1. Let $G$ be a finite cyclic group of order $n$ and fix a generator $\sigma$. Let $A$ be a $G$-module (i.e., abelian group with $G$-action). Consider the maps $N : A \to A$ and $\sigma - 1 : A \to A$ defined by

\[ N(x) = \sum_{i=0}^{n-1} \sigma^i(x) \quad \text{and} \quad (\sigma - 1)(x) = \sigma(x) - x. \]

(a) Verify that the $\mathbb{Z}[G]$-module $\mathbb{Z}$ has a free resolution

\[ \cdots \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma - 1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma - 1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0 \]

where $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is the usual augmentation map or counit sending every group element to 1. It might help to learn about homotopy retractions.

(b) Show that this resolution gives the following periodicity on the level of cohomology

\[ H^0(G, A) = A^G \quad \text{and} \quad H^i(G, A) = \begin{cases} \mathbb{N}A/(\sigma - 1)A & \text{if } i \text{ is odd} \\ A^G/NA & \text{if } i \text{ is even} \end{cases} \]

for $i > 0$, where $\mathbb{N}A = \ker(N : A \to A)$.

(c) Give formulas for $H^i(G, A)$ when $G$ acts trivially on $A$.

2. Let $L/K$ be a finite Galois extension with cyclic Galois group $G$.

(a) Use the cohomology of cyclic groups to show that the cohomological form of Hilbert’s Theorem 90, namely $H^1(G, L^\times) = 1$, is equivalent to the classical form: that $x \in L^\times$ satisfies $N_{L/K}(x) = 1$ if and only if $x = \sigma(y)/y$ for some $y \in L^\times$.

(b) Recall that $\text{Br}(L/K) = \ker(\text{Br}(K) \to \text{Br}(L))$. Use the cohomology of cyclic groups to prove that $\text{Br}(L/K) \cong K^\times/N_{L/K}(L^\times)$.

(c) Let $K$ have characteristic $\neq 2$ and $L = K(\sqrt{a})$ a quadratic extension of $K$. Prove that every 2-torsion element of $\text{Br}(L/K)$ is represented by a quaternion algebra of the form $(a, b)$ for some $b \in K^\times$ and that $(a, b) \cong (a, b')$ if and only if $b/b = x^2 - ay^2$ for $x, y \in K$. Explicitly classify the 2-torsion elements in $\text{Br}(\mathbb{Q}(i)/\mathbb{Q})$. 

Guidelines. You may use any external sources, but work by yourself.

Notations. A global field is a finite extension of either $\mathbb{Q}$ or $\mathbb{F}_q(t)$. A quadratic form $q$ over a field $K$ is called isotropic if it has a nontrivial zero defined over $K$, and otherwise, is called anisotropic.
3. Let $K$ be a field of characteristic 0 with algebraic closure $\overline{K}$. Assume that the absolute Galois group $G_K = \text{Gal}(\overline{K}/K)$ is cyclic of prime order $p$.

(a) Prove that $\text{Br}(K) \cong K^\times/K^{\times p}$.

**Hint.** Use the long exact sequence in Galois cohomology associated to the Kummer sequence, along with Hilbert’s Theorem 90, and the cohomology of cyclic groups.

(b) Conclude that $N_K/K(K^\times) = K^{\times p}$ and hence that the only possibility is $p = 2$ and $\overline{K} = K(\sqrt{-1})$.

**Hint.** Show that $K$ contains a primitive $p$th root of unity (if not try adjoining it), hence that the cyclic extension $\overline{K}/K$ is a Kummer extension.

(c) Show that declaring the squares in $K^\times$ to be positive will equip $K$ with the structure of an ordered field.

(d) (Artin–Schreier) Prove that if $K$ is a field of characteristic 0 whose absolute Galois group is a nontrivial finite group, then $\overline{K} = K(\sqrt{-1})$ and $K$ is an ordered field where the squares are positive.

**Hint.** Take a $p$-Sylow subgroup of the Galois group and use the fact that $p$-groups are solvable, then iteratively apply the previous results.

**Remark.** Such fields are called **real closed**. In fact, Artin and Schreier proved that in positive characteristic, the absolute Galois group is either trivial or infinite.

4. Let $K$ be a global field of characteristic $\neq 2$ and $\Omega_K$ its set of places.

(a) Let $q$ be a nondegenerate quadratic form in 3 variables over $K$. Prove that if $q$ is isotropic over $K_v$ for all but possibly a single $v \in \Omega_K$, then $q$ is isotropic over $K$.

(b) Let $q$ be a nondegenerate quadratic form in 4 variables over $K$ with square discriminant. Prove that if $q$ is isotropic over $K_v$ for all but possibly a single $v \in \Omega_K$, then $q$ is isotropic over $K$.

(c) Consider the quadratic form $q = x_1^2 + x_2^2 + x_3^2 + 7x_4^2$ defined over $\mathbb{Q}$. Prove that $q$ is isotropic over $\mathbb{Q}_p$ for every prime $p$, but that $q$ is anisotropic over $\mathbb{R}$.