Problem Set # 8 (due in class on Thursday April 5)

**Notation:** You can use the fact, which we will prove later, that a finite extension is separable and normal if and only if it is Galois.

**Reading:** GT 9, 17.4–17.5.

**Problems:**

1. Let $K/F$ be a finite extension of fields of characteristic $p > 0$. Prove that $\alpha \in K$ is separable over $F$ if and only if $F(\alpha) = F(\alpha^p)$. **Hint.** If $\alpha \in K$ is inseparable over $F$, find a nonzero $F(\alpha^p)$-derivation on $F(\alpha)$ to deduce something useful about the extension $F(\alpha)/F(\alpha^p)$. Conversely, what would the minimal polynomial of $\alpha \in K$ be over $F(\alpha^p)$.

2. Let $K/F$ be an algebraic extension of fields of characteristic $p > 0$. Prove that the following are equivalent.
   
   (a) Every element $\alpha \in K \setminus F$ is inseparable over $F$.
   
   (b) For every $\alpha \in K$, there exists $n \geq 1$ such that $\alpha^{p^i} \in F$.

   We call such extensions $K/F$ purely inseparable. **Warning:** Just because $\alpha \in K$ is inseparable over $F$, it does not mean that every element of $F(\alpha)$ is inseparable over $F$. You might try to even find an example just to make sure!

3. Let $K/F$ be a finite extension of fields of characteristic $p > 0$. Prove that $K/F$ is purely inseparable if and only if $K = F(\alpha_1, \ldots, \alpha_n)$ and for each $1 \leq i \leq n$ there exists $n_i \geq 1$ such that $\alpha_i^{p^{n_i}} \in F$. **Remark.** We might call the elements $\alpha \in K$ that satisfy condition (b) in Problem 2 “purely inseparable” elements. In comparison to the warning in Problem 2, an extension generated by purely inseparable elements is actually purely inseparable.

   Prove that $\mathbb{F}_p(t)[x]/(x^p - t)$ is a purely inseparable extension of $\mathbb{F}_p(t)$ of degree $p$.

4. Let $K/F$ be a finite extension. Prove that there exists an intermediate extension $K/M/F$ such that $M/F$ is separable and $K/M$ is purely inseparable. **Hint.** Use the condition (b) in Problem 2 to construct $M$.

5. Let $K/F$ be a finite Galois extension, and $F'/F$ be any extension. Let $K' = K.F'$ be the compositum of $K$ and $F'$. Prove that $K'/F'$ is a Galois extension whose Galois group is isomorphic to a subgroup of $\text{Gal}(K/F)$.

6. Let $\gamma = \sqrt{2} + \sqrt{2} \in \mathbb{R}$.
   
   (a) Show that $\mathbb{Q}(\gamma)/\mathbb{Q}$ is Galois with cyclic Galois group.
   
   (b) Show that $\mathbb{Q}(\gamma, i) = \mathbb{Q}(\zeta_{16})$ and is Galois over $\mathbb{Q}$.

7. The 12th roots of unity. Let $\zeta = \zeta_{12}$.
   
   (a) Prove that $x^4 - x^2 + 1$ is the minimal polynomial of $\zeta$ over $\mathbb{Q}$ and that the other zeros are $\zeta^5, \zeta^7, \zeta^{11}$.
   
   (b) Prove that $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension and that there is an isomorphism of groups
   
   $$\left(\mathbb{Z}/12\mathbb{Z}\right) \to \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$$
   
   $$j \mapsto (\varphi_j : \zeta \mapsto \zeta^j)$$

   so that the Galois group is a Klein four group.