1. Some basic constructions with central simple algebras of degree 2.

(a) The classical adjoint \( \alpha : M_2 \rightarrow M_2 \) defined by

\[
\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

is a \( k \)-linear determinant-preserving involution of symplectic type. Prove that \( SL_2 = Sp(M_2, \alpha) \). Recall that the symplectic group \( Sp(A, \sigma) \) of an algebra \( (A, \sigma) \) with symplectic involution has group of \( R \)-points \( \{ x \in A \otimes_k R : x \sigma(x) = 1 \} \) for \( R \in \text{Alg}_k \). Prove that \( PSL_2 \cong PGL_2 \) is isomorphic to the group scheme \( \text{Aut}(M_2, \alpha) \) of algebra automorphisms of \( M_2 \) preserving \( \alpha \).

(b) Let \( A \) be a central simple algebra of degree 2 over \( k \). Recall the reduced norm \( N_{A/k} : A \rightarrow k \) and reduced trace \( T_{A/k} : A \rightarrow k \). Define the standard involution \( \sigma_A \) of \( A \) by \( \sigma_A(x) = T_{A/k}(x) 1_A - x \). Prove that \( x \sigma_A(x) = N_{A/k}(x) \) for \( x \in A \). (Hint: Remember that the reduced norm and trace were defined as coefficients of the characteristic polynomial of \( x \) in \( A \).) Prove that \( \sigma_A \) is an involution of symplectic type on \( A \). (Hint: Extending scalars so that \( A \) becomes isomorphic to \( M_2 \), what does \( \sigma_A \) become?) In fact, it’s the unique such involution!

2. In this problem you will give a classification of forms of the semisimple linear algebraic group \( SL_2 \times SL_2 \).

(a) Compute the automorphism group scheme \( \text{Aut}(SL_2 \times SL_2) \). Clearly, the map \( s \) switching the factors of \( SL_2 \) is an outer automorphism. Are there others?

(b) Prove that \( \text{Aut}(SL_2 \times SL_2) \) is isomorphic to the group scheme \( \text{Aut}(M_2 \times M_2) \) of algebra automorphisms of \( M_2 \times M_2 \). (Hint: How does an algebra automorphism of \( M_2 \times M_2 \) restrict to its center?)

(c) Conclude that the set of isomorphism classes of forms of \( SL_2 \times SL_2 \) are in bijection with set of isomorphism classes of pairs \( (A, K) \) where \( K \) is an étale quadratic algebra of degree 2 over \( k \) and \( A \) is a central simple algebra of degree 2 over \( K \). Given \( (A, K) \) what is the associated form of \( SL_2 \times SL_2 \)? Use the inner automorphism exact sequence

\[ 1 \rightarrow \text{Inn}(SL_2 \times SL_2) \rightarrow \text{Aut}(SL_2 \times SL_2) \rightarrow \text{Out}(SL_2 \times SL_2) \rightarrow 1 \]

to describe the inner forms. Notice that the map \( H^1(k, \text{Inn}(SL_2 \times SL_2)) \rightarrow H^1(k, \text{Aut}(SL_2 \times SL_2)) = \text{Forms}(SL_2 \times SL_2) \) is not injective!

(d) Describe the forms when \( k = \mathbb{R} \).
3. In this problem, you can assume, for simplicity, that the characteristic of $k$ is $\neq 2$.
Consider the determinant map as a quadratic form $\det : M_2 \to k$ on the space of $2 \times 2$ matrices. Let $\text{SO}(M_2, \det)$ be its special orthogonal groups. Over $\mathbb{R}$, this is called $\text{SO}_{2,2}$.

(a) Prove that the classical adjoint $\alpha : M_2 \to M_2$ defines an element of $\text{O}(M_2, \det)$ that is not in $\text{SO}(M_2, \det)$. Conclude that (conjugation by) $\alpha$ generates the outer automorphism group $\text{Out}(\text{SO}(M_2, \det))$.

(b) Show that the map
$$\text{SL}_2 \times \text{SL}_2 \longrightarrow \text{SO}(M_2, \det)$$
defined on $R$-points by $(A, B) \mapsto (X \mapsto AXB^{-1})$ for $R \in \text{Alg}_k$, is a central isogeny with kernel the diagonally embedded $\mu_2$. Conclude that this map yields an isomorphism of group schemes $\text{SL}_2 \times \text{SL}_2 \cong \text{Spin}(M_2, \det)$.

(c) Prove that the above map induces an isomorphism
$$\text{Aut}(\text{SL}_2 \times \text{SL}_2) \longrightarrow \text{Aut}(\text{SO}(M_2, \det)).$$
First, show that the diagonal $\mu_2$ is fixed by all automorphisms of $\text{SL}_2 \times \text{SL}_2$; this induces a homomorphism $\text{Aut}(\text{SL}_2 \times \text{SL}_2) \to \text{Aut}(\text{SO}(M_2, \det))$. (Hint: To show this is an isomorphism, match up the inner automorphism exact sequences for the two groups; on the level of outer automorphisms, verify that the switch map is taken to conjugation by the classical adjoint.) As an interesting side note, conclude that the explicit map $\varphi \mapsto (X \mapsto \varphi(X^t)^t)$ is an outer automorphism of $\text{SO}(M_2, \det)$.

(d) Describe the resulting bijection (from taking nonabelian $H^1$)
$$\text{Forms}(\text{SL}_2 \times \text{SL}_2) \longrightarrow \text{Forms}(\text{SO}(M_2(k), \det))$$
using our previous description of forms of $\text{SL}_2 \times \text{SL}_2$ and the description of forms for a special orthogonal group given in class in terms of central simple algebras (here of degree 4) with orthogonal involution.

You will need the following construction. Given $(A, K)$ as before, let $\tau$ be the nontrivial element of the Galois group of $K/k$, and define $^\tau A$ to be the same underlying $k$-algebra as $A$ but with the $K$-action twisted by $\tau$. The naive switch map on $A \otimes_K ^\tau A$ is a $\tau$-semilinear algebra automorphism. Define the algebra norm $N_{K/k} A$ of $A$ from $K$ down to $k$ to be the $k$-subalgebra of elements of $A \otimes_K ^\tau A$ invariant under the naive switch map. This turns out to be a central simple algebra of degree 4 over $k$. You should verify that restricting the involution $\sigma_A \otimes \sigma_A^\tau$ from $A \otimes_K ^\tau A$ to $N_{K/k} A$ yields an involution of orthogonal type. This verification can be performed over the separable closure of $k$, where $K$ is split and $A$ becomes isomorphic to $M_2 \times M_2$. When $K = k \times k$ and $A = A' \times A''$, then this construction yields $(A' \otimes_k A'', \sigma_{A'} \otimes \sigma_{A''})$.

(e) Finally, describe this bijection when $k = \mathbb{R}$. Recall that a quadratic form over $\mathbb{R}$ is uniquely determined up to isometry by its dimension and signature.