1. Quadratic forms over finite fields. Let $q$ be an odd prime power.
   (a) Show that $\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}$ has order 2.
   (b) Show than any element of $\mathbb{F}_q$ is a sum of two squares.
   (c) Show that every binary quadratic form over $\mathbb{F}_q$ represents every nonzero element.
   (d) Show that if $q \equiv 1 \pmod{4}$ then $W(\mathbb{F}_q)$ is isomorphic to the ring $\mathbb{Z}/2\mathbb{Z}[\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}]$.
   (e) Show that if $q \equiv 3 \pmod{4}$ then $W(\mathbb{F}_q)$ is isomorphic to the ring $\mathbb{Z}/4\mathbb{Z}$.
   (f) Show that the isomorphic type of $GW(\mathbb{F}_q)$ as a ring does not depend on $q$.

2. Characteristic 2, scary! Let $F$ be a field of characteristic 2 and $a, b \in F$. A quadratic form $q : V \to F$ is nondegenerate if its associated bilinear form $b_q : V \times V \to F$ defined by $b_q(v, w) = q(v + w) - q(v) - q(w)$ has a radical of dimension at most 1. Define the quadratic form $[a, b]$ on $F^2$ by $(x, y) \mapsto ax^2 + xy + by^2$. Let $\mathbb{H}$ be the hyperbolic form on $F^2$ defined by $(x, y) \mapsto xy$.
   (a) Prove that $\langle a \rangle$ is nondegenerate for any $a \in F^\times$ but that $\langle a, b \rangle$ is always degenerate.
   (b) Prove that $[a, b]$ is nondegenerate for any $a, b \in F$.
   (c) Prove that any nondegenerate quadratic form of dimension 2 over $F$ is isometric to a binary quadratic form $[a, b]$ for some $a, b \in F$.
   (d) Prove that $\mathbb{H} \cong [0, 0] \cong [0, a]$ for any $a \in F$.
   (e) Let $\varphi : F \to F$ be the Artin–Schreier map $x \mapsto x^2 + x$. For $a \in F$ prove that $[1, a]$ is isotropic if and only if $a \in \varphi(F)$. The group $F/\varphi(F)$ plays the role of the group of square classes.
   (f) Prove that a nondegenerate quadratic form of dimension 2 over $F$ is isometric to $\mathbb{H}$ if and only if it is isotropic.
   (g) Let $q$ be a nondegenerate quadratic form over $F$. Prove that $q$ is isotropic if and only if $q \cong \mathbb{H} \perp q'$.
   (h) Prove that any nondegenerate quadratic form over $F$ can be written as an orthogonal sum
   \[
   \bigoplus_{i=1}^m [a_i, b_i] \quad \text{or} \quad \langle c \rangle \perp \bigoplus_{i=1}^m [a_i, b_i]
   \]
   depending on whether the dimension is even or odd. This is “diagonalization” in characteristic 2.
3. *Prime ideals in the Witt ring.* Let $F$ be a field of characteristic not 2 and $P$ a prime ideal of the Witt ring $W(F)$. Prove the following.

(a) If $W(F)/P$ has characteristic zero, then $W(F)/P \cong \mathbb{Z}$.

(b) If $W(F)/P$ has characteristic $p > 0$, then $W(F)/P \cong \mathbb{Z}/p\mathbb{Z}$.

(c) If $W(F)/P$ has characteristic 2, then $P = I$ is the fundamental ideal.

(d) If $P$ is not the fundamental ideal, then \( \{ a \in F \mid \langle a \rangle \equiv 1 \pmod{P} \} \) is the set of positive elements of an ordering on $F$.

**Hint.** For any $a \in F^\times$, prove the relation $(\langle a \rangle + 1)(\langle a \rangle - 1) = 0$ in $W(F)$. 