, looking at the coefficients of L_1, L_2 and L_3, we also get $a_1 = a_2 = a_3 = 0$, from which we have that M is a multiple of ω.

Over k^s, the block G contains ω as part of an exceptional sequence (according to Table 1), hence after tensoring by M, the block G contains a multiple $\omega^{\otimes m}$ for some integer m. The category generated by this $\omega^{\otimes m}$ is a proper admissible subcategory of G. If G descends to an

admissible subcategory $\mathsf{D}^{\mathrm{b}}(K, A)$ of $\mathsf{D}^{\mathrm{b}}(S)$, then this subcategory admits $\langle \omega^{\otimes m} \rangle$ as a proper admissible subcategory. Then Lemma 1.16 shows that K cannot be a field.

Degree 3, case (ii). The block E is generated by a rank 2 vector bundle T_6 with $c_1(T_6) = H$. We have $c_1(T_6 \otimes M) = H + 2M$, and for the block to descend, we need $H + 2M = r\omega$. Taking M (up to multiples of ω) with $a_1 = 0$, we conclude that $r = 0$. As a result, $a_i = 0$ for all i, so that $M = nH$ is a multiple of H, and then $H + 2M = (2n + 1)H$. We conclude that the only multiple of T_6 that can descend is the trivial one. This contradicts Corollary 2.13.

Degree 2, case (i). This case is very similar to the case of *Degree 3, case* (ii). Indeed, the block F is generated by a single rank 2 vector bundle with first Chern class H.

Degree 2, case (ii). In this case, F is generated by $\mathscr{O}(L_4), \ldots, \mathscr{O}(L_7)$. Arguing as in the case of *Degree 4*, we deduce that if there exists a linear combination of $L_4 + M, \ldots, L_7 + M$ (a necessary condition for F to descend) then $r = 0$ and $M = \sum_{i=4}^{7} a_i L_i$. The block E is generated by two rank 2 vector bundles of first Chern classes $H + \omega$ and $-H + L_{\geq 4}$, respectively. As a necessary condition for E to descend, we are looking for a pair of integers a and b such that

$$a(H + \omega + 2M) + b(-H + L_{\geq 4} + 2M) = r\omega.$$

Since $M = a_4 L_4 + \ldots + a_7 L_7$, considering the coefficient of L_1, we arrive at $a = r$, from which it follows that $r(H + 2M) + b(-H + L_{\geq 4} + 2M) = 0$, by subtracting $r\omega$ on both sides. Now, looking at the coefficient of H, we arrive at $b = r$, hence we conclude that $r(4M + L_{\geq 4}) = 0$. Once again considering the coefficient of H, we see that $r \neq 0$ is impossible. It follows that $a = 0$ and hence $b = 0$, as well. This contradicts Corollary 2.13.

Degree 2, case (iii). In this case, the block E is generated, over k^s, by a rank 3 vector bundle with first Chern class $L_{\geq 4}$. A necessary condition for E to descend is that there exists an integer a such that $a(L_{\geq 4} + 3M) = r\omega$. Taking M with $a_1 = 0$, we easily get $r = 0$ and $M = a_4 L_4 + \ldots + a_7 L_7$. As in the case of *Degree 2, case* (ii) we find a contradiction to Corollary 2.13.

Degree 1, case (i). This case is similar to that of *Degree 3, case* (ii). Indeed, in both cases the block E is generated by a single vector bundle. In this case, its first Chern class is $-H + 2\omega$. Arguing as in the previous case, we arrive at a contradiction to Corollary 2.13.

Degree 1, case (ii). In this case the block E is generated by a single vector bundle of rank 4 and first Chern class $2H - L_1 - L_2 - L_3$. Modifying M by multiples of ω, we can choose $a_4 = 0$, and we can then conclude that $a(2H - L_1 - L_2 - L_3 + 4M) = r\omega$ implies that $r = 0$ and $M = nH + \sum_{i=4}^{7} a_i L_i$. Looking at the coefficients of H, we see that if $a \neq 0$, there is no n such that the latter equation holds. Hence $a = 0$ and we find a contradiction to Corollary 2.13.

Degree 1, case (iii). In this case, the block E is generated by two rank 3 vector bundles with first Chern classes $L_4 + L_5 + L_6 + L_7$ and $L_4 + L_5 + L_6 + L_8$, respectively. Hence a necessary condition for E to descend is that there exist integers a and b such that

$$a(L_4 + L_5 + L_6 + L_7 + 3M) + b(L_4 + L_5 + L_6 + L_8 + 3M) = r\omega.$$

As before, we can assume that a and b are both nonzero and coprime. Choosing $M = nH + \sum_{i=1}^{8} a_i L_i$ with $a_1 = 0$, we arrive at $r = 0$ (by considering the coefficient of L_1), and hence also $n = a_1 = a_2 = a_3 = 0$. Now, looking to the coefficients of L_7 and L_8, we arrive at a system of equations

$$\begin{cases} a + 3(a + b)a_7 = 0 \\ b + 3(a + b)a_8 = 0 \end{cases}$$

However, this system has no integer solutions when a and b are coprime, as seen by reducing modulo 3. This contradicts Corollary 2.13.

Degree 1, case (iv). In this case, E is generated by a single rank 5 vector bundle with Chern class $-2\omega + L_{\geq 4}$. Similarly as in the case *Degree 3, case* (ii), the descent of E yields a contradiction to Corollary 2.13.

Thus in each case of degree ≤ 4, the assumptions that all three blocks can descend to simple categories leads to a contradiction and the proof is complete. □

Corollary 5.2. *Let S be a del Pezzo surface of degree $d \leq 4$ and Picard rank one. Then S is not categorically representable in dimension 0.*

Proof. By Corollary 3.7, categorical representability of S would be given by a semiorthogonal decomposition as (5.1) with $\alpha_i = 0$. □

On the other hand, given any geometrically rational surface S, the line bundle $\mathscr{O}_S$ (or the line bundle ω_S) defines an exceptional object in $\mathsf{D}^{\mathrm{b}}(S)$, hence we always have a nontrivial semiorthogonal decomposition.

Proposition 5.3. *Let S be a geometrically rational surface over k. Then there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{A}_S \rangle.$$

If $\rho(S) = 1$, then $K_0(\mathsf{A}_S)_{\mathbb{Q}} = \mathbb{Q}^{\oplus 2}$. Furthermore, if S is a del Pezzo surface of degree ≤ 4, then there is no semiorthogonal decomposition

$$\mathsf{A}_S = \langle \mathsf{D}^{\mathrm{b}}(l_1/k, \alpha_1), \mathsf{D}^{\mathrm{b}}(l_2/k, \alpha_2) \rangle$$

with l_i fields and α_i in $\mathrm{Br}(l_i)$.

Proof. The admissible subcategory $\mathsf{D}^{\mathrm{b}}(k)$ is generated by the exceptional object $\mathscr{O}_S$, hence the semiorthogonal decomposition exists. The calculation of $K_0(\mathsf{A}_S)$ is straightforward, and the last statement is a consequence of Theorem 5.1. □

Proposition 5.4. *Let S be a del Pezzo surface of degree $d \leq 4$ and Picard rank 1, and suppose that, if $d = 4$, then $\mathrm{ind}(S) > 1$. Let S' be birational to S. Then there is a semiorthogonal decomposition*

$$\mathsf{A}_{S'} = \langle \mathsf{T}, \mathsf{A}_S \rangle,$$

where T is representable in dimension 0, and $\mathsf{T} = 0$ if and only if $S \simeq S'$.

Proof. Under the assumptions, S is birationally rigid. Hence, if S' is minimal, $S' \simeq S$, and if S' is not minimal, there is a blow-up $S' \to S$. Then we conclude by the blow-up formula. □

Remark 5.5. If S is a del Pezzo surface of degree ≤ 4, notice from Table 1 that there is no 3 block decomposition with a block generated by a single exceptional line bundle. It follows that the category A_S does not base-change to a category generated by two exceptional blocks.

We end this section with a conjecture on the structure of the category A_S for a del Pezzo surface of degree ≤ 4. The highly non-trivial noncommutative structure of A_S should be a reflection of the more complicated arithmetic behavior of S.

Conjecture 5.6. *If S is a del Pezzo surface of degree ≤ 4 and $\rho(S) = 1$, then the category A_S has no nontrivial semiorthogonal decomposition.*

We now describe the few known facts about A_S in degrees 3 and 4. If S has degree 4, then the anticanonical embedding $S \to \mathbb{P}^4_k$ realizes S as the intersection of two quadric hypersurfaces. The following result was shown in [11], as a consequence of Homological Projective Duality (see also [71]).

Theorem 5.7. *Let S be a del Pezzo surface of degree 4 and $X \to \mathbb{P}^1$ be the associated pencil of quadrics of $\mathbb{P}^4$ containing S, with associated even Clifford algebra $\mathscr{C}_0$ over $\mathbb{P}^1$. Then there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, \mathscr{C}_0) \rangle.$$

If $S(k) \neq \varnothing$ then the quadric threefold fibration $X \to \mathbb{P}^1$ can be reduced by hyperbolic splitting (cf. [11, §1.3]) to a conic bundle $Y \to \mathbb{P}^1$ with Clifford algebra $\mathscr{C}_0'$, and there is an equivalence between $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, \mathscr{C}_0)$ and $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, \mathscr{C}_0')$ by [11, Thm. 3]. We will see another (more explicit) way to describe the orthogonal complement A_S via a conic bundle in Appendix B.

If S has degree 3, then the anticanonical embedding $S \to \mathbb{P}^3_k$ realizes S as a cubic hypersurface, so that we can use Kuznetsov's calculation (see [72]).

S	$\mathrm{ind}(S)$	A_1	Z	ind	c_2	$(V_1)_{k^s}$	A_2	Z	ind	c_2	$(V_2)_{k^s}$
$\mathrm{SB}(A)$	3	A	k	3	3	$\mathscr{O}(1)^{\oplus 3}$	A^{-1}	k	3	12	$\mathscr{O}(2)^{\oplus 3}$
$\mathbb{P}^2_k$	1	k	k	1	0	$\mathscr{O}(1)$	k	k	1	0	$\mathscr{O}(2)$

TABLE 2. The invariants of a Severi–Brauer surface S (a del Pezzo surface of degree 9). Here, the algebras $A_i = \mathrm{End}(V_i)$ are listed up to Morita equivalence; Z and ind refer to the center and index of A_i; and c_2 refers to the 2nd Chern class of V_i.

Theorem 5.8. *If S is a del Pezzo surface of degree 3, there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{A}_S \rangle,$$

with A_S a Calabi–Yau category of dimension $\frac{4}{3}$.

The category A_S can be described via Matrix Factorization as in [86, Thm. 40] and via Homological Projective Duality as in [15].

6. Severi–Brauer surfaces (del Pezzo surfaces of degree 9)

If S is a del Pezzo surface of degree 9 over k, then S is the Severi–Brauer surface associated to a degree 3 central simple algebra A over k. Denote by $\alpha \in \mathrm{Br}(k)$ the Brauer class of A.

Proposition 6.1. *Let $S = \mathrm{SB}(A)$ be a Severi–Brauer surface. Then the following are equivalent:*

(i) S has a k-point,
(ii) S is k-rational,
(iii) S is categorically representable in dimension 0,
(iv) A_S is categorically representable in dimension 0.

Proof. It's a result of Châtelet (cf. Example 3.2) that *(i)* is equivalent to *(ii)* is equivalent to $S \cong \mathbb{P}^2$ is equivalent to the triviality of A in the Brauer group. In turn, this implies *(iii)* and *(iv)* by considering the full exceptional collection $\{\mathscr{O}, \mathscr{O}(1), \mathscr{O}(2)\}$ of $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^2)$ described by Beilinson [17] and Bernšteĭn–Gelfand–Gelfand [24]. It then suffices to prove that *(iii)* implies that A has trivial Brauer class.

In general, a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k, A), \mathsf{D}^{\mathrm{b}}(k, A^{-1}) \rangle \tag{6.1}$$

was constructed in [19], which base changes to the semiorthogonal decomposition $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^2_{k^s}) = \langle \mathscr{O}, \mathscr{O}(1), \mathscr{O}(2) \rangle$, see Example 2.14.

Assuming *(iii)*, then by Corollary 3.7, there are fields l_i and a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(l_1/k), \mathsf{D}^{\mathrm{b}}(l_2/k), \mathsf{D}^{\mathrm{b}}(l_3/k) \rangle, \tag{6.2}$$

which by Proposition 4.1, base changes to a 3-block exceptional collection, unique up to mutation and tensoring by line bundles on S. Hence, up to mutations, the decomposition (6.2) base changes to the decomposition $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^2_{k^s}) = \langle \mathscr{O}(i), \mathscr{O}(i+1), \mathscr{O}(i+2) \rangle$. Twisting by powers of the canonical bundle and performing one more mutation, we can assume that $i = 0$. Hence the decomposition (6.2) is equivalent to the decomposition (6.1). In particular, we get that $\mathsf{D}^{\mathrm{b}}(k, A) \simeq \mathsf{D}^{\mathrm{b}}(l_i/k)$ for some $i = 1, 2$, which by Corollary 1.18, implies that $l_i = k$ and that A is split. □

Now we verify that the Griffiths–Kuznetsov component GK_S is well defined.

Proposition 6.2. *Let $S = \mathrm{SB}(A)$ be a nonrational Severi–Brauer surface. Then the Griffiths–Kuznetsov component is a well defined birational invariant in the following sense: if $S_1 \dashrightarrow S$ is a birational map then there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(S_1) = \langle \mathsf{T}, \mathsf{D}^{\mathrm{b}}(k, \alpha), \mathsf{D}^{\mathrm{b}}(k, \alpha^{-1}) \rangle$$

where T is representable in dimension 0.

Proof. If S_1 is minimal then $S_1 = \mathrm{SB}(B)$ is a Severi–Brauer surface by Proposition 3.10. Amitsur's theorem [2] implies that $B = A$ or $B = A^{-1}$. Indeed, in the Appendix (see Proposition A.1), we can show how the decomposition of the birational map $S_1 \dashrightarrow S$ gives either $\mathsf{D}^{\mathrm{b}}(k, A) \simeq \mathsf{D}^{\mathrm{b}}(k, B)$ or $\mathsf{D}^{\mathrm{b}}(k, A) \simeq \mathsf{D}^{\mathrm{b}}(k, B^{-1})$ only using the description of tilting bundles and their behavior under birational maps. Then the statement follows, possibly up to a mutation, from the semiorthogonal decomposition (6.1). If S_1 is not minimal, there is a minimal model $S_1 \to S_0$, and we conclude using Lemma 1.15 and the first part of the proof. □

Remark 6.3. Recall the vector bundle V of rank 3 on S constructed by Quillen [91, §8.4] (see Example 2.14) such that $V_{k^s} = \mathscr{O}(1)^{\oplus 3}$. More explicitly, if K/k is a separable degree 3 extension splitting A, then $V_i = \mathrm{tr}_{K/k}\mathscr{O}_{\mathbb{P}^2_K}(i)$ by Theorem 2.9. We set $V_1 = V^{\min}$ and $V_2 = (V^\vee \otimes \omega_S^\vee)^{\min}$. Remark that V_{2,k^s} is either $\mathscr{O}(2)$ or $\mathscr{O}(2)^{\oplus 3}$. These vector bundles are tilting bundles for the blocks F and G, respectively, and $A_1 = \mathrm{End}(V_1)$ is Morita equivalent to A, $A_2 = \mathrm{End}(V_2)$ is Morita equivalent to A^{-1}, and and V_i are indecomposable. We list the ranks and 2nd Chern classes of the vector bundles V_i on Table 2.

The calculation of the second Chern classes of the vector bundles V_1 and V_2 is easily obtained by their description over k^s. In particular, we notice that $\mathrm{ind}(S) = \gcd(c_2(V_1), c_2(V_2))$ when S is nonsplit and $\gcd(c_2(V_1^{\oplus 2}), c_2(V_2^{\oplus 2})) = \mathrm{ind}(S)$.

Remark 6.4. The second Chern classes of the generators of $\mathsf{D}^{\mathrm{b}}(k, A)$ and $\mathsf{D}^{\mathrm{b}}(k, A^{-1})$ are not stable under mutations. The values 3 and 12 are obtained for the specific choices of bundles listed in Table 2. This same remark applies to all other degrees.

7. Involution varieties (del Pezzo surfaces of degree 8)

If the degree of S is 8, either S is an involution surface (cf. Example 3.3), or S is the blow-up of $\mathbb{P}^2_{k^s}$ at a k-rational point (cf. Example 3.4). In the latter case, S is rational, not minimal, and has a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathscr{O}, \mathscr{O}(H), \mathscr{O}(2H), \mathscr{O}(L_1)\rangle = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k)\rangle$$

where H is the pull back of the hyperplane class from $\mathbb{P}^2_k$ to S and L_1 is the exceptional divisor of the blow-up. There is no 3-block decomposition in this case.

So we focus our attention on involution surfaces. In this case, S is associated to a degree 4 central simple k-algebra (A, σ) with quadratic pair. The even Clifford algebra $C_0(A, \sigma)$ (see Example 3.3) is a quaternion algebra over its center l, which is the étale quadratic discriminant extension of S. Denote by α the Brauer class of A and γ the Brauer class of $C_0(A, \sigma)$ over the discriminant extension l. When $l = k^2$, we write $\gamma = (\gamma_+, \gamma_-) \in \mathrm{Br}(k^2) = \mathrm{Br}(k) \times \mathrm{Br}(k)$.

The fundamental relations for the Clifford algebra of an algebra with involution [67, Thm. 9.14] imply that $\alpha = \mathrm{cor}_{l/k}\gamma \in \mathrm{Br}(k)$. We record another important fact from the algebraic theory of quadratic forms.

Lemma 7.1. *Let S be an involution surface over a field k. Then $S(k) \neq \varnothing$ if and only if $\gamma \in \mathrm{Br}(l)$ is trivial.*

Proof. If $S(k) \neq \varnothing$ then $\mathrm{SB}(A)(k) \neq \varnothing$, hence α is split. Also, if $\gamma \in \mathrm{Br}(l)$ is trivial, then $\alpha \in \mathrm{Br}(k)$ is trivial by the fundamental relations. Thus we can reduce to the case when (A, σ) is adjoint to a quadratic form q of dimension 4 over k, which in this case q is isotropic if and only if $C_0(q)$ is split over its center, i.e., $\gamma \in \mathrm{Br}(l)$ is trivial, by [68, Thm. 6.3] (also see [94, 2 Thm. 14.1, Lemma 14.2] in characteristic $\neq 2$ and [14, II Prop. 5.3] in characteristic 2). □

Proposition 7.2. *Let S be an involution surface over a field k. Then the following are equivalent:*

(i) S has a k-point,
(ii) S is k-rational,
(iii) S is categorically representable in dimension 0,
(iv) A_S is categorically representable in dimension 0.

	S	$\mathrm{ind}(S)$	$\rho(S)$	A_1	Z	ind	c_2	rk	A_2	Z	per	ind	c_2	rk
8.1	$S \subset \mathrm{SB}(A)$	4	1	C_0	l	2	4	4	A	k	2	4	12	4
8.2	$\mathrm{SB}(B) \times \mathrm{SB}(B')$	4	2	$B \times B'$	k^2	2	4	4	$B \otimes B'$	k	2	4	12	4
8.3	$S \subset \mathrm{SB}(A)$	2	1	C_0	l	2	4	4	A	k	2	2	2	2
8.4	$S \subset \mathbb{P}^3_k$	2	1	C_0	l	2	4	4	k	k	1	1	0	1
8.5	$\mathrm{SB}(B) \times \mathrm{SB}(B')$	2	2	$B \times B'$	k^2	2	4	4	$B \otimes B'$	k	2	2	2	2
8.6	$\mathrm{SB}(B) \times \mathrm{SB}(B)$	2	2	$B \times B$	k^2	2	4	4	k	k	1	1	0	1
8.7	$\mathrm{SB}(B) \times \mathbb{P}^1$	2	2	$B \times k$	k^2	2	4	4	B	k	2	2	2	2
8.8	$S \subset \mathbb{P}^3_k$	1	1	l	l	1	1	2	k	k	1	1	0	1
8.9	$\mathbb{P}^1 \times \mathbb{P}^1$	1	2	k^2	k^2	1	1	2	k	k	1	1	0	1

TABLE 3. The invariants of an involution surface S (a minimal del Pezzo of degree 8). Here: the algebras $\mathrm{End}(V_1)$ and $\mathrm{End}(V_2)$ are Morita equivalent to $A_1 = C_0$ and $A_2 = A$; also Z, per, and ind refer to the center, period, and index of A_i; and c_2 and rk refer to the 2nd Chern class and rank of V_i. Note that $(V_1)_{k^s}$ is a direct sum of spinor bundles while $(V_2)_{k^s}$ is a direct sum of hyperplane classes. Also l stands for the separable quadratic discriminant extension of k. Recall that S is rational if and only if $\mathrm{ind}(S) = 1$. In all these cases, S is minimal. We give a brief geometric description of all cases in Remark 7.4.

Proof. Our aim is to produce a semiorthogonal decomposition that base changes to the 3-block decomposition from Table 1. In characteristic $\neq 2$, this is a result of Blunk [26, §7], who works with tilting bundles and does not explicitly mention semiorthogonal decompositions. We will give a sketch of an alternate proof that works in any characteristic. First, under the closed embedding $S \subset \mathrm{SB}(A) = X$, we get V_2 as the pull back of the indecomposable vector bundle on X of pure type $\mathscr{O}_{\mathbb{P}^3}(1)$. Then V_2 has rank dividing 4 and $\mathrm{End}(V_2)$ is Morita equivalent to A. Second, the fully faithful embedding of $\mathsf{D}^{\mathrm{b}}(k, C_0)$ can be seen as the twisted version of Kuznetsov's result [71] (see [11, Thm. 2.2.1] for the case of a quadric over a general field). More explicitly, $\mathscr{O}_{S_{k^s}}(1,0) \oplus \mathscr{O}_{S_{k^s}}(0,1)$ is a Galois invariant vector bundle, with the Galois group of l/k acting by switching the factors (when the discriminant is nontrivial), and there is a unique indecomposable vector bundle V_1 of this pure type by §2.2. Hence over k^s, by comparing with the usual decomposition (cf. [71, Lemma 4.14], which is non other than the 3-block decomposition in Table 1), we find a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k, C_0), \mathsf{D}^{\mathrm{b}}(k, A) \rangle. \tag{7.1}$$

The fact that the endomorphism algebra of $\mathscr{O}_{S_{k^s}}(1,0) \oplus \mathscr{O}_{S_{k^s}}(0,1)$ is Morita-equivalent to the even Clifford algebra C_0 goes back to Kapranov [60, §4.14].

Now we proceed with the proof of the equivalences. It's a classical result (cf. Example 3.3) that *(i)* is equivalent to *(ii)*. By Lemma 7.1, condition *(i)* is equivalent to the triviality of $\gamma \in \mathrm{Br}(l)$ (and also $\alpha \in \mathrm{Br}(k)$). In particular, *(i)* implies *(iii)* and *(iv)*, since then the semiorthogonal decomposition just constructed is of the form $\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(l), \mathsf{D}^{\mathrm{b}}(k) \rangle$, with the first block generated by $\mathscr{O}_S$. Hence both S and A_S is categorically representable in dimension 0 by Lemma 1.19.

Finally, we are left to proving that *(iii)* or *(iv)* implies the triviality of $\gamma \in \mathrm{Br}(l)$.

First, assume that S has Picard rank 1. If S is categorically representable in dimension 0, then by Corollary 3.7, there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(l_1/k), \mathsf{D}^{\mathrm{b}}(l_2/k), \mathsf{D}^{\mathrm{b}}(l_3/k) \rangle, \tag{7.2}$$

which by Proposition 4.1, base-changes to a 3-block exceptional collection. Hence, up to mutation, tensoring by line bundles on S, and the Weyl group action, we can assume that $\mathsf{D}^{\mathrm{b}}(l_1/k)$ base changes to $\langle \mathscr{O} \rangle$ and $\mathsf{D}^{\mathrm{b}}(l_3/k)$ base changes to $\langle \mathscr{O}(1,1) \rangle$. Hence the decomposition (7.2) base changes to the decomposition (7.1). In particular, we get that $\mathsf{D}^{\mathrm{b}}(k, C_0) \simeq \mathsf{D}^{\mathrm{b}}(l_2/k)$ and $\mathsf{D}^{\mathrm{b}}(k, A) \simeq \mathsf{D}^{\mathrm{b}}(l_3/k)$, which by Corollary 1.18, implies that $l = l_2$ and C_0 is split over l and that A is split over $l_3 = k$.

Second, assume that S has Picard rank 2. In this case, we have $S = C \times C'$ for Severi–Brauer curves $C = \mathrm{SB}(B)$ and $C' = \mathrm{SB}(B')$, and then $C_0 = B \times B'$ and $A = B \otimes B'$, cf. Example 3.3. Hence we have a semiorthogonal decomposition

$$\mathsf{A}_S = \langle \mathsf{D}^{\mathrm{b}}(k, B), \mathsf{D}^{\mathrm{b}}(k, B'), \mathsf{D}^{\mathrm{b}}(k, A) \rangle. \tag{7.3}$$

If A_S is representable in dimension 0, then by Corollary 3.7, there is a semiorthogonal decomposition

$$\mathsf{A}_S = \langle \mathsf{D}^{\mathrm{b}}(l_1/k), \mathsf{D}^{\mathrm{b}}(l_2/k), \mathsf{D}^{\mathrm{b}}(l_3/k) \rangle, \tag{7.4}$$

Since the Picard rank of S is stable under base change, it follows that $l_i = k$ for $i = 1, 2, 3$. Thus A_S is generated by three k-exceptional objects, hence $\mathsf{D}^{\mathrm{b}}(S)$ is generated by four k-exceptional objects. Over k^s, we can appeal to Rudakov [92] to mutate the semiorthogonal decomposition (7.4) into the decomposition (7.3). Using Theorem 1.17, we deduce that the Brauer classes of A, B, B', and hence also C_0, are trivial. □

We now want to show that the Griffiths–Kuznetsov component is well defined, except possibly when S has index 2 and Picard rank 1. In the Appendix B, we will see how this case should be thought of as a conic bundle of degree 6 from the categorical point of view.

Proposition 7.3. *Let S be an involution surface. Then the Griffiths–Kuznetsov component GK_S is well defined as a birational invariant in the following cases. Letting $S_1 \dashrightarrow S$ be a birational map, we have:*

- $\mathrm{ind}(S) = 4$ *if and only if $\alpha \in \mathrm{Br}(k)$ has index 4; there is a semiorthogonal decomposition* $\mathsf{A}_{S_1} = \langle \mathsf{T}, \mathsf{D}^{\mathrm{b}}(l, \gamma), \mathsf{D}^{\mathrm{b}}(k, \alpha) \rangle$
- $\mathrm{ind}(S) = 2$ *and $\rho(S) = 2$ then γ is never trivial; if α is trivial then there is a semiorthogonal decomposition* $\mathsf{A}_{S_1} = \langle \mathsf{T}, \mathsf{D}^{\mathrm{b}}(l, \gamma) \rangle$ *and if α is nontrivial then there is a semiorthogonal decomposition* $\mathsf{A}_{S_1} = \langle \mathsf{T}, \mathsf{D}^{\mathrm{b}}(l, \gamma), \mathsf{D}^{\mathrm{b}}(k, \alpha) \rangle$

where T always denotes a category representable in dimension 0.

Proof. By Lemma 7.1, γ is trivial if and only if $S(k) \neq \varnothing$. Furthermore, we can always find a quadratic extension l'/l that splits C_0 over l. Hence $S(l') \neq \varnothing$. Since l'/k has degree 4, it follows that S has a closed point of degree 4 so that $\mathrm{ind}(S)$ divides 4.

Since $S \subset \mathrm{SB}(A)$, we have that $\mathrm{ind}(S)$ must be a multiple of $\mathrm{ind}(\mathrm{SB}(A)) = \mathrm{ind}(A)$. In particular, $\mathrm{ind}(A) = 4$ if and only if $\mathrm{ind}(S) = 4$. Also if $\mathrm{ind}(A) = 2$, then a generalization of Albert's result on common splitting fields for quaternion algebras, cf. [67, Cor. 16.28], shows that there is a quadratic extension of k splitting C_0, hence also A by the fundamental relations, and thus $\mathrm{ind}(S) = 2$.

Now suppose that $\mathrm{ind}(S) = 4$. Then both γ and α are nontrivial. If $\rho(S) = 1$ then S is birationally rigid by Proposition 3.10. If $\rho(S) = 2$, then $S \cong \mathrm{SB}(B) \times \mathrm{SB}(B')$ such that $A = B \otimes B'$ has index 4 and $C_0 = B \times B'$. In §C, we show that S_1 admits a birational morphism $S_1 \to S_0$ where S_0 is a conic bundle over either $\mathrm{SB}(B)$ or $\mathrm{SB}(B')$ and has the required semiorthogonal decomposition.

Now suppose that $\mathrm{ind}(S) = 2$ and $\rho(S) = 2$, then S_1 admits a birational morphism $S_1 \to S_0$ where S_0 is a conic bundle of degree 8, and in the Appendix §C, we show that it has the required semiorthogonal decomposition. □

Remark 7.4. We now describe the geometry of the all possible cases listed in Table 3. Any involution surface is minimal. If $\rho(S) = 1$ (equivalently, l is a field) then $\mathrm{Pic}(S)$ is generated either by the anticanonical bundle or its square root. In the second case, $\mathrm{ind}(S)|2$ and $S \subset \mathbb{P}^3$ is a quadric surface, which can either be isotropic (case 8.8) or anisotropic (case 8.4). If the Picard group is generated by the anticanonical bundle, then $S \subset \mathrm{SB}(A)$ is a degree 2 divisor of a Severi–Brauer threefold and $\mathrm{ind}(S) = \mathrm{ind}(A)$ (cases 8.1 and 8.3). In the case when $\rho(S) = 2$ then S is isomorphic to a product of Severi–Brauer curves $\mathrm{SB}(B)$ and $\mathrm{SB}(B')$. In this case, $C_0 = B \times B'$ and $A = B \otimes B'$ and the possible cases are: B not equivalent to B' and both nontrivial (cases 8.2 and 8.5 according to the index of A); $B = B'$ nontrivial (case 8.6); and B

nontrivial and B' trivial (case 8.7); and both B and B' trivial (case 8.9). We remark that cases 8.6 and 8.7 are k-birational to each other for any given B.

Remark 7.5. There is a natural vector bundle W of rank 4 on S such that $W_{k^s} = \mathscr{O}(1,0)^{\oplus 2} \oplus \mathscr{O}(0,1)^{\oplus 2}$. There is also a natural rank 4 vector bundle U on $\mathrm{SB}(A)$ such that $U_{k^s} = \mathscr{O}(1)^{\oplus 4}$. We let $V_1 = W^{\min}$ and $V_2 = U|_S^{\min}$ (recall Definition 2.10). These vector bundles are tilting bundles for the blocks F and G, respectively, and have the following properties: $A_1 = \mathrm{End}(V_1)$ is Morita equivalent to C_0 and $A_2 = \mathrm{End}(V_2)$ is Morita equivalent to A; V_1 is indecomposable if and only if l is a field; and V_2 is always indecomposable. We list the ranks and 2nd Chern classes of the vector bundles V_i on Table 3.

The calculation of the second Chern classes of the vector bundles V_1 and V_2 is easily obtained by their description over k^s. In particular, we notice that $\mathrm{ind}(S) = \gcd(c_2(V_1), c_2(V_2))$, except when S is an anisotropic quadric surface, in which case $\gcd(c_2(V_1), c_2(V_2^{\oplus 2})) = \mathrm{ind}(S)$. Here, we use the convention that $\gcd(a, 0) = a$.

8. del Pezzo surfaces of degree 7

A del Pezzo surface S of degree 7 is the blow up of $\mathbb{P}^2_k$ along a point of degree 2, see Example 3.4. If the residue field of the center of blow up is l, then there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathscr{O}, \mathscr{O}(H), \mathscr{O}(2H), V\rangle = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(l)\rangle$$

where H is the pull back of the hyperplane class from $\mathbb{P}^2_k$ to S, and V is a rank 2 vector bundle on S such that $\mathrm{End}(V) = l$ and $V \otimes k^s = \mathscr{O}(L_1) \oplus \mathscr{O}(L_2)$, where L_1 and L_2 are the exceptional divisors on k^s. In particular, S is k-rational and is categorically representable in dimension 0 by Lemma 1.15. However, there is no 3-block decomposition of S.

9. del Pezzo surfaces of degree 6

Let S be a del Pezzo surface of degree 6. There is an associated quaternion Azumaya algebra Q over a cubic étale k-algebra L and an associated degree 3 Azumaya algebra B over a quadratic étale k-algebra K. Blunk [25] gives an interpretation of these algebras in terms of a toric presentation of S, building on the geometric construction of Colliot-Thélène, Karpenko, and Merkurjev [41]. Let $\kappa \in \mathrm{Br}(L)$ and $\beta \in \mathrm{Br}(K)$ denote the Brauer classes of Q and B, respectively. Blunk, Sierra, and Smith [27] provide a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k, Q), \mathsf{D}^{\mathrm{b}}(k, B)\rangle = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(L/k, \kappa), \mathsf{D}^{\mathrm{b}}(K/k, \beta)\rangle, \tag{9.1}$$

Our first task is to prove that the semiorthogonal decomposition (9.1) descends the minimal three block decomposition from [64]. This will give an alternative description of the tilting bundles generating the blocks.

Proposition 9.1. *The base change of the semiorthogonal decomposition* (9.1) *coincides with the minimal 3-block decomposition of* $\mathsf{D}^{\mathrm{b}}(S_{k^s})$.

Remark 9.2. We can appeal to Proposition 4.1 for the fact that the semiorthogonal decomposition (9.1) coincides, up to mutation, tensoring by a line bundle, and the Weyl group action, with the minimal 3-block decomposition. Hence in Proposition 9.1, we prove slightly more: using Blunk's work [25], we explicitly describe the generators of the semiorthogonal components of (9.1) that base change to the 3-block collection from Table 1. Aside from being necessary for the sequel, we believe that the direct proof clarifies the connection between the beautiful geometry and arithmetic of del Pezzo surfaces of degree 6 and its derived category.

Before giving the proof, we recall the construction in Blunk [25], and Blunk, Sierra, and Smith [27], of certain vector bundles on S. The del Pezzo surface S_{k^s} of degree 6 over k^s is the blow-up of $\mathbb{P}^2_{k^s}$ in three noncolinear points p_1, p_2, p_3. There are six exceptional lines, coming in two pairs of three lines, say L_1, L_2, L_3 and M_1, M_2, M_3. The intersection products are $M_i.M_j = L_i.L_j = -\delta_{ij}$, and $M_i.L_j = \delta_{ij}$. So there is a map $\pi : S_{k^s} \to \mathbb{P}^2_{k^s}$, whose exceptional divisors are the L_i (with the convention that L_i is over p_i). The other three exceptional lines M_i

are the strict transform of the lines in $\mathbb{P}^2_{k^s}$ joining pairs of the three points (with the convention that M_i corresponds to the line not going through p_i). There is another birational morphism $\eta : S_{k^s} \to \mathbb{P}^2_{k^s}$ contracting the M_i to three points q_1, q_2, q_3, and sending L_i to lines joining two of those three points. We end up with the following diagram

$$\begin{tikzcd} & S_{k^s} \arrow[dl, "\eta"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2_{k^s} \arrow[rr, dashed, "\phi"] & & \mathbb{P}^2_{k^s}, \end{tikzcd} \tag{9.2}$$

where ϕ is the well-known Cremona involution, a birational self-map of the projective plane of degree 2.

This description allows us to present the Picard group of S_{k^s} in a way convenient to compare the base change of the semiorthogonal decompositions of Blunk–Sierra–Smith and Karpov–Nogin 3-block decomposition. Indeed, the Picard group of S_{k^s} has rank 4 and is generated by the exceptional lines L_i and M_i, with the relations $L_i + M_j = L_j + M_i$. If we denote by $H = \pi^*\mathscr{O}_{\mathbb{P}^2_{k^s}}(1)$, we have that $H = L_1 + L_2 + M_3$. The anticanonical divisor $K_{S_{k^s}}$ is then $K_{S_{k^s}} = L_1 + L_2 + L_3 + M_1 + M_2 + M_3$. On the other hand, if we denote by $H' = \eta^*\mathscr{O}_{\mathbb{P}^2_{k^s}}(1)$, we have $H' = M_1 + M_2 + L_3 = -K_{S_{k^s}} - H$.

To describe the semiorthogonal decomposition (9.1), Blunk, Sierra, and Smith construct vector bundles over S_{k^s} that descend to S in the following way [27]. The first one is just $\mathscr{O}_{S_{k^s}}$.

To define the second vector bundle, consider the following rank 2 vector bundles

$$\begin{aligned} J_1 &= \mathscr{O}(L_3 + M_2) \oplus \mathscr{O}(L_2 + M_3), \\ J_2 &= \mathscr{O}(L_1 + M_3) \oplus \mathscr{O}(L_3 + M_1), \\ J_3 &= \mathscr{O}(L_1 + M_2) \oplus \mathscr{O}(L_2 + M_1), \end{aligned} \tag{9.3}$$

on S_{k^s}. The presentation (9.3) shows that $\overline{J} = J_1 \oplus J_2 \oplus J_3$ is Galois invariant. Blunk, Sierra, and Smith assert that $\overline{J}$ descends to a vector bundle J of rank 6 on S and they consider $Q = \mathrm{End}(J)$. On S_{k^s}, we remark that $J_i = \mathscr{O}(H - L_i)^{\oplus 2}$. Thus, by base change, we get that $Q \otimes k^s = \mathrm{End}(\mathscr{O}(H - L_1)^{\oplus 2} \oplus \mathscr{O}(H - L_2)^{\oplus 2} \oplus \mathscr{O}(H - L_3)^{\oplus 2})$, which is Morita equivalent to $\mathrm{End}\big(\bigoplus_{i=1}^3 \mathscr{O}(H - L_i)\big)$.

To define the third vector bundle, consider the two rank 3 vector bundles

$$\begin{aligned} I_1 &= \mathscr{O}(L_1 + M_2 + M_3) \oplus \mathscr{O}(M_1 + L_2 + M_3) \oplus \mathscr{O}(M_1 + M_2 + L_3), \\ I_2 &= \mathscr{O}(L_1 + L_2 + M_3) \oplus \mathscr{O}(L_1 + M_2 + L_3) \oplus \mathscr{O}(M_1 + L_2 + L_3), \end{aligned} \tag{9.4}$$

on S_{k^s}. The presentation (9.4) shows that the sum $\overline{I} := I_1 \oplus I_2$ is Galois invariant. Blunk, Sierra, and Smith assert that $\overline{I}$ descends to a vector bundle I of rank 6 on S and they consider $B = \mathrm{End}(I)$. On S_{k^s}, we remark that $I_1 = \mathscr{O}(H')^{\oplus 3} = \mathscr{O}(-K_{S_{k^s}} - H)^{\oplus 3}$, and $I_2 = \mathscr{O}(H)^{\oplus 3}$. In particular, by base change, we get that $B \otimes k^s = \mathrm{End}(\mathscr{O}(H)^{\oplus 3} \oplus \mathscr{O}(-K_{S_{k^s}} - H)^{\oplus 3})$, which is Morita equivalent to $\mathrm{End}(\mathscr{O}(H) \oplus \mathscr{O}(-K_{S_{k^s}} - H))$.

Proof of Proposition 9.1. Let us now recall the construction of the 3 block decomposition over S_{k^s} described by Karpov and Nogin [64]. We provide a slightly different presentation. Consider the birational morphism $\pi : S_{k^s} \to \mathbb{P}^2_{k^s}$, which is the blow up of three points with exceptional divisors L_1, L_2 and L_3. A semiorthogonal decomposition of $\mathsf{D}^{\mathrm{b}}(S_{k^s})$ is given by Orlov's formula (see [84]) as follows:

$$\mathsf{D}^{\mathrm{b}}(S_{k^s}) = \langle \pi^*\mathsf{D}^{\mathrm{b}}(\mathbb{P}^2_{k^s}), \mathscr{O}_{L_1}, \mathscr{O}_{L_2}, \mathscr{O}_{L_3} \rangle.$$

Choosing the full exceptional collection $\{\mathscr{O}(-1), \mathscr{O}, \mathscr{O}(1)\}$ on $\mathbb{P}^2_{k^s}$, we get the semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S_{k^s}) = \langle \mathscr{O}(-H), \mathscr{O}, \mathscr{O}(H), \mathscr{O}_{L_1}, \mathscr{O}_{L_2}, \mathscr{O}_{L_3} \rangle.$$

Mutating $\mathscr{O}(-H)$ to the left with respect to the whole orthogonal complement we get

$$\mathsf{D}^{\mathrm{b}}(S_{k^s}) = \langle \mathscr{O}, \mathscr{O}(H), \mathscr{O}_{L_1}, \mathscr{O}_{L_2}, \mathscr{O}_{L_3}, \mathscr{O}(-K_{S_{k^s}} - H) \rangle,$$

	S	$\mathrm{ind}(S)$	$\rho(S)$	A_1	Z	ind	c_2	rk	A_2	Z	ind	c_2	rk
6.1	$S \subset R_{K/k}\mathrm{SB}(B)$	6	1	Q	L	2	12	6	B	K	3	24	6
6.2	$S \subset R_{K/k}\mathrm{SB}(B)$	3	1	L	L	1	3	3	B	K	3	24	6
6.3	$S \subset \mathrm{SB}(A) \times \mathrm{SB}(A^{-1})$	3	2	L	L	1	3	3	$A \times A^{-1}$	k^2	3	24	6
6.4	$S \subset R_{K/k}\mathbb{P}^2$	2	1	Q	L	2	12	6	K	K	1	2	2
6.5	$S \subset R_{K/k}\mathbb{P}^2$	2	2	$Q'' \times Q'$	$k \times L'$	2	12	6	K	K	1	2	2
6.6	$S \subset R_{K/k}\mathbb{P}^2$	2	2	$k \times Q'$	$k \times L'$	2	8	5	K	K	1	2	2
6.7	$S \subset R_{K/k}\mathbb{P}^2$	2	3	$Q' \times Q'' \times Q'''$	k^3	2	12	6	K	K	1	2	2
6.8	$S \subset R_{K/k}\mathbb{P}^2$	2	3	$k \times Q' \times Q'$	k^3	2	8	5	K	K	1	2	2
6.9	$S \subset R_{K/k}\mathbb{P}^2$	1	1	L	L	1	3	3	K	K	1	2	2
6.10	$S \subset R_{K/k}\mathbb{P}^2$	1	2	$k \times L'$	$k \times L'$	1	3	3	K	K	1	2	2
6.11	$S \subset \mathbb{P}^2 \times \mathbb{P}^2$	1	2	L	L	1	3	3	k^2	k^2	1	2	2
6.12	$S \subset R_{K/k}\mathbb{P}^2$	1	3	k^3	k^3	1	3	3	K	K	1	2	2
6.13	$S \subset \mathbb{P}^2 \times \mathbb{P}^2$	1	3	$k \times L'$	$k \times L'$	1	3	3	k^2	k^2	1	2	2
6.14	$S \subset \mathbb{P}^2 \times \mathbb{P}^2$	1	4	k^3	k^3	1	3	3	k^2	k^2	1	2	2

TABLE 4. The invariants of a del Pezzo surface S of degree 6. Here, the algebras $\mathrm{End}(V_1) = A_1 = Q$ and $\mathrm{End}(V_2) = A_2 = B$ up to Morita equivalence; $l_1 = Z(A_1) = L$ and $l_2 = Z(A_2) = K$ are separable cubic and quadratic extensions of k; the columns Z and ind refer to the center and index of A_i; and the columns c_2 and rk refer to the 2nd Chern class and rank of V_i. Note that $(V_1)_{k^s}$ is a direct sum of $\oplus(\mathscr{O}(H - L_i))$, while $(V_2)_{k^s}$ is a direct sum of $\mathscr{O}(H) \oplus \mathscr{O}(H')$. Recall that S is rational if and only if $\mathrm{ind}(S) = 1$, see [43, §2]. See Remark 9.9 for a geometric description of each case.

using [29, Prop. 3.6]. Now we mutate the three exceptional objects $\mathscr{O}_{L_i}$ to the left with respect to $\mathscr{O}(H)$. An easy calculation with the evaluation sequences for L_i gives

$$\mathsf{D}^{\mathrm{b}}(S_{k^s}) = \langle \mathscr{O}, \mathscr{O}(H - L_1), \mathscr{O}(H - L_2), \mathscr{O}(H - L_3), \mathscr{O}(H), \mathscr{O}(-K_{S_{k^s}} - H) \rangle. \tag{9.5}$$

The decomposition (9.5) is a mutation of the 3-block decomposition [64, (3)]. The latter is indeed obtained mutating the three exceptional objects $\mathscr{O}_{L_i}$ to the right with respect to $\mathscr{O}(-K_{S_{k^s}} - H) = \mathscr{O}(2H - L_1 - L_2 - L_3)$, as Karpov and Nogin do. The presentation (9.5) allows the following description of the three blocks:

$$\mathsf{E} = \langle \mathscr{O} \rangle, \quad \mathsf{G} = \langle \mathscr{O}(H - L_1), \mathscr{O}(H - L_2), \mathscr{O}(H - L_3) \rangle, \quad \mathsf{F} = \langle \mathscr{O}(H), \mathscr{O}(-K_{S_{k^s}} - H) \rangle.$$

So, the block E corresponds to the first component of (9.1). Recall that $\mathrm{End}\big(\bigoplus_{i=1}^3 \mathscr{O}(H - L_i)\big)$ is Morita equivalent to $Q \otimes k^s$, and that $\mathrm{End}(\mathscr{O}(H) \oplus \mathscr{O}(-K_{S_{k^s}} - H))$ is Morita equivalent to $B \otimes k^s$. The claim follows now by Proposition 1.6. □

Remark 9.3. A consequence of [25, Thm. 4.1] is that B comes with a natural K/k-unitary involution, equivalently, the corestriction of B from K to k is split. This involution on B was already constructed by Colliot-Thélène, Karpenko, and Merkurjev [41]. Furthermore, the corestriction of Q from L to k is split. Also, B is split by L and Q is split by K. Otherwise, any choices of K and L are possible and any choices of algebras B/K and Q/L are possible, subject to the above restrictions, see [25, Thm. 2.2].

Remark 9.4. Given a del Pezzo surface S of degree 6, Blunk constructs the triple (Q, B, KL) and shows that a toric presentation of S is uniquely determined by the equivalence class under pairwise L-isomorphisms of Q and K-isomorphisms of B, see [25, Thm. 2.4]. On the other hand, a consequence of Blunk's work is that the isomorphism class of S is uniquely determined by the equivalence class under pairwise k-isomorphisms of Q and B, see [25, Prop. 3.2]. By Theorem 1.17, the semiorthogonal decomposition (9.1) determines Q and B up to pairwise k-linear Morita equivalence, hence k-isomorphism, since the algebras involved are semi-simple of finite rank. We conclude that the semiorthogonal decomposition (9.1) identifies the isomorphism class of S.

Now we prove that rationality is equivalent to categorical representability in dimension 0.

Proposition 9.5. *Let S be a del Pezzo surface S of degree 6. The following are equivalent:*

(i) S has a k-rational point

(ii) S is k-rational
(iii) S is categorically representable in dimension 0
(iv) A_S is representable in dimension 0

Proof. By [43, §2], S is rational if and only if $S(k) \neq \varnothing$ if and only if $\mathrm{ind}(S) = 1$. By [25, Cor. 3.5], $S(k) \neq \varnothing$ is equivalent to the triviality of the Brauer classes κ and β and so (9.1) becomes a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(L/k), \mathsf{D}^{\mathrm{b}}(K/k)\rangle.$$

Hence *(i)* is equivalent to *(ii)*, which implies *(iii)*.

On the other hand, suppose that S has Picard rank 1. If S is categorical representable in dimension 0, then there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(l_1/k), \mathsf{D}^{\mathrm{b}}(l_2/k), \mathsf{D}^{\mathrm{b}}(l_3/k)\rangle, \tag{9.6}$$

base-changing to a 3-block exceptional collection over k^s. Hence, up to mutation, the decomposition (9.6) equals the decomposition (9.1) and hence A_S is representable in dimension 0 and both the Brauer classes of Q and B are trivial, hence $S(k) \neq \varnothing$ by [25, Cor. 3.5]. Thus *(iii)* implies *(iv)*, which implies *(i)*. If the Picard rank of S is > 1, then S is not minimal and we can consider its minimal model S', which is a del Pezzo surface of degree ≥ 7. We conclude in this case by appealing to the results from the previous sections. □

Remark 9.6. Colliot-Thélène, Karpenko, and Merkurjev [41, Rem. 4.5] provide a geometric argument to show that the splitting of K implies the non-minimality of S.

This can also be seen via the description of $B = \mathrm{End}(I)$, where I is the vector bundle constructed above. Indeed, the splitting of K, i.e., $B \cong A \times A^{\mathrm{op}}$, means that both $H^{\oplus 3}$ and $(H')^{\oplus 3}$ descend to vector bundles of rank 3 on S, hence both $\overline{H}$ and $\overline{H}'$ are Galois invariant line bundles, with Brauer obstruction to descent being the Brauer classes of A and A^{op}, respectively. Hence (cf. [100, Prop. 8]) we get birational morphisms $S \to \mathrm{SB}(A)$ and $S \to \mathrm{SB}(A^{-1})$ (compare with [41, Rem. 4.5]). As an interesting consequence, the birational Cremona involution ϕ from diagram (9.2) descends to a k-birational map $\mathrm{SB}(A) \dashrightarrow \mathrm{SB}(A^{-1})$.

Lemma 9.7. *Let S be a del Pezzo surface of degree 6. The index of X divides 6 and there is always a closed point whose degree equals the index.*

Proof. There is always a closed point of degree 6 on S. Indeed, the 6 points of intersection of the hexagon of exceptional curves defined over k^s is Galois invariant, hence defines a point of degree 6. Since the index is the greatest common divisor of the degrees of all closed points, the index of S divides 6.

Considering the Galois action on the hexagon of exceptional curves defined over k^s, there exists a quadratic extension K/k (resp. cubic extension) such that S_K has a triple (resp. pair) of skew exceptional curves, cf. [39, Lemme 1]. Blowing down, we have $S_K \to S_0$ where S_0 is a Severi–Brauer surface defined over K and also $S_L \to S_1$ where S_1 is a del Pezzo of degree 8 defined over L. In this later case, a lattice computation over k^s shows that S_1 is actually an involution surface. If we assume that S has index 2, then so does S_0, hence $S_0 \cong \mathbb{P}^2_K$. In particular, $S(K) \neq \varnothing$ so S has a closed point of degree 2. Similarly, if we assume that S has index 3, then so does S_1, hence $S_1(L) \neq 0$ by Springer's theorem. In particular, $S(L) \neq \varnothing$ so S has a closed point of degree 3. Finally, if we assume that S has index 1, then it must have a point of degree relatively prime to 6, hence $S(k) \neq \varnothing$ by [43]. □

We now want to show that the Griffiths–Kuznetsov component is well defined. This also gives a strengthening of [41, Lemma 4.6].

Proposition 9.8. *Let S be a del Pezzo surface of degree 6. Then the Griffiths–Kuznetsov component GK_S is well defined as a birational invariant as follows. Letting $S_1 \dashrightarrow S$ be a birational map, we have:*

- $\mathrm{ind}(S) = 6$ *if and only if both κ and β are nontrivial; there is a semiorthogonal decomposition* $\mathsf{A}_{S_1} = \langle \mathsf{T}, \mathsf{D}^{\mathrm{b}}(L/k, \kappa), \mathsf{D}^{\mathrm{b}}(K/k, \beta)\rangle$

- $\mathrm{ind}(S) = 3$ *if and only if* κ *is trivial and* β *is nontrivial; there is a semiorthogonal decomposition* $\mathsf{A}_{S_1} = \langle \mathsf{T}, \mathsf{D}^{\mathrm{b}}(K/k, \beta)\rangle$
- $\mathrm{ind}(S) = 2$ *if and only if* κ *is nontrivial and* β *trivial; there is a semiorthogonal decomposition* $\mathsf{A}_{S_1} = \langle \mathsf{T}, \mathsf{D}^{\mathrm{b}}(L/k, \kappa)\rangle$

where T *always denotes a category representable in dimension 0.*

Proof. We can reduce to the case when S is minimal, equivalently, has Picard rank 1. Indeed, if S is not minimal (and not rational), then there is a birational morphism $S \to S_0$ where S_0 is either a nonsplit involution surface (when $\mathrm{ind}(S) = 2$) or S' is a nonsplit Severi–Brauer surface (when $\mathrm{ind}(S) = 3$). We have already treated the Griffiths–Kuznetsov component in these cases, see §6 and § 7. For the interpretations of κ and β in the nonminimal cases, see Remark 9.9.

First, we remark that if β is trivial then $S(K) \neq \varnothing$, hence $\mathrm{ind}(S)|2$ by [41, Rem. 4.5]. Similarly, we will argue that if κ is trivial, then $S(L) \neq \varnothing$, hence $\mathrm{ind}(S)|3$. Indeed, β_L is split by Remark 9.3, so assuming that κ is split implies that S_L is categorically representable in dimension 0, hence is rational by Proposition 9.5. As a consequence, if $\mathrm{ind}(S) = 6$, then both β and κ are nontrivial and also S is birationally rigid by Proposition 3.10.

Now assume that S has index 3. Then S has a degree 3 point x by Lemma 9.7, and we can appeal to the description of elementary links detailed in §A.3. There is only one elementary link $\phi : S \dashrightarrow S'$ defined by degree three points x on S and x' on S'. Let X be the del Pezzo surface of degree 3 obtained as a resolution of ϕ with the following diagram (over k^s):

$$\begin{tikzcd}
 & X \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\
S \arrow[rr, dashed, "\phi"] \arrow[d, "\sigma_0"'] & & S' \arrow[d, "\tau_0"] \\
\mathbb{P}^2 \arrow[rr, dashed, "\phi_0"] & & \mathbb{P}^2
\end{tikzcd}$$

Denote by $G = \tau_0^* \mathscr{O}_{\mathbb{P}^2}(1)$ and F_i be the exceptional divisors of τ_0 and by abuse of notations, we write L for $\sigma^* L$ and G for $\tau^* G$. Finally, L_4, L_5, L_6 are the exceptional divisors of σ and F_4, F_5, F_6 the exceptional divisors of τ (over k^s we are blowing up three points). As calculated in Appendix A, we have that

$$\begin{aligned} G &= 5H - \textstyle\sum_{j=1}^{6} 2E_j \\ F_i &= 2H - \textstyle\sum_{j \neq i+3} E_j \quad \text{for } i = 1,2,3 \\ F_i &= 2H - \textstyle\sum_{j \neq i-3} E_j \quad \text{for } i = 4,5,6. \end{aligned} \tag{9.7}$$

We perform a series of mutations in $\mathsf{D}^{\mathrm{b}}(X)$ over k^s in order to compare $\mathrm{End}(I)$, $\mathrm{End}(J)$, $\mathrm{End}(I')$, $\mathrm{End}(J')$ and the residue fields $k(x)$ and $k(x')$. Let us choose the following 3-block semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S') = \langle \mathscr{O}_{S'} | \mathscr{O}(G), \mathscr{O}(-K_{S'} - G) | \mathscr{O}(-K_{S'} - G + F_1), \mathscr{O}(-K_{S'} - G + F_2), \mathscr{O}(-K_{S'} - G + F_3)\rangle, \tag{9.8}$$

which is the original 3-block decomposition of Karpov–Nogin [64]. We Denote E', F' and G' the three blocks in the order given in (9.8) as in Table 1. In particular, E' descends to $\mathsf{D}^{\mathrm{b}}(k)$, F' descends to $\mathsf{D}^{\mathrm{b}}(k, B')$ and G' descends to $\mathsf{D}^{\mathrm{b}}(k, Q')$. We mutate the second and the third block of (9.8) to the left with respect to $\mathscr{O}_{S'}$ to obtain

$$\mathsf{D}^{\mathrm{b}}(S') = \langle \mathscr{O}(K_{S'} + G), \mathscr{O}(-G) | \mathscr{O}(-G + F_1), \mathscr{O}(-G + F_2), \mathscr{O}(-G + F_3) | \mathscr{O}_{S'}\rangle, \tag{9.9}$$

since the top left mutation amounts to tensoring with $\omega_{S'} = \mathscr{O}(K_{S'})$. Via the blow-up τ, we get the following four-block decomposition of X:

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathscr{O}(K_{S'}+G), \mathscr{O}(-G) | \mathscr{O}(-G+F_1), \mathscr{O}(-G+F_2), \mathscr{O}(-G+F_3) | \mathscr{O}_X | \mathscr{O}_{F_4}, \mathscr{O}_{F_5}, \mathscr{O}_{F_6}\rangle. \tag{9.10}$$

The decomposition (9.10) is made of the four blocks F', G', E', and a new block H' arising from the blow up, descending to $\mathsf{D}^{\mathrm{b}}(k(x')/k)$. Finally, if we mutate H' to the left with respect to $\mathscr{O}_X$,

the evaluation sequence holds:
(9.11)
$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathscr{O}(K_{S'}+G), \mathscr{O}(-G) | \mathscr{O}(-G+F_1), \mathscr{O}(-G+F_2), \mathscr{O}(-G+F_3) | \mathscr{O}(-F_4), \mathscr{O}(-F_5), \mathscr{O}(-F_6) | \mathscr{O}_X \rangle.$$

Now we rewrite all line bundles in terms of H and L_i using the relations (9.7). This makes (9.11) into:
(9.12)
$$\begin{aligned}\mathsf{D}^{\mathrm{b}}(X) = \ & \langle \mathscr{O}(2K_X+2H-L_1-L_2-L_3), \mathscr{O}(2K_X+H) | \mathscr{O}(K_X+L_4), \mathscr{O}(K_X+L_5), \mathscr{O}(K_X+L_6) | \\ & \mathscr{O}(K_X+H-L_1), \mathscr{O}(K_X+H-L_2), \mathscr{O}(K_X+H-L_3) | \mathscr{O}_X \rangle\end{aligned}$$

where the blocks are now F', G', H', and E'. We apply the autoequivalence $\otimes\omega_X^\vee$ and mutate the first block F' to the right with respect to its right orthogonal to obtain:
(9.13)
$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathscr{O}(L_4), \mathscr{O}(L_5), \mathscr{O}(L_6) | \mathscr{O}(H-L_1), \mathscr{O}(H-L_2), \mathscr{O}(H-L_3) | \mathscr{O}(K_X) | \mathscr{O}(-\sigma^* K_S - H), \mathscr{O}(H) \rangle,$$

where the blocks are now G', H', E', and F'.

Now consider the semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathscr{O}_S | \mathscr{O}(H-L_1), \mathscr{O}(H-L_2), \mathscr{O}(H-L_3) | \mathscr{O}(H), \mathscr{O}(-K_S-H) \rangle, \tag{9.14}$$

as in Proposition 9.1. This decomposition has blocks E (descending to $\mathsf{D}^{\mathrm{b}}(k)$), G (descending to $\mathsf{D}^{\mathrm{b}}(k,Q)$) and F (descending to $\mathsf{D}^{\mathrm{b}}(k,B)$) in the order presented in (9.14), as in Table 1. This presentation provides, via σ^*, equivalences $\mathsf{F} \simeq \mathsf{F}'$, whence $\mathsf{D}^{\mathrm{b}}(k,B) \simeq \mathsf{D}^{\mathrm{b}}(k,B')$. On the other hand, $\mathsf{G} \simeq \mathsf{H}'$, whence $\mathsf{D}^{\mathrm{b}}(k,Q) \simeq \mathsf{D}^{\mathrm{b}}(k(x')/k)$. By symmetry, we have $\mathsf{D}^{\mathrm{b}}(k,Q') \simeq \mathsf{D}^{\mathrm{b}}(k(x)/k)$. Using Theorem 1.17, we conclude that $\kappa \in \mathrm{Br}(L)$ is trivial. If in addition $\beta \in \mathrm{Br}(K)$ is trivial, then $S(k) \neq \varnothing$ and S is rational. Otherwise, if $\beta \in \mathrm{Br}(K)$ is nontrivial, the category $\mathsf{D}^{\mathrm{b}}(K/k,\beta)$ is a birational invariant. Indeed, in this case, the index of S is 3 so can have no point of degree 2 and all birational maps $S \dashrightarrow S'$ decompose into elementary links of type $M_{6,3}$. We have proved that $\mathrm{ind}(S)|3$ implies that κ is trivial.

Now we handle the case where S has index 2. Then S has a degree 2 point x (cf. Lemma 9.7), and we can appeal to the description of elementary links detailed in §A.3. Let $\phi : S \dashrightarrow S'$ be the elementary link defined by the two degree two points x on S and x' on S'. Let X be the del Pezzo surface of degree 4 obtained as a resolution of ϕ with the following diagram (over k^s):

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ S \arrow[rr, dashed, "\phi"] \arrow[d, "\sigma_0"'] & & S' \arrow[d, "\tau_0"] \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi_0"] & & \mathbb{P}^2 \end{tikzcd}$$

Denote by $G = \tau_0^* \mathscr{O}_{\mathbb{P}^2}(1)$ and F_i be the exceptional divisors of τ_0 and by abuse of notations, we write L for $\sigma^* L$ and G for $\tau^* G$. Finally, L_4 and L_5 are the exceptional divisors of σ and F_4 and F_5 the exceptional divisors of τ (over k^s we are blowing up two points). As in the index 3 case, we perform a series of mutations in $\mathsf{D}^{\mathrm{b}}(X)$ over k^s in order to compare $\mathrm{End}(I)$, $\mathrm{End}(J)$, $\mathrm{End}(I')$, $\mathrm{End}(J')$ and the residue fields $k(x)$ and $k(x')$. We perform the same first mutation as in the index 3 case, and consider the decomposition (9.9) of $\mathsf{D}^{\mathrm{b}}(S'_{k^s})$. Via the blow-up τ, we get the following four-block decomposition of X:

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathscr{O}(K_{S'}+G), \mathscr{O}(-G) | \mathscr{O}(-G+F_1), \mathscr{O}(-G+F_2), \mathscr{O}(-G+F_3) | \mathscr{O}_X | \mathscr{O}_{F_4}, \mathscr{O}_{F_5} \rangle. \tag{9.15}$$

The decomposition (9.15) is made of the four blocks F', G', E', and H'. The latter arises from the blow-up and descends to $\mathsf{D}^{\mathrm{b}}(k(x')/k)$. Finally, if we mutate $\mathscr{O}_{F_4}$ and $\mathscr{O}_{F_5}$ to the left with respect to $\mathscr{O}_X$, the evaluation sequence holds:
(9.16)
$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathscr{O}(K_{S'}+G), \mathscr{O}(-G) | \mathscr{O}(-G+F_1), \mathscr{O}(-G+F_2), \mathscr{O}(-G+F_3) | \mathscr{O}(-F_4), \mathscr{O}(-F_5) | \mathscr{O}_X \rangle.$$

Now we rewrite all line bundles in terms of H and the L_i using the relations in §A.3 for the link $M_{6,2}$:

$$\begin{aligned} G &= 3H - L_1 - L_2 - L_3 - L_4 - 2L_5 \\ F_i &= H - L_i - L_5, \qquad i = 1, \dots, 4 \\ F_5 &= 2H - L_1 - L_2 - L_3 - L_4 - L_5 \end{aligned}$$

This makes (9.16) into:

(9.17)
$$\begin{aligned} \mathsf{D}^{\mathrm{b}}(X) \;=\; & \langle \mathscr{O}(K_X + L_4), \mathscr{O}(K_X + L_5) | \mathscr{O}(K_X + H - L_1), \mathscr{O}(K_X + H - L_2), \mathscr{O}(K_X + H - L_3) | \\ & \mathscr{O}(-H + L_4 + L_5), \mathscr{O}(K_X + H) | \mathscr{O}_X \rangle, \end{aligned}$$

where the blocks are F', G', H', and E'. Now we mutate the left orthogonal to $\mathscr{O}_X$ to the right to obtain:

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathscr{O}_X | \mathscr{O}(L_4), \mathscr{O}(L_5) | \mathscr{O}(H - L_1), \mathscr{O}(H - L_2), \mathscr{O}(H - L_3) | \mathscr{O}(-\sigma^* K_S - H), \mathscr{O}(H) \rangle \tag{9.18}$$

where the blocks are E', F', G', and H'. Now consider the semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathscr{O}_S | \mathscr{O}(H - L_1), \mathscr{O}(H - L_2), \mathscr{O}(H - L_3) | \mathscr{O}(H), \mathscr{O}(-K_S - H) \rangle, \tag{9.19}$$

as in Proposition 9.1. This decomposition has blocks E (descending to $\mathsf{D}^{\mathrm{b}}(k)$), G (descending to $\mathsf{D}^{\mathrm{b}}(k, Q)$) and F (descending to $\mathsf{D}^{\mathrm{b}}(k, B)$) in the order presented in (9.19), as in Table 1.

This presentation provides, via σ^*, equivalences $\mathsf{E} \simeq \mathsf{E}'$, and $\mathsf{G} \simeq \mathsf{G}'$, whence $\mathsf{D}^{\mathrm{b}}(k, Q) \simeq \mathsf{D}^{\mathrm{b}}(k, Q')$. On the other hand, $\mathsf{F} \simeq \mathsf{H}'$, whence $\mathsf{D}^{\mathrm{b}}(k, B) \simeq \mathsf{D}^{\mathrm{b}}(k(x')/k)$. By symmetry, we have $\mathsf{D}^{\mathrm{b}}(k, B') \simeq \mathsf{D}^{\mathrm{b}}(k(x)/k)$. Using Theorem 1.17, we conclude that $\beta \in \mathrm{Br}(K)$ is trivial. If in addition $\kappa \in \mathrm{Br}(L)$ is trivial, then $S(k) \neq \varnothing$ and S is rational. Otherwise, if $\kappa \in \mathrm{Br}(L)$ is nontrivial, the category $\mathsf{D}^{\mathrm{b}}(L/k, Q)$ is a birational invariant. Indeed, in this case, the index of S is 2 so can have no point of degree 3 and all birational maps $S \dashrightarrow S'$ decompose into elementary links of type $M_{6,2}$. We have proved that $\mathrm{ind}(S)|2$ implies that β is trivial. □

Remark 9.9. We now describe the geometry of the all possible cases listed in Table 4. In particular, for the nonminimal cases, we describe how the classes κ and β are related to the Brauer classes arising from their minimal model.

Cases 6.1, 6.2, 6.4, and 6.9 are minimal since they have Picard rank 1. Cases 6.9–6.14 are rational by Proposition 9.5.

Case 6.3 is the blow up of the Severi–Brauer surface $\mathrm{SB}(A)$ in a point x of degree 3 with residue field L, via the natural projection of $S \subset \mathrm{SB}(A) \times \mathrm{SB}(A^{-1})$. In fact, S resolves the standard Cremona quadratic transformation $\mathrm{SB}(A) \dashrightarrow \mathrm{SB}(A^{-1})$. Recall that $B = \mathrm{End}(I) = \mathrm{End}(I_1 \oplus I_2)$ and notice that I_1 and I_2 are the pull-backs of the natural rank 3 vector bundles on $\mathrm{SB}(A)$ and $\mathrm{SB}(A^{-1})$, respectively. Over k^s, the block G is obtained by right mutation of the category generated by the exceptional divisors of $S_{k^s} \to \mathbb{P}^2_{k^s}$. From this, we see that $\mathsf{D}^{\mathrm{b}}(L/k, \kappa) \simeq \mathsf{D}^{\mathrm{b}}(k(x)/k)$. Hence κ is trivial and $k(x) = L$.

We now argue that cases 6.5–6.8 are blow ups of an involution variety associated to (A, σ) in a point x of order 2 with residue field K, where the center of C_0 is isomorphic to L' (or k^2). The minimal model $\pi : S \to S_0$ has index 2 and degree > 6, hence must be an involution surface of index 2. Over k^s, consider the diagram:

$$\begin{tikzcd} & S \arrow[d, "\eta"] \arrow[ddl, "\pi"'] & \\ & S_1 \arrow[dl, "\sigma"] \arrow[dr, "\tau"] & \\ S_0 = \mathbb{P}^1 \times \mathbb{P}^1 & & \mathbb{P}^2 \end{tikzcd}$$

where σ blows up a point q with exceptional divisor F, τ blows up two points p_1 and p_2 with exceptional divisors L_1 and L_2, and η blows up a point p_3 with exceptional divisor L_3. Here, only π is defined over k and the exceptional divisor is $L_3 + F$. Let us denote by H_1 and H_2

the two ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ and by H the hyperplane section of $\mathbb{P}^2$, and by abuse of notations, all their pullbacks. Then we have $H_i = H - L_i$ for $i = 1, 2$ and $F = H - L_1 - L_2$. Now we consider the 3-block decomposition

$$\mathsf{D}^{\mathrm{b}}(\mathbb{P}^1 \times \mathbb{P}^1) = \langle \mathscr{O}(-H_1 - H_2) | \mathscr{O} | \mathscr{O}(H_1), \mathscr{O}(H_2) \rangle$$

where the first block descends to $\mathsf{D}^{\mathrm{b}}(k, A)$ and the third block descends to $\mathsf{D}^{\mathrm{b}}(k, C_0)$. Via the blow up π, we obtain

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathscr{O}(-L_3), \mathscr{O}(-F) | \mathscr{O}(-H_1 - H_2) | \mathscr{O} | \mathscr{O}(H_1), \mathscr{O}(H_2) \rangle$$

where the first block, call it H, descends to $\mathsf{D}^{\mathrm{b}}(k(x)/k)$. Mutating this block to the right with respect to $\mathscr{O}(-H_1 - H_2)$ and mutating $\mathscr{O}(-H_1 - H_2)$ to the right with respect to its right orthogonal, and substituting the previous relations, we obtain:

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{H} | \mathscr{O} | \mathscr{O}(H - L_1), \mathscr{O}(H - L_2) | \mathscr{O}(H - L_3) \rangle$$

so that the last two blocks descend together to $\mathsf{D}^{\mathrm{b}}(L/k, \kappa)$. We conclude that $\mathsf{D}^{\mathrm{b}}(L/k, \kappa) = \mathsf{D}^{\mathrm{b}}(k, C_0) \times \mathsf{D}^{\mathrm{b}}(k, A)$. It also follows that H descends to $\mathsf{D}^{\mathrm{b}}(K/k, \beta)$ and we conclude that β is trivial and $K = k(x)$.

By comparing with the index 2 cases on Table 3, we see that: case 6.5 is the blow up of case 8.3, Q' is Morita equivalent to $C_0(A, \sigma)$ and also Q'' is the corestriction of Q' from L'/k and is Morita equivalent to A; case 6.6 is the blow up of case 8.4 (a quadric of Picard rank 1) and Q' is Morita equivalent to $C_0(A, \sigma)$; case 6.7 is the blow up of case 8.5 and $Q' \otimes Q'' \otimes Q'''$ is trivial; and case 6.8 is the blow up of either cases 8.6 or 8.7 (which are anyway birational to each other), in fact, it is the resolution of this birational map.

Finally, when S is rational, we can see that: case 6.10 is the blow up of a rational quadric of Picard rank 1 along a point of degree 2 or and is not the blow up of $\mathbb{P}^2$; case 6.11 is the blow up of $\mathbb{P}^2$ in a point of degree 3 with residue field L and cannot be the blow up of a quadric; case 6.12 is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point of degree 2 or the blow up of a rational quadric of Picard rank 1 in two rational points (this resolves the rational map between these two); case 6.13 is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point of degree 2 with residue field L' or of $\mathbb{P}^2$ in a union of a rational point and a point of degree 2 with residue field L' (this resolves a birational map between the quadric and the Hirzebruch surface of degree 1). Case 6.14 is totally split.

Remark 9.10. Let $V_1 = J^{\min}$ and $V_2 = I^{\min}$ (recall Definition 2.10). These vector bundles are tilting bundles for the blocks F and G, respectively, and have the following properties: $A_1 = \mathrm{End}(V_1)$ is Morita equivalent to Q and $A_2 = \mathrm{End}(V_2)$ is Morita equivalent to B; V_1 is indecomposable if and only if L is a field; and V_2 is indecomposable if and only if K is a field. In particular, both V_i are indecomposable if and only if S is minimal. We list the ranks and 2nd Chern classes of the vector bundles V_i on Table 4.

The calculation of the second Chern classes of the vector bundles V_1 and V_2 is easily obtained by their description over k^s. In particular, we notice that $\mathrm{ind}(S) = \gcd(c_2(V_1), c_2(V_2))$ unless $\mathrm{ind}(S) = 6$, in which case $\gcd(c_2(V_1), c_2(V_2)) = 12$. When $\mathrm{ind}(S) = 6$, we have to appeal to a particular generator ($\omega_S^{\oplus 2}$ works) of the remaining block to obtain a bundle with second Chern class equal to 6.

10. del Pezzo surfaces of degree 5

Let S be a Del Pezzo surface of degree 5. It is a classical fact (announced by Enriques [48] and proved first by Swinnerton-Dyer [99]), that $S(k) \neq \varnothing$ (see [95] for a different proof).

The base change S_{k^s} is the blow-up of $\mathbb{P}^2_{k^s}$ at four points in general position. Such a surface has 10 exceptional lines. Fix $x \in S(k)$. If x lies in the intersection of two exceptional lines, then S is not minimal (see [82, 29.4.4.(v)]). So we can suppose that x does not lie on the intersection of two exceptional lines and consider the geometric construction described by Manin to show that S is rational: let $X \to S$ be the blow-up of x, and D its exceptional divisor. Then there are 5 pairwise nonintersecting exceptional lines $L_1, \ldots, L_5$, where $L_5 = D$, on X_{k^s}.

Manin shows that on X there is an exceptional divisor $Z \subset X$ whose contraction gives a birational map onto a del Pezzo surface of degree 9. Since the target also has a rational point, we have a birational map $\pi : X \to \mathbb{P}^2_k$. We have a diagram of birational morphisms

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\varepsilon"] & \\ \mathbb{P}^2_k & & S, \end{tikzcd}$$

where $\pi : X \to \mathbb{P}^2_k$ is the blow-up of a closed point of degree 5 in general position.

Passing to the algebraic closure, we can describe these birational maps in the following way: let $p_1, \dots, p_5$ be five points in general position on $\mathbb{P}^2_{k^s}$. Then X_{k^s} is a del Pezzo of degree four which has 16 exceptional lines: five of them are the exceptional divisors $E_1, \dots, E_5$ of $\overline{\pi}$, ten of them are the strict transforms of the lines $L_{i,j}$ passing through the points p_i and p_j. The last one is the strict transform D of the conic through the five points p_i. This is, indeed, the exceptional divisor of the blow-up $\overline{\varepsilon} : X_{k^s} \to S_{k^s}$. On the other hand, the lines $E_1, \dots, E_5$ all meet twice D, and it can be checked that they are the only exceptional lines on X_{k^s} with such property. Since D is defined over k; it follows that $E_1, \dots, E_5$ are Galois-invariant and hence the cycle Z is the descent of the disjoint union of the E_i's. Conversely, given any point of degree 5 (geometrically) in general position on $\mathbb{P}^2_k$, we can blow it up, and then blow down the strict transform of the conic through the point to obtain a del Pezzo surface of degree 5.

Moreover, the surface S_{k^s} is a del Pezzo surface of degree 5 and can therefore be realized as the blow up of $\mathbb{P}^2_{k^s}$ in four points in general positions. Given S_{k^s}, this can be realized by the choice of 4 pairwise nonintersecting exceptional lines $L_1, \dots, L_4$. It is easy to see, via the previous construction, that we have five choices, one for each of the points p_i. Once we fix such a point, it is then enough to consider all the lines joining it to the other four points, which are blown up by $\overline{\pi}$ to exceptional lines (call them loosely $L_1, \dots, L_4$), which are, in turn, blown down by $\overline{\varepsilon}$ to 4 exceptional nonintersecting lines. Let us then fix p_5, so that L_i is the (strict transform of) the line through p_5 and p_i, and consider the blow-down $\eta : S_{k^s} \to \mathbb{P}^2_{k^s}$. This latter map is not, in general, defined over k, so we will avoid the "overline" notation to mark this difference.

We end up with the following diagram:

$$\begin{tikzcd} & \{E_1, \dots, E_5\} \arrow[r, hook] & X_{k^s} \arrow[dl, "\overline{\pi}"'] \arrow[dr, "\overline{\varepsilon}"] & \{D, L_1, \dots, L_4\} \arrow[l, hook'] & \\ & & & S_{k^s} \arrow[d, "\eta"] & \{x, L_1, \dots, L_4\} \arrow[l, hook'] \\ \{p_1, \dots, p_5\} \arrow[r, hook] & \mathbb{P}^2_{k^s} \arrow[rr, dashed, "\phi"] & & \mathbb{P}^2_{k^s} & \{x, q_1, \dots, q_4\}, \arrow[l, hook'] \end{tikzcd} \tag{10.1}$$

where q_i are the points blown-up by η, and ϕ the birational map obtained by composition.

Let us denote by $\mathscr{O}_{S_{k^s}}(H) = \eta^* \mathscr{O}_{\mathbb{P}^2_{k^s}}(1)$. We can assume that this line bundle is not defined over k, since we can suppose that S is minimal. Otherwise, S is the blow-up of $\mathbb{P}^2_k$.

We will explicitly use this construction to show that the three-block exceptional collection described by Karpov and Nogin [64] over k^s descends to a zero-dimensional semiorthogonal decomposition of $\mathsf{D}^{\mathrm{b}}(S)$.

Proposition 10.1. *Any del Pezzo surface of degree* 5 *is k-rational and is categorically representable in dimension 0. In particular, there is a degree* 5 *étale k-algebra l and a semiorthogonal decomposition*

$$\mathsf{A}_S = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(l/k) \rangle$$

hence A_S is also representable in dimension 0. Moreover, l is a field if and only if $\rho(S) = 1$.

S	ind(S)	A_1	c_2	rk	A_2	c_2	rk
$S \subset \mathbb{P}^5_k$	1	k	2	2	l	20	5

TABLE 5. The invariants of a del Pezzo surface S of degree 5 of Picard rank 1. Here, the algebras $\mathrm{End}(V_1) = k$ and $\mathrm{End}(V_2) = l$ are listed up to Morita equivalence; l is an étale k-algebra of degree 5, and is a field if and only if $\rho(S) = 1$; and Z, per, and ind refer to the center, period, and index of A_i; and c_2 and rk refer to the 2nd Chern class and rank of V_i. Note that $(V_1)_{k^s}$ is the unique extension of $\mathscr{O}(H)$ by $\mathscr{O}(K_S - H)$, while $(V_2)_{k^s}$ is the direct sum $\mathscr{O}(H) \oplus \bigoplus_{i=1}^4 \mathscr{O}(L_i - K_S - H)$.

Proof. Over k^s, Karpov and Nogin [64, §4] provide the following 3-block decomposition:

$$\mathsf{D}^{\mathrm{b}}(S_{k^s}) = \langle \mathscr{O}_{S_{k^s}} \,|\, F \,|\, \mathscr{O}_{S_{k^s}}(H), \mathscr{O}_{S_{k^s}}(L_1 - K_{S_{k^s}} - H), \ldots, \mathscr{O}_{S_{k^s}}(L_4 - K_{S_{k^s}} - H)\rangle, \tag{10.2}$$

where F is the rank 2 vector bundle given by the nontrivial extension

$$0 \longrightarrow \mathscr{O}_{S_{k^s}}(-K_{S_{k^s}} - H) \longrightarrow F \longrightarrow \mathscr{O}_{S_{k^s}}(H) \longrightarrow 0. \tag{10.3}$$

These blocks are denoted E, F, and G. Consider the rank 5 vector bundle on S_{k^s}

$$V = \bigoplus_{i=1}^{4} \mathscr{O}_{S_{k^s}}(L_i - K_{S_{k^s}} - H) \oplus \mathscr{O}_{S_{k^s}}(H), \tag{10.4}$$

and $B_V = \mathrm{End}_{S_{k^s}}(V)$. Since the five line bundles form an exceptional block, we have that $B_V \simeq (k^s)^5$ and V is a tilting bundle for the block G. We are going to show that V descends to k, hence B_V descends to a degree 5 extension l of k, and G descends to k. To this end, recall that the functors $\varepsilon^* : \mathsf{D}^{\mathrm{b}}(S) \to \mathsf{D}^{\mathrm{b}}(X)$ and $\overline{\varepsilon}^* : \mathsf{D}^{\mathrm{b}}(S_{k^s}) \to \mathsf{D}^{\mathrm{b}}(X_{k^s})$ are fully faithful. We analyze the pull-back of the three-block collection (10.2) as an exceptional collection in $\mathsf{D}^{\mathrm{b}}(X_{k^s})$ to deduce the descent. In order to do that, we first stress the structure of the Picard group of X_{k^s}.

On one hand, we have the line bundle $\mathscr{O}_{X_{k^s}}(H) = \overline{\varepsilon}^*\mathscr{O}_{S_{k^s}}(H) = \overline{\varepsilon}^*(\eta^*\mathscr{O}_{\mathbb{P}^2_{k^s}}(1))$. We have the five exceptional lines D (the exceptional locus of ε) and $L_1, \ldots, L_4$ (by abuse of notation, we denote $L_i = \overline{\varepsilon}^* L_i$). We have that $K_{X_{k^s}} = \overline{\varepsilon}^* K_{S_{k^s}} - D$.

On the other hand, we have the line bundle $\mathscr{O}_{X_{k^s}}(G) = \overline{\pi}^*\mathscr{O}_{\mathbb{P}^2_{k^s}}(1)$, and the five exceptional lines $E_1, \ldots, E_5$ of the blow-up $\overline{\pi}$. We have that $K_{X_{k^s}} = -3G + \sum_{i=1}^5 E_i$.

The divisor D is the strict transform via π of the conic passing through the five points p_i. Hence $D = 2G - \sum_{i=1}^5 E_i$, so that $-K_{X_{k^s}} = D + G$. For each $1 \le i \le 4$, the divisor L_i is the strict transform of the line through p_5 and p_i, so that $L_i = G - E_5 - E_i$.

Finally, the birational map ϕ is given by the system of cubics through the five points p_i which have multiplicity 2 in p_5. Indeed, the map ϕ is not defined over the curve given by the conic D and the four lines joining p_i to p_5. This is a curve C of degree 6. The map ϕ can be written as three homogeneous polynomials of the same degree d, which we proceed to determine. The degeneracy locus C of ϕ is then the zero locus of the determinant of the Jacobian matrix. This is a 3×3 matrix with entries of degree $d-1$, hence the polynomial defining C has degree $3(d-1)$. Thus $6 = \deg(C) = 3(d-1)$, from which we deduce that $d = 3$. This implies that ϕ is given by a linear system of cubics. The hyperplane section H of the target $\mathbb{P}^2_{k^s}$ corresponds then to the linear system $3G - \sum_{i=1}^n m_i x_i$, where x_i are the points of multiplicity $m_i > 0$ of the map ϕ. By construction, it is clear that the points of positive multiplicity are exactly the p_i (they are transformed via ϕ into lines), so we get the linear system $3G - \sum_{i=1}^5 m_i p_i$. This linear system must have degree 1, so we get $9 - \sum_{i=1}^5 m_i^2 = 1$. Since ϕ is given by the cubics passing through the points p_i, with multiplicity two in p_5, we deduce that $m_5 = 2$ and $m_i = 1$ for $1 \le i \le 4$.

From this, we get that

$$\overline{\varepsilon}^* H = 3G - 2E_5 - \sum_{i=1}^{4} E_i = 3G - \sum_{i=1}^{5} E_i - E_5 = -K_{X_{k^s}} - E_5. \tag{10.5}$$

On the other hand, consider, for any $1 \leq i \leq 4$, the divisor $L_i - K_{S_{k^s}} - H$ and pull it back via $\overline{\varepsilon}$. Recall that $\overline{\varepsilon}^* K_{S_{k^s}} = K_{X_{k^s}} + D$, and that $L_i = H - E_i - E_5$ over X_{k^s}. We then get

$$\overline{\varepsilon}^*(L_i - K_{S_{k^s}} - H) = G - E_5 - E_i - K_{X_{k^s}} + D - H.$$

using equation (10.5), we substitute L to get:

$$\overline{\varepsilon}^*(L_i - K_{S_{k^s}} - H) = G + D - E_i = -K_{X_{k^s}} - E_i. \tag{10.6}$$

Using equations (10.5) and (10.6), the block G pulls back via $\overline{\varepsilon}$ to the exceptional block

$$\langle \mathscr{O}_{X_{k^s}}(-K_{X_{k^s}} - E_1), \dots \mathscr{O}_{X_{k^s}}(-K_{X_{k^s}} - E_5)\rangle, \tag{10.7}$$

in $\mathsf{D}^{\mathrm{b}}(X_{k^s})$, and where we have performed a mutation of the completely orthogonal bundles in the block to arrive at this ordering. It follows that

$$\overline{\varepsilon}^* V = \bigoplus_{i=1}^{5} \mathscr{O}_{X_{k^s}}(-K_{X_{k^s}} - E_i) = \omega_{X_{k^s}}^{\vee} \otimes \bigoplus_{i=1}^{5} \mathscr{O}_{X_{k^s}}(-E_i),$$

hence $\overline{\varepsilon}^* V$ descends to a vector bundle of rank 5 on X and V descends to a vector bundle (again denoted by V) of rank 5 on S since ε^* is fully faithful. We see that $\mathrm{End}(V)$ is then isomorphic to the structure sheaf l of the degree 5 point in $\mathbb{P}^2_k$ that is blown up by π, which is an k-étale algebra of degree 5. We conclude that G descends to a block over k equivalent to $\mathsf{D}^{\mathrm{b}}(l/k)$.

It is now sufficient to prove that F descends to k, which would imply that $\mathsf{F} \simeq \mathsf{D}^{\mathrm{b}}(k)$. Since E and G descend to blocks of $\mathsf{D}^{\mathrm{b}}(S)$ defined over k, so does F, being the orthogonal complement. Hence by Theorem 2.11, F descends to a block equivalent to $\mathsf{D}^{\mathrm{b}}(k, \alpha)$ for some $\alpha \in \mathrm{Br}(k)$. We proceed to show that α is trivial. To this end, consider the semiorthogonal decomposition (10.2). Orlov's formula applied to the blow up $\overline{\varepsilon}$ gives the following 4-block semiorthogonal decomposition:

$$\mathsf{D}^{\mathrm{b}}(X_{k^s}) = \langle \mathscr{O}_{X_{k^s}}(-D) | \mathscr{O}_{X_{k^s}} | \overline{\varepsilon}^* F | \mathscr{O}_{X_{k^s}}(-K_{X_{k^s}} - E_1), \dots, \mathscr{O}_{X_{k^s}}(-K_{X_{k^s}} - E_5)\rangle \tag{10.8}$$

where we used the identifications (10.5) and (10.6) in writing G. Mutating G to the left with respect to its left orthogonal, we obtain:

$$\mathsf{D}^{\mathrm{b}}(X_{k^s}) = \langle \mathscr{O}_{X_{k^s}}(-E_1), \dots, \mathscr{O}_{X_{k^s}}(-E_5) | \mathscr{O}_{X_{k^s}}(-D) | \mathscr{O}_{X_{k^s}} | \overline{\varepsilon}^* F\rangle. \tag{10.9}$$

As the first block, the mutation of G, is generated by the exceptional divisors of the blow up $\overline{\pi}$, by Orlov's blow up formula, it follows that $\langle \mathscr{O}_{X_{k^s}}(-D) | \mathscr{O}_{X_{k^s}} | \overline{\varepsilon}^* F\rangle$ can be identified with $\overline{\pi}^* \mathsf{D}^{\mathrm{b}}(\mathbb{P}^2_{k^s})$. Since π, as well as the line bundles $\mathscr{O}_X(-D)$ and $\mathscr{O}_X$, is defined over k, we arrive at a 3-block decomposition $\pi^* \mathsf{D}^{\mathrm{b}}(\mathbb{P}^2_k) = \langle \mathscr{O}_X(-D), \mathscr{O}_X, \mathsf{D}^{\mathrm{b}}(k, \alpha)\rangle$. By the uniqueness of 3-block decompositions on $\mathbb{P}^2$ (see [52] or Proposition 4.1), and by Corollary 1.18, we conclude that α is trivial. Moreover, $\overline{\varepsilon}^* F$ can be mutated into an exceptional line bundle $\overline{\pi}^* \mathscr{O}_{\mathbb{P}^2_{k^s}}(i)$ via a sequence of mutations inside $\overline{\pi}^* \mathsf{D}^{\mathrm{b}}(\mathbb{P}^2_{k^s})$, which are a posteriori, all defined over k. It follows that $\varepsilon^* F$ can be mutated to $\pi^* \mathscr{O}_{\mathbb{P}^2}(i)$, hence F descends to a k-exceptional vector bundle of rank 2. □

Remark 10.2. Let $V_1 = F$ and $V_2 = V$. These vector bundles are tilting bundles for the blocks F and G, respectively, and have the following properties: $A_1 = \mathrm{End}(V_1)$ is Morita equivalent to k and $A_2 = \mathrm{End}(V_2)$ is Morita equivalent to l; V_1 is always indecomposable; and V_2 is indecomposable if and only if l is a field. In particular, both V_i are indecomposable if and only if S is minimal. We list the ranks and 2nd Chern classes of the vector bundles V_i on Table 5.

The calculation of the second Chern classes of the vector bundles V_1 and V_2 is easily obtained by their description over k^s. In particular, we notice that $1 = \mathrm{ind}(S) = \gcd(c_2(V_1), c_2(V_2), c_2(\omega_S^{\oplus 2}))$ showing that we must include a generator of each block.

Part 3. Appendix: Explicit calculations with elementary links

Appendix A. Elementary links for non-rational minimal del Pezzo surfaces

In this appendix, we consider all possible links between two non-rational minimal del Pezzo surfaces S and S'. Let us briefly sketch the notion of elementary link in Sarkisov's program from [58]. We consider $\pi : S \to Z$ to be a minimal geometrically rational surface with an extremal

contraction. Hence one obtains that either Z is a point and S a minimal surface with Picard number 1, or Z is a Severi–Brauer curve and $S \to Z$ is a minimal conic bundle, and the Picard number of S is 2.

If $S \to Z$ and $S' \to Z'$ are such extremal contractions, an *elementary link* is a birational map $\phi : S \dashrightarrow S'$ of one of the following types:

Type I) There is a commutative diagram

$$\begin{tikzcd} S \arrow[d] & S' \arrow[l, "\sigma"'] \arrow[d] \\ Z & Z' \arrow[l, "\psi"'] \end{tikzcd}$$

where $\sigma : S' \to S$ is a Mori divisorial elementary contraction and $\psi : Z' \to Z$ is a morphism. In this case, $Z = \mathrm{Spec}(k)$, $\rho(S) = 1$, S is a minimal del Pezzo, and $S' \to Z'$ is a conic bundle over a Severi–Brauer curve.

type II) There is a commutative diagram

$$\begin{tikzcd} S \arrow[d] & X \arrow[l, "\sigma"'] \arrow[r, "\tau"] & S' \arrow[d] \\ Z & & Z' \arrow[ll, "\cong"'] \end{tikzcd}$$

where $\sigma : X \to S$ and $\tau : X \to S'$ are Mori divisorial elementary contractions. In this case, S and S' have the same Picard number.

type III) There is a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d] & S' \arrow[d] \\ Z \arrow[r, "\psi"] & Z' \end{tikzcd}$$

where $\sigma : S \to S'$ is a Mori divisorial elementary contraction and $\psi : Z \to Z'$ is a morphism. These links are inverse to links of type I.

type IV) There is a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "\cong"] \arrow[d] & & S' \arrow[d] \\ Z \arrow[dr, "\psi"] & & Z' \arrow[dl, "\psi'"'] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

where Z and Z' are Severi–Brauer curves and ψ and ψ' are the structural morphisms. This link amounts to a change of conic bundle structure on S.

Iskovskikh shows that any birational map $S \dashrightarrow S'$ between minimal geometrically rational surfaces factors into a finite sequence of elementary links [58]. We are interested in the case where S and S' are both non k-rational del Pezzo surfaces of Picard rank 1.

Thanks to Iskovskikh's classification, a link of type I (resp. III) can happen in the non k-rational cases only if S (resp. S') has either degree 8 and a point of degree 2, or has degree 4 and a rational point [58, Thm. 2.6]. It follows that if we assume S to not be of this type, then we only have to consider links of type II where $\rho(S) = \rho(S') = 1$. By Iskovskikh's classification, there is a finite list of such links. In particular, if we assume S to not be k-rational, and S' not isomorphic to S, then we have that $\deg(S) = \deg(S')$ can be only 6, 8 or 9, and we are left with five possible links.

$\deg(S)$	$\deg(x)$	Transformation Matrix
9	3	$M_{9,3} = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$
	6	$M_{9,6} = \begin{pmatrix} 5 & 4 \\ -6 & -5 \end{pmatrix}$
8	4	$M_{8,4} = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$
6	2	$M_{6,2} = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$
	3	$M_{6,3} = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$

TABLE 6. The possible links of type II between nonisomorphic non-k-rational Del Pezzo surfaces. The transformation matrix expresses the base change from $\sigma^*\omega_S, E$ to $\tau^*\omega_{S'}, F$.

Let $\phi : S \dashrightarrow S'$ be a link of type II between non-k-rational non-isomorphic surfaces, and recall that we assume that if S has degree 8 (resp. 4), there is no degree 2 (resp. rational) point on S. Then $\deg(S) = \deg(S')$ and there is a closed point x in S of degree d such that ϕ is resolved as

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ S & & S' \end{tikzcd}$$

where σ is the blow-up of x and τ is the blow up of a point x' of degree d on S'. Let E be the exceptional divisor of σ and F be the exceptional divisor of τ. If one considers the $\mathbb{Z}$-bases $(\sigma^*\omega_S, E)$ and $(\tau^*\omega_{S'}, F)$ of $\mathrm{Pic}(X)$, the birational map ϕ can be described by the transformation matrix between these two bases. We list all possibilities from [58, Thm. 2.6] in Table 6.

In order to understand the behavior of the semiorthogonal decompositions of S and S' under birational maps, it is enough to consider the links listed in Table 6. We will proceed as follows: given a link $\phi : S \dashrightarrow S'$, we describe the birational map $\overline{\phi} : S_{k^s} \dashrightarrow S'_{k^s}$. Notice that $\overline{\phi}$ is not a link, since over k^s, we can factor σ into a finite sequence of blow-ups (actually, $\deg(x)$ of them).

In order to describe $\overline{\phi}$ we will consider the description of the Picard group of S_{k^s}. If $\deg(S) = 9$, then $\overline{\phi}$ is described by a linear system on $\mathbb{P}^2_{k^s}$, the so-called *homaloidal system* of $\overline{\phi}$. If $\deg(S) = 8$, we find similarly a homaloidal system on the quadric $S_{k^s} \subset \mathbb{P}^3_{k^s}$. Finally, if $\deg(S) = 6$, we have to choose models $S_{k^s} \to \mathbb{P}^2_{k^s}$ and $S'_{k^s} \to \mathbb{P}^2_{k^s}$, and describe how $\overline{\phi}$ corresponds to a homaloidal system on $\mathbb{P}^2_{k^s}$.

In general, let us consider a linear system on $\mathbb{P}^2$ of the form $G = nH - \sum_{i=1}^r m_i p_i$, where $n > 0$ and $m_i > 0$ are integers, H denotes the hyperplane divisor, and p_i are points on $\mathbb{P}^2$. Such a linear system defines a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ if and only if $\deg(G) = 1$ and the curves in the linear system are rational. We can resolve the birational map by blowing up the points p_i, and call X the blow up. We obtain then the following conditions on n and m_i:

$$\begin{cases} 3n - 3 = \sum_{i=1}^r m_i & \text{since } G.K_X = 3 \\ n^2 - 1 = \sum_{i=1}^r m_i^2 & \text{since } G^2 = 1 \end{cases} \tag{A.1}$$

In order to describe the system, we will extensively use the Cauchy–Schwartz inequality

$$\Big(\sum_{i=1}^r m_i\Big)^2 \leq r \sum_{i=1}^r m_i^2$$

to bound the possible values of n. In particular, we obtain that $9(n-1) \leq r(n+1)$. Moreover, it is clear that if $n = 1$, then $G = H$ defines an isomorphism. Hence we have that $r \geq 3$. Let us spell out all the possible birational transforms with $3 \leq r \leq 6$.

If $r = 3$, then $n = 2$. Conditions (A.1) give $m_i = 1$. These are the standard quadratic transformations $\phi_2 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

If $r = 4$, then $n = 2$. Conditions (A.1) give $\sum_{i=1}^4 m_i^2 = 3$, which is impossible since we only consider $m_i > 0$. It follows in particular that $n = 2$ if and only if $r = 3$.

If $r = 5$, then $n = 3$. Conditions (A.1) give $m_1 = \ldots = m_4 = 1$ and $m_5 = 2$, so that there is only one possibility (up to renumbering the points). We denote these birational maps by $\phi_3 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

If $r = 6$, then $n \leq 5$. Conditions (A.1) give two possibilities. The first one is $n = 5$, $m_i = 2$. We denote these birational maps by $\phi_5 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. The second possibility is $n = 4$, $m_1 = m_2 = m_3 = 1$ and $m_4 = m_5 = m_6 = 2$ (up to renumbering the points). One can check that the birational map $\phi_4 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the composition of two standard quadratic transforms, the first one $\phi_2 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ blows up p_1, p_2, and p_3, so that p_4, p_5, and p_6 belong to the target $\mathbb{P}^2$. The second standard quadratic transform blows-up p_4, p_5, and p_6.

With this calculation in mind, we are able to describe the homaloidal systems of $\mathbb{P}^2_{k^s}$ for the links in degree 6, 8, and 9 del Pezzo surfaces in Table 6.

A.1. **Degree 9.** If $\deg(S) = 9$, then $S_{k^s} \simeq \mathbb{P}^2_{k^s}$. Hence we are considering a birational map $\overline{\phi} : \mathbb{P}^2_{k^s} \to \mathbb{P}^2_{k^s}$.

The link $M_{9,3}$ base changes to the following diagram (we omit the overlines for ease of notations):

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi"] & & \mathbb{P}^2 \end{tikzcd}$$

where σ blows up three points p_1, p_2, and p_3. Hence ϕ is the standard quadratic transformation. Since $G = 2H - L_1 - L_2 - L_3$, we have that $\sigma^*\mathscr{O}(-1) = \tau^*\mathscr{O}(1) \otimes \omega_X$.

The link $M_{9,6}$ base changes to the following diagram (we omit the overlines for ease of notations):

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi"] & & \mathbb{P}^2 \end{tikzcd}$$

where σ blows up six points $p_1 \ldots, p_6$. As explained above, we have two possibilities: ϕ is either of type ϕ_5 or ϕ_4. Checking the action of the matrix $M_{9,6}$ one gets that ϕ is of type ϕ_5, since all m_i must be equal. Since $G = 5H - 2L_1 - 2L_2 - 2L_3 - 2L_4 - 2L_5 - 2L_6$, we have that $\sigma^*\mathscr{O}(-1) = \tau^*\mathscr{O}(1) \otimes \omega_X^{\otimes 2}$.

These considerations lead to a simple proof of Amitsur's theorem in the case of degree 3 central simple algebras.

Proposition A.1. *Let S be a non-k-rational minimal surface of degree 9, and let T_S^+ (resp. T_S^-) be the category generated by the descent of a hyperplane section $\mathscr{O}_{S_{k^s}}(1)$ (resp. $\mathscr{O}_{S_{k^s}}(-1)$). For any birational map $\phi : S \dashrightarrow S'$ to a minimal surface S' (which must be of degree 9), we have either an equivalence $\mathsf{T}_S^+ \simeq \mathsf{T}_{S'}^+$, or an equivalence $\mathsf{T}_S^+ \simeq \mathsf{T}_{S'}^-$.*

Proof. It is easy to see that any elementary link $S \dashrightarrow S'$ gives an equivalence between T_S^+ and $\mathsf{T}_{S'}^-$, just by pull-back to X and tensor by either ω_X or $\omega_X^{\otimes 2}$. □

A.2. **Degree 8.** If $\deg(S) = 8$ and S is an involution surface, then $S_{k^s} \subset \mathbb{P}^3_{k^s}$ is a quadric surface. We are interested in the hyperplane section $\mathscr{O}(1)$ and in $\mathscr{O}(2) = \omega_{S_{k^s}}^\vee$, the anticanonical divisor. The latter is always defined on S.

The link $M_{8,4}$ base changes to the following diagram (we omit the overlines for ease of notations):

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ S \arrow[rr, dashed, "\phi"] & & S' \end{tikzcd}$$

where σ blows up four points $p_1, \ldots, p_4$. Using the action of the matrix on the Picard group of X, we get that $\sigma^*\mathscr{O}(1) = \tau^*\mathscr{O}(1) \otimes \omega_X^{\otimes 2}$.

As a corollary, we can see that involution surfaces with Picard rank one and no point of degree ≤ 2 are birationally semirigid. We can give a further refinement of this result purely using the algebraic theory of quadratic forms. Recall that an involution surface has Picard rank 2 or 1 depending on whether it has trivial or nontrivial discriminant, respectively, see Example 3.3.

Proposition A.2. *Let S and S' be k-birational involution surfaces over an arbitrary field k. Then S and S' are k-isomorphic in the following cases:*

(i) S and S' are anisotropic quadrics in $\mathbb{P}^3$,
(ii) S (and hence S') has nontrivial discriminant and no rational point,
(iii) S (and hence S') has index 4.

Proof. We know that S is rational as soon as it has a rational point, in which case it can have k-birational yet nonisomorphic partners. Hence we can assume that $S(k) = \varnothing$. By considering the Galois action on the the Picard groups, we see that k-birational involution surfaces have the same discriminant. Now let S and S' be associated to quadratic pairs (A, σ) and (A', σ').

Part *(i)* is a classical result in the theory of quadratic forms, see [78, XII.2.2]. This handles the case when both A and A' are split, hence we may assume that A is not split.

We recall a result proved by Arason for quadric surfaces and generalized by Tao [104, Thm. 4.8(b)] to involution surfaces:

$$\ker(\mathrm{Br}(k) \to \mathrm{Br}(k(S))) = \begin{cases} \langle A \rangle & \text{if the discriminant is nontrivial} \\ \langle C_0^+, C_0^- \rangle & \text{if the discriminant is trivial} \end{cases} \tag{A.2}$$

Any k-birational isomorphism between S and S' induces a k-isomorphism of function fields $k(S) \cong k(S')$, under which the Brauer group kernels $\ker(\mathrm{Br}(k) \to \mathrm{Br}(k(S)))$ and $\ker(\mathrm{Br}(k) \to \mathrm{Br}(k(S')))$ coincide. We now proceed according to cases.

For part *(ii)*, S and S' have nontrivial discriminant, so (A.2) implies that the cyclic subgroups of the Brauer group generated by A and A' are the same. However, since both algebras carry involutions of the first kind, they are of period 2 in the Brauer group (we are assuming they are not split). Thus A and A' are Brauer equivalent, hence are k-isomorphic since they have the same degree. We will now show that the quadratic pairs (A, σ) and (A', σ') become adjoint to anisotropic quadratic forms over $F = k(\mathrm{SB}(A)) \cong k(\mathrm{SB}(A'))$ (and S_F and S'_F are still F-birational), which by case *(i)* implies that they are isomorphic over F, which in turn, by the following Lemma A.3, implies that they are isomorphic over k. This anisotropicity statement follows, in the case when S (hence S') has index 4, i.e., that $A \cong A'$ is division, from Tao [104, Prop. 4.18, Cor. 4.20, Cor. 4.21] (see also Karpenko [61, Thm. 5.3] in greater generality). In the case when S (hence S') has index 2, which under our assumptions implies that $A \cong A'$ has index 2, a result of Karpenko [62, Thm. 3.3], stating that the Witt index of a quadratic pair (A, σ) over F is divisible by the index of the A, implies that the quadratic pairs σ and σ' are either anisotropic or hyperbolic over F (see also [16]). The later is impossible, since by assumption the quadratic pairs have nontrivial discriminant, which remains nontrivial over F. Another way to see the index 2 case, at least when the characteristic is not 2, is by writing $A = M_2(H)$, where H is a quaternion algebra, and interpreting the involution (A, σ) as a (-1)-hermitian form of rank 2 over (H, τ), where τ is the standard involution, and similarly for (A', σ'), then invoking the result of Parimala, Sridharan, and Suresh [89] that the involutions remain anisotropic over $k(\mathrm{SB}(H))$, hence over F, since $F/k(\mathrm{SB}(H))$ is purely transcendental.

To finish part *(iii)*, we need only deal with the case of index 4 and trivial discriminant, in which case (A.2) implies an equality of two Klein four subgroups of the Brauer group. By the fundamental relations for Clifford algebras [67, Thm. 9.14], we have the equality of Brauer classes $[A] = [C_0^+] + [C_0^-]$, and we can rule out $[A] = [C_0^\pm]$ since $\mathrm{ind}(A) = 4$ while $\mathrm{ind}(C_0^\pm) \leq 2$ (in fact, we see that $\mathrm{ind}(C_0^\pm) = 2$), and similarly for A'. We deduce that A and A' are each the unique element of index 4 in their respective Klein four Brauer group kernels. Hence A and A' are Brauer equivalent, thus are k-isomorphic since they have the same degree. Also, the unordered pairs of Clifford algebra components C_0^+ and C_0^-, associated to A and A', are isomorphic. Thus, by the classification of quadratic pairs of degree 4 and trivial discriminant [67, §15.B], both (A, σ) and (A', σ') are isomorphic to $(C_0^+, \tau_0^+) \otimes (C_0^-, \tau_0^-)$, where $\tau_0^\pm$ is the standard involution on the quaternion algebra $C_0^\pm$. □

The following result, in characteristic $\neq 2$, can be seen as a consequence of general hyperbolicity results for orthogonal involutions due to Karpenko [63]. The following direct argument in the case of degree 4 algebras, using the results of [67, §15.B], was communicated to us by Anne Quéguiner-Mathieu and works over any field.

Lemma A.3. *Let σ_1 and σ_2 be quadratic pairs on a central simple algebra A of degree 4 over a field k and let $F = k(\mathrm{SB}(A))$. If σ_1 and σ_2 become isomorphic quadratic pairs over F then they are isomorphic over k.*

Proof. Since (A, σ_1) and (A, σ_2) become isomorphic over F, their discriminants coincide over F, hence coincide over k, since the map $H^1(k, \mathbb{Z}/2\mathbb{Z}) \to H^1(F, \mathbb{Z}/2\mathbb{Z})$ is injective. Let K/k be the discriminant extension. By the low dimension classification of algebras of degree 4 with quadratic pair [67, §15.B], we have that $(A, \sigma_i) = N_{K/k}(H_i, \tau_i)$, where τ_i is the standard involution on the quaternion algebra H_i over K.

Over K, we get $(A, \sigma_i)_K = (H_i, \tau_i) \otimes ({}^\iota H_i, {}^\iota\tau_i)$, where ι is the the non trivial automorphism of K/k. Let KF be the compositum of K and F. Over KF, we have that H_i and ${}^\iota H_i$ are isomorphic, hence $A \cong \mathrm{End}(H_i)$ and σ_i is adjoint to the quadratic norm form of H_i. Therefore, if the quadratic pairs σ_i are isomorphic over F, then the norm forms of H_i over KF are isomorphic. In particular, $H_1 \otimes H_2$ is split over KF. Hence, either $H_1 \otimes H_2$ is already split over K, or, by Amitsur's theorem, $H_1 \otimes H_2 = H_1 \otimes {}^\iota H_1$. Thus H_2 is either isomorphic to H_1 or ${}^\iota H_1$. In both cases, we get an isomorphism of quadratic pairs σ_1 and σ_2. □

In the remaining cases, we will classify all possible nonisomorphic birational involution surface partners in Proposition C.3.

A.3. **Degree 6.** Let S be a degree 6 non-k-rational del Pezzo surface. Then, as recalled in Table 6, there are two possible types of elementary links. Consider S_{k^s}, and recall that there are six exceptional lines, coming into two triples of non-intersecting lines. Each of these triples gives a map $S_{k^s} \to \mathbb{P}^2_{k^s}$, so that we have two ways of blowing down S_{k^s} to $\mathbb{P}^2_{k^s}$. The previous considerations show that S_{k^s} can be seen as the resolution of a standard quadratic transformation. Let H' and H denote the pull back of the generic line from each of the $\mathbb{P}^2$, and let $\{L_i\}$ and $\{L'_i\}$ be the two sets of exceptional divisors.

In particular, we have two $\mathbb{Z}$-bases for $\mathrm{Pic}(S_{k^s})$, one given by H and the L_i, and the other given by H' and the L'_i. We have $H' = 2H - L_1 - L_2 - E$, and $L'_i = H - L_j - L_k$ for $i \neq j \neq k \neq i$. If $\phi : S \dashrightarrow S'$ is an elementary link, we suppose that we have chosen the triple L_i (and hence L'_i) to describe $\overline{\phi}$ as coming from a homaloidal system on $\mathbb{P}^2_{k^s}$.

The link $M_{6,2}$ base changes to the following diagram (we omit the overlines for ease of notations):

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ S \arrow[rr, dashed, "\phi"] \arrow[d, "\sigma_0"'] & & S \arrow[d, "\tau_0"] \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi_0"] & & \mathbb{P}^2 \end{tikzcd}$$

where σ_0 blows up three points p_1, p_2, and p_3, and σ blows up two points p_4 and p_5. Hence ϕ_0 is resolved by blowing up 5 points. In this case, we should calculate which one of the five points has coefficient 2 in the homaloidal system of ϕ_0. Let us then denote $H = \sigma^*\sigma_0^*\mathscr{O}(1)$, $G = \tau^*\tau_0^*\mathscr{O}(1)$, L_i the exceptional divisor over p_i, and F_i the exceptional divisor over q_i.

The matrix $M_{6,2}$ in Table 6 is the transformation matrix from $\sigma^*\omega_S, E$ to $\tau^*\omega_{S'}, F$. Since $\omega_S = -3H + L_1 + L_2 + L_3$ and $\omega_{S'} = -3G + F_1 + F_2 + F_3$, we get the following conditions:

$$\begin{cases} 3G - F_1 - F_2 - F_3 = 6H - 2L_1 - 2L_2 - 2L_3 - 3L_4 - 3L_5 \\ F_4 + F_5 = 3H - L_1 - L_2 - L_3 - 2L_4 - 2L_5. \end{cases} \tag{A.3}$$

Since X is a del Pezzo of degree 4, there are only 16 exceptional lines on X, which can be described, in the base H, L_i as follows:

- The five exceptional lines L_i,
- The ten strict transforms $L_{i,j}$ of the lines through two of the p_i's. They are of the form $H - L_j - L_i$ for any $i \neq j$.
- The strict transform D of the conic through the p_i's. It is of the form $2H - \sum_{i=1}^{r} L_i$.

Using the above description (and the fact that F_i is not of type L_j), it is easy to check that (up to switching 4 and 5) we get that $F_5 = D$ and $F_i = L_{i,5}$ for $i \neq 5$. Hence the homaloidal system of ϕ_0 is $3H - L_1 - \ldots - L_4 - 2L_5$, which is indeed the only linear system with 5 base points. Our calculation aims at finding the point with coefficient -2. Notice that while we could have switched 4 and 5, the coefficients of L_1, L_2, and L_3 must be 1.

The link $M_{6,3}$ base changes to the following diagram (we omit the overlines for ease of notations):

$$\begin{tikzcd} & X \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ S \arrow[rr, dashed, "\phi"] \arrow[d, "\sigma_0"'] & & S \arrow[d, "\tau_0"] \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi_0"] & & \mathbb{P}^2 \end{tikzcd}$$

where σ_0 blows-up three points p_1, p_2 and p_3, and σ blows-up three points p_4, p_5 and p_6. Hence ϕ_0 is resolved by blowing-up 6 points. In this case we have two possibilities for the homaloidal system of ϕ_0, and we appeal to the form of the matrix $M_{6,3}$ to understand which one we are indeed considering. Let us then denote by $H := \sigma^*\sigma_0^*\mathscr{O}(1)$, and $G := \tau^*\tau_0^*\mathscr{O}(1)$, by L_i the exceptional divisor over p_i, and by F_i the exceptional divisor over q_i.

The matrix $M_{6,3}$ in Table (6) is the transformation matrix from $(\sigma^*\omega_S, E)$ to $(\tau^*\omega_{S'}, F)$. Since $\omega_S = -3H + L_1 + L_2 + L_3$ and $\omega_{S'} = -3G + F_1 + F_2 + F_3$, we get the following conditions:

$$\begin{cases} 3G - F_1 - F_2 - F_3 = 9H - 3L_1 - 3L_2 - 3L_3 - 4L_4 - 4L_5 - 4L_6 \\ F_4 + F_5 + F_6 = 6H - 2L_1 - 2L_2 - 2L_3 - 3L_4 - 3L_5 - 3L_6. \end{cases} \tag{A.4}$$

Since X is a del Pezzo of degree 3, there are only 27 exceptional lines on X, which can be described, in the basis (H, L_i), as follows:

- The six exceptional lines L_i.
- The fifteen strict transforms $L_{i,j}$ of the lines through two of the p_i's. They are of the form $H - L_j - L_i$ for any $i \neq j$.

- The six strict transforms D_j of the conic through the p_i's for $i \neq j$. It is of the form $2H - \sum_{i\neq j} L_i$.

Now we use that ϕ_0 is a birational map resolved by a cubic surface, we have hence two possibilities to write G in the basis H, L_i. The first one is $G = 4H-2L_1-2L_2-2L_3-L_4-L_5-L_6$, in which case ϕ_0 is the map we called ϕ_4 above, and we observed that this map is the composition of two standard quadratic tranformations. It is easy to check that in this case S would not be minimal, because it would admit a birational morphism onto a Severi–Brauer surfce.

We are left with the case where $G = 5H - \sum_{i=1}^{6} 2L_i$, in which case ϕ_0 is the map we called ϕ_5 above. Using the above description of the exceptional lines on the cubic and the action of the matrix $M_{3,6}$ (and the fact that F_i is not of type L_j), it is easy to check that (up to internal permutations of $1, 2, 3$ and of $4, 5, 6$) we get that $F_1 = D_4$, $F_2 = D_5$, and $F_3 = D_6$, while $F_4 = D_1$, $F_5 = D_2$, and $F_6 = D_3$.

Hence, if S is minimal with a point of degree 3, the birational map ϕ corresponds over k^s to the homaloidal system of quintics passing twice to six points in general position in $\mathbb{P}^2_{k^s}$, three of which are Galois-conjugate and correspond to the closed point of degree 3 on S.

Appendix B. Links of type I, and minimal del Pezzo surfaces of "conic bundle type"

Let S be a minimal non-rational del Pezzo surface, which is not deg-rigid. Then S has either degree 8 and a point of degree 2, or degree 4 and a point of degree 1. In both cases, blowing up the given point gives a conic bundle $S' \to C$ over a conic, of degree either 6 or 3, respectively.

In this appendix, we would like to show how these two special cases should be thought of, from a derived categorical (or a noncommutative) point of view as conic bundles instead of del Pezzo surfaces. The main point is describing a semiorthogonal decomposition of S whose nonrepresentable components can be seen as the "natural" nonrepresentable components of some conic bundle S'. First of all, let us recall a result of Kuznetsov on the derived category of a conic bundle, as a special case of a quadric fibration (see [71], and [11] for a statement over any field).

Proposition B.1. *Let $\pi : X \to C$ be a conic bundle over a genus zero curve, A the Azumaya algebra associated to C, and C_0 the even Clifford algebra associated to π. Then there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k, A), \mathsf{D}^{\mathrm{b}}(C, C_0)\rangle,$$

where the two first components are the pull-back via π of the natural semiorthogonal decomposition of $\mathsf{D}^{\mathrm{b}}(C)$. In particular, the first one is generated by $\mathscr{O}_X = \pi^*\mathscr{O}_C$ and the second one by the local form of $\pi^*\mathscr{O}_C(1)$.

In particular, given a conic bundle, we have two natural potentially nonrepresentable components, one of which is representable in dimension 0 if and only if $C \simeq \mathbb{P}^1$.

Del Pezzo of degree 4 with a rational point. Let S be a degree 4 del Pezzo surface with a rational point. Any such surface can be realized in $\mathbb{P}^4$ as an intersection of two quadrics. In Theorem 5.7, we recalled how $\mathsf{A}_S \simeq \mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, C_0)$, where C_0 is the Clifford algebra of the quadratic form spanned by the two quadrics. Since S has a point, the fibration has a regular section, so that it can be reduced by hyperbolic splitting (see [11, §1.3]) to a conic bundle with Clifford algebra C_0' over $\mathbb{P}^1$, so that $\mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, C_0) \simeq \mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, C_0')$.

On the other hand, one can blow up the point and obtain a degree 3 surface S', with a structure of conic bundle $S' \to \mathbb{P}^1$ (see, e.g., [58, Thm. 2.6(i)]). Let us denote by B_0 the Clifford algebra of such a conic bundle.

Theorem B.2 ([11, §4])**.** *For S a del Pezzo of degree 4 with a rational point, the $\mathscr{O}_{\mathbb{P}^1}$-algebras C_0, C_0', and B_0 described above are all Morita equivalent. In particular, $\mathsf{A}_S \simeq \mathsf{D}^{\mathrm{b}}(\mathbb{P}^1, B_0)$.*

Del Pezzo of degree 8 with a degree 2 point. In the case where S is involution surface with a point of degree 2, the nonrepresentable component can also be described by a Clifford algebra over $\mathbb{P}^1$.

Theorem B.3. *Let S be a minimal nonrational del Pezzo surface of degree 8, with a closed point x of degree 2. Let $S' \to S$ the blow-up of S along x and $\pi : S' \to C$ the associated conic bundle. Write $C = \mathrm{SB}(A')$ and C'_0 for the even Clifford algebra of π. Recall the semiorthogonal decomposition*

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k, A), \mathsf{D}^{\mathrm{b}}(k, C_0)\rangle, \tag{B.1}$$

where A is the underlying degree 4 central simple algebra defining S, i.e., $S \hookrightarrow \mathrm{SB}(A)$, and C_0 is the even Clifford algebra associated to S. Then A and A' are Brauer equivalent and there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(C, C'_0) = \langle \mathsf{D}^{\mathrm{b}}(l/k), \mathsf{D}^{\mathrm{b}}(k, C_0)\rangle,$$

where l is the residual field of x.

Proof. Consider the diagram:

$$\begin{tikzcd} & S' \arrow[dl, "\sigma"'] \arrow[d, "\pi"] \\ S \arrow[r, dashed, "\phi"] & C, \end{tikzcd}$$

where ϕ is a rational map that is resolved by the blow up $\sigma : S' \to S$ of x to the conic bundle $\pi : S' \to C$. Denote by E the exceptional divisor of σ. Over k^s, we have that $C \cong \mathbb{P}^1_{k^s}$ and $S \cong \mathbb{P}^1_{k^s} \times \mathbb{P}^1_{k^s}$ and x decomposes into points x_1 and x_2. Let $G = \sigma^*\mathscr{O}(1,1)$ and $H = \pi^*\mathscr{O}(1)$.

As one can check, comparing parts a) and b) of the case $K^2 = 8$ of [58, Thm. 2.6(i)], the rational map ϕ is defined, over k^s, by the linear system $\mathscr{O}_{S_{k^s}}(1,1) - x_1 - x_2$. In particular, since π resolves ϕ, we have $H = G - L_1 - L_2$ over k^s, where $E = L_1 + L_2$. The semiorthogonal decomposition (B.1) can then be written:

$$\mathsf{D}^{\mathrm{b}}(S_{k^s}) = \langle \mathscr{O}, \mathscr{O}(1,1), \mathsf{F}\rangle,$$

where F is the block descending to $\mathsf{D}^{\mathrm{b}}(k, C_0)$. Consider the semiorthogonal decompositions of $\mathsf{D}^{\mathrm{b}}(S')$ given respectively by the blow up and by the conic bundle formulas:

$$\mathsf{D}^{\mathrm{b}}(S'_{k^s}) = \langle \mathscr{O}_{S'_{k^s}}, \mathscr{O}_{S'_{k^s}}(G), \sigma^*\mathsf{F}, \mathscr{O}_{L_1}, \mathscr{O}_{L_2}\rangle \tag{B.2}$$

Now consider the decomposition (B.2), and mutate $\sigma^*\mathsf{F}$ to the right with respect to $\langle \mathscr{O}_{L_1}, \mathscr{O}_{L_2}\rangle$, then mutate $\langle \mathscr{O}_{L_1}, \mathscr{O}_{L_2}\rangle$ to the left with respect to $\mathscr{O}_{S'_{k^s}}(G)$, then we mutate $\langle \mathscr{O}_{S'_{k^s}}(G), \sigma^*\mathsf{F}\rangle$ to the left with respect to its left orthogonal, then we mutate $\sigma^*\mathsf{F}$ to the left with respect to its left orthogonal, then we mutate $\sigma^*\mathsf{F}$ to the right with respect to its right orthogonal to arrive at a decomposition:

$$\mathsf{D}^{\mathrm{b}}(S'_{k^s}) = \langle \mathscr{O}_{S'_{k^s}}(-G + L_1 + L_2), \mathscr{O}_{S'_{k^s}}, \mathscr{O}_{S'_{k^s}}(G - L_1), \mathscr{O}_{S'_{k^s}}(G - L_2), \sigma^*\mathsf{F}\rangle \tag{B.3}$$

where we have used the evaluation sequence and the fact that $\omega_{S'_{k^s}} = \mathscr{O}_{S'_{k^s}}(-2G + L_1 + L_2)$. Rewriting in terms of H, we have the decomposition

$$\mathsf{D}^{\mathrm{b}}(S'_{k^s}) = \langle \mathscr{O}_{S'_{k^s}}(-H), \mathscr{O}_{S'_{k^s}}, \mathscr{O}_{S'_{k^s}}(H + L_2), \mathscr{O}_{S'_{k^s}}(H + L_1), \sigma^*\mathsf{F}\rangle \tag{B.4}$$

in which we can identify $\langle \mathscr{O}_{S'_{k^s}}(-H), \mathscr{O}_{S'_{k^s}}\rangle$ with $\pi^*\mathsf{D}^{\mathrm{b}}(C_{k^s})$. Since $\mathscr{O}_{S'_{k^s}}(G)$ is the mutation of $\mathscr{O}_{S'_{k^s}}(-H)$, and these mutations are defined over k, we conclude that they $\langle \mathscr{O}_{S'_{k^s}}(G)\rangle$ and $\langle \mathscr{O}_{S'_{k^s}}(-H)\rangle$ descend to equivalent categories over k. It follows that $\mathsf{D}^{\mathrm{b}}(k, A) \simeq \mathsf{D}^{\mathrm{b}}(k, A')$, hence by Corollary 1.18, A and A' are Brauer equivalent.

The block $\langle \mathscr{O}_{S'_{k^s}}(H + L_2), \mathscr{O}_{S'_{k^s}}(H + L_1)\rangle$ has tilting bundle $\mathscr{O}_{S'_{k^s}}(H + L_2) \oplus \mathscr{O}_{S'_{k^s}}(H + L_1) = \mathscr{O}_{S'_{k^s}}(H) \otimes (\mathscr{O}_{S'_{k^s}}(L_2) \oplus \mathscr{O}_{S'_{k^s}}(L_1))$. There is a vector bundle V on C of rank 2 such that $V_{k^s} = \mathscr{O}(H)^{\oplus 2}$ and there is a vector bundle W on S' of rank 2 such that $W_{k^s} = \mathscr{O}(L_1) \oplus \mathscr{O}(L_2)$. Then $\pi^*V \otimes W$ is a tilting bundle for a category base changing to the block $\langle \mathscr{O}_{S'_{k^s}}(H + L_2), \mathscr{O}_{S'_{k^s}}(H +$

$L_1)\rangle$. It follows that the later descends to a category equivalent to $\mathsf{D}^{\mathrm{b}}(k(x)/k, A)$. Since $k(x)$ is the residue field of a point of $S \subset \mathrm{SB}(A)$, we have that $k(x)$ splits the algebra A. When $\mathsf{D}^{\mathrm{b}}(k(x)/k, A) \simeq \mathsf{D}^{\mathrm{b}}(k(x)/k)$. In conclusion, we get a decomposition $\pi^*\mathsf{D}^{\mathrm{b}}(C, C_0') = \langle \mathsf{D}^{\mathrm{b}}(l/k), \mathsf{F}\rangle$ and we recall that $\mathsf{F} \simeq \mathsf{D}^{\mathrm{b}}(k, C_0)$. □

Appendix C. Links of type II between conic bundles of degree 8

In this section, we consider conic bundles of degree 8 and study their semiorthogonal decompositions under links of type II and IV. The upshot is to show that the Griffiths–Kuznetsov component is well defined in these cases, which include in particular all involution surfaces of Picard rank 2.

Let us first consider a conic bundle $\pi : S \to C$ over a Severi–Brauer curve $C = \mathrm{SB}(A)$. Such a conic bundle has an associated Clifford algebra $\mathscr{C}_0$, a locally free sheaf over C. Kuznetsov provides a semiorthogonal decomposition:

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(C), \mathsf{D}^{\mathrm{b}}(C, \mathscr{C}_0)\rangle,$$

and shows that there is a root stack structure $\widehat{C}$, obtained by the natural $\mathbb{Z}/2\mathbb{Z}$-action on points of C where the fiber is degenerate, and a Brauer class β in $\mathrm{Br}(C)$ such that $\mathscr{C}_0$ pulls back to $\widehat{C}$ to an Azumaya algebra with class β. Recall that a conic bundle over C has degree $8 - r$, where r is the number of degenerate fibers. If S has degree 8, then it has no degenerate fibers and hence $\widehat{C} \cong C$ since the root stack structure is trivial. Recall moreover that, denoting by $\alpha \in \mathrm{Br}(k)$ the class of A, there is a semiorthogonal decomposition $\mathsf{D}^{\mathrm{b}}(C) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k, \alpha)\rangle$. We finally obtain a semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k, \alpha), \mathsf{D}^{\mathrm{b}}(k, \beta), \mathsf{D}^{\mathrm{b}}(k, \alpha \otimes \beta)\rangle. \tag{C.1}$$

We can show that the nontrivial components of this decomposition are a birational invariant, which allows us to conclude that the Griffiths–Kuznetsov component is well defined.

Proposition C.1. *Suppose that $S \to C$ is a conic bundle of degree 8 and that $S_1 \dashrightarrow S$ is a birational map. Then there is a semiorthogonal decomposition*

$$\mathsf{A}_{S'} = \langle \mathsf{T}, \mathsf{D}^{\mathrm{b}}(k, \alpha), \mathsf{D}^{\mathrm{b}}(k, \alpha'), \mathsf{D}^{\mathrm{b}}(k, \alpha \otimes \beta)\rangle$$

Remark C.2. Notice that if $\alpha = 0$ (i.e. $C = \mathbb{P}^1$), we can include $\mathsf{D}^{\mathrm{b}}(k, \alpha)$ in T, and similarly for $\beta = 0$ (i.e. π has a section), we can include $\mathsf{D}^{\mathrm{b}}(k, \beta)$ in T

Proof. Notice that, over k^s, the conic bundle S_{k^s} is isomorphic to a Hirzebruch surface $\mathbb{F}_n$, that is, a $\mathbb{P}^1$-bundle $\overline{\pi} : S_{k^s} \to \mathbb{P}^1_{k^s}$. One can check that the semiorthogonal decomposition (C.1) base-changes to the semiorthogonal decomposition

$$\mathsf{D}^{\mathrm{b}}(S_{k^s}) = \langle \mathscr{O}_{S_{k^s}}, \mathscr{O}_{S_{k^s}}(F), \mathscr{O}_{S_{k^s}}(\Sigma), \mathscr{O}_{S_{k^s}}(\Sigma + F)\rangle, \tag{C.2}$$

where we denoted by F and Σ the fiber and the section of $\overline{\pi}$ respectively.

Suppose first that S_1 is minimal, so that S can be decomposed in a series of links of type either II or IV (a link of type III is a blow-down, so S_1 is not minimal). Let us first consider links of type II, which are described by Iskovskikh [58, Thm. 2;6 (ii)]. Let $\pi : S \to C$ be a conic bundle of degree 8, and

$$\begin{tikzcd} S \arrow[d] & X \arrow[l, "\sigma"'] \arrow[r, "\tau"] & S' \arrow[d] \\ C & & C' \arrow[ll, "\cong"'] \end{tikzcd}$$

where $\sigma : X \to S$ and $\tau : X \to S'$ are blow-ups in a point x and x' respectively of the same degree d, and $S' \to C'$ is a conic bundle. We denote by E and E' the exceptional divisors of the blow ups. We first notice that S' has also degree 8 and that $C' \simeq C$. The degree d of the point can be either 1, 2, or 4. It is easy to check that, over k^s, the link is just a composition of d elementary transformations of the Hirzebruch surface S_{k^s}. In particular, if there is a point of degree 1, then we obtain a birational map between S and a quadric with a rational point, so that S is already a Hirzebruch surface and there is nothing to prove. So we assume that

d is either 2 or 4. Moreover, as one can check over k^s, we have $-K_S = 2\Sigma - (n-2)F$ and $-K_{S'} = 2\Sigma' - (n+d-2)F'$. In the Picard group of X_{k^s} we have the following relations (we omit the pull-back notation):

$$F = F', \tag{C.3}$$

$$E = d\Sigma' - E', \tag{C.4}$$

$$-K_S = -K_{S'} + dF' + 2E', \tag{C.5}$$

$$\Sigma = \Sigma' - E', \tag{C.6}$$

where the last equality, is obtained combining the first and the third one. In particular, we obtain an identification $\mathscr{O}(F)$ with $\mathscr{O}(F')$ and that the the equivalence $\otimes\mathscr{O}(-E')$ of $\mathsf{D}^{\mathrm{b}}(X_{k^s})$ sends the exceptional line bundle $\mathscr{O}(\Sigma)$ to the exceptional line bundle $\mathscr{O}(\Sigma')$ and the exceptional line bundle $\mathscr{O}(\Sigma + F)$ to the exceptional line bundle $\mathscr{O}(\Sigma' + F')$. We notice that, since E' is defined over k, the latter equivalence is defined over k as well. From this we obtain equivalences $\mathsf{D}^{\mathrm{b}}(k,\alpha) \simeq \mathsf{D}^{\mathrm{b}}(k,\alpha')$ (the identity), $\mathsf{D}^{\mathrm{b}}(k,\beta) \simeq \mathsf{D}^{\mathrm{b}}(k,\beta')$ and $\mathsf{D}^{\mathrm{b}}(k,\alpha\otimes\beta) \simeq \mathsf{D}^{\mathrm{b}}(k,\alpha'\otimes\beta')$ (induced by $\otimes\mathscr{O}(-E')$).

Let us now consider links of type IV. Thanks to Iskovskikh, this is possible only in the cases where $S = S' = C \times C'$ is the product of two Severi–Brauer curves and the conic bundle structures are given by the two projections. Let α and α' in $\mathrm{Br}(k)$ be the classes of C and C' respectively. Then there are natural decompositions:

$$\begin{aligned} &\mathsf{D}^{\mathrm{b}}(C) = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k,\alpha)\rangle \qquad \mathsf{D}^{\mathrm{b}}(C,\mathscr{C}_0) = \langle \mathsf{D}^{\mathrm{b}}(k,\alpha'), \mathsf{D}^{\mathrm{b}}(\alpha\otimes\alpha')\rangle \\ &\mathsf{D}^{\mathrm{b}}(C') = \langle \mathsf{D}^{\mathrm{b}}(k), \mathsf{D}^{\mathrm{b}}(k,\alpha')\rangle \qquad \mathsf{D}^{\mathrm{b}}(C,\mathscr{C}_0') = \langle \mathsf{D}^{\mathrm{b}}(k,\alpha), \mathsf{D}^{\mathrm{b}}(\alpha'\otimes\alpha)\rangle \end{aligned} \tag{C.7}$$

It follows that the decompositions (C.1) obtained by the conic bundle structures are composed by equivalent categories.

This proves the Proposition for S_1 minimal. If S_1 is not minimal, then just consider a minimal model $S_1 \to S_0$ and use the blow-up formula. □

We finally classify birational equivalent involution surfaces based on algebraic methods. This can be seen as the algebraic interpretation of the classification of all type IV links for such surfaces.

Proposition C.3. *Let S and S' be nonrational involution varieties over a field k. Then S and S' are k-birational if and only if either they are isomorphic or $S = \mathrm{SB}(B_1)\times\mathrm{SB}(B_2)$ and $S' = \mathrm{SB}(B_1')\times\mathrm{SB}(B_2')$ and the Klein four subgroups of the Brauer group generated by B_i and B_i' coincide and consist solely of quaternion algebras.*

Proof. Let S and S' are be k-birational involution varieties associated to quadratic pairs (A,σ) and (A',σ'). The only case not ruled out by Proposition A.2 is that of S (hence S') of index 2, trivial discriminant, and with (at least) (A,σ) nonsplit. By the classification of degree 4 algebras with orthogonal involution of trivial discriminant [67, §15.B], (A,σ) is isomorphic to $(C_0^+,\tau_0^+)\otimes(C_0^-,\tau_0^-)$, where $\tau_0^\pm$ is the standard involution on the quaternion algebra $C_0^\pm$, and similarly for (A',σ'). By (A.2) and the considerations in the proof of Proposition A.2, the Klein four subgroups of $\mathrm{Br}(k)$ generated by $C_0^\pm$ and $C_0'^{\pm}$ coincide and consist solely of quaternion algebras (since A and A' have index at most 2).

Now we verify that if B_1, B_2 and B_1', B_2' are Brauer classes generating the same Klein four subgroup consisting of classes of index at most 2, then $S = \mathrm{SB}(B_1)\times\mathrm{SB}(B_2)$ and $S' = \mathrm{SB}(B_1')\times\mathrm{SB}(B_2')$ are k-birational. If the unordered pairs (B_1,B_2) and (B_1',B_2') are isomorphic then $S \cong S'$. Otherwise, without loss of generality, we can write $B_1' = B_1$ and $B_2' = H$, where $B_1\otimes B_2 \cong M_2(H)$. However, since $[H] = [B_1]+[B_2] \in \mathrm{Br}(k)$ and $\ker(\mathrm{Br}(k)\to\mathrm{Br}(\mathrm{SB}(B_1))) = \langle [B_1]\rangle$, the images of $[H]$ and $[B_2]$ coincide in $\mathrm{Br}(k(\mathrm{SB}(B_1)))$, and thus we conclude that $k(S') = k(\mathrm{SB}(B_1))\otimes k(\mathrm{SB}(H)) \cong k(\mathrm{SB}(B_1))\otimes k(\mathrm{SB}(B_2)) = k(S)$ so that S and S' are k-birational. □

As a special case, this result says that the involution surfaces $C\times C$ and $\mathbb{P}^1\times C$ are k-birational for a any Severi–Brauer curve C over k.

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Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, CT 06511
E-mail address: asher.auel@yale.edu

Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 9, France
E-mail address: marcello.bernardara@math.univ-toulouse.fr